ON SYMMETRIC GROUP $S_3$ ACTIONS ON SPIN 4-MANIFOLDS

XIMIN LIU and HONGXIA LI

Abstract. Let $X$ be a smooth, closed, connected spin 4-manifold with $b_1(X) = 0$ and non-positive signature $\sigma(X)$. In this paper we use Seiberg-Witten theory to prove that if $X$ admits an odd type symmetric group $S_3$ action preserving the spin structure, then $b_2^+(X) \geq |\sigma(X)|/8 + 3$ under some non-degeneracy conditions. We also obtain some information about $\text{Ind}_{\tilde{S}_3} D$, where $\tilde{S}_3$ is the extension of $S_3$ by $\mathbb{Z}_2$.

1. Introduction

Let $X$ be a smooth, closed, connected spin 4-manifold. We denote by $b_2(X)$ the second Betti number and denote by $\sigma(X)$ the signature of $X$. In [11], Y. Matsumoto conjectured the following inequality

$$b_2(X) \geq \frac{11}{8} |\sigma(X)|.$$ (1)

This conjecture is well known and has been called the $\frac{11}{8}$-conjecture. All complex surfaces and their connected sums satisfy the conjecture (see [13]). From the classification of unimodular even integral quadratic forms and the Rochlin’s theorem, for the choice of orientation with non-positive signature the intersection form of a closed spin 4-manifold $X$ is

$$-2kE_8 \oplus mH, \quad k \geq 0,$$

where $E_8$ is the $8 \times 8$ intersection form matrix and $H$ is the hyperbolic matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ (2)

Thus, $m = b_2^+(X)$ and $k = -\sigma(X)/16$ and so the inequality (1) is equivalent to $m \geq 3k$. Since $K3$ surface satisfies the equality with $k = 1$ and $m = 3$, the coefficient $\frac{11}{8}$ is optimal, if the $\frac{11}{8}$-conjecture is true.

Donaldson has proved that if $k > 0$ then $m \geq 3$ [4]. In early 1995, using the Seiberg-Witten theory introduced by Seiberg and Witten [15], Furuta [7] proved
that

\[ b_2(X) \geq \frac{5}{4} |\sigma(X)| + 2. \]

This estimate has been dubbed the \( \frac{10}{8} \)-theorem. In fact, if the intersection form of \( X \) is definite, i.e., \( m = 0 \), then Donaldson proved that \( b_2(X) \) and \( \sigma(X) \) are zero [4, 5]. Thus, Furuta assumed that \( m \) is not zero. Inequality (2) follows by a surgery argument from the non-positive signature, \( b_1(X) = 0 \) case:

**Theorem 1.1** (Furuta [7]). Let \( X \) be a smooth spin 4-manifold with \( b_1(X) = 0 \) with non-positive signature. Let \( k = -\sigma(X)/16 \) and \( m = b_2^+(X) \). Then,

\[ 2k + 1 \leq m \]

if \( m \neq 0 \).

His key idea is to use a finite dimensional approximation of the monopole equation. Later Furuta and Kametani [7] used equivariant \( c \)-invariants and improved the above \( \frac{10}{8} \)-theorem as following.

**Theorem 1.2** (Furuta and Kametani [7]). Suppose that \( X \) is a closed oriented spin 4-manifold. If \( \sigma(X) < 0 \),

\[
\begin{align*}
\text{if } -\sigma(X)/16 \equiv 0 \text{ mod } 4, & \\
\text{if } -\sigma(X)/16 \equiv 2 \text{ mod } 4, & \\
\text{if } -\sigma(X)/16 \equiv 3 \text{ mod } 4, &
\end{align*}
\]

The above inequality was also proved by N. Minami [12] by using an equivariant join theorem to reduce the inequality to a theorem of Stolz [14].

Throughout this paper we will assume that \( m \) is not zero and \( b_1(X) = 0 \), unless stated otherwise.

A \( \mathbb{Z}/2^p \)-action is called a spin action if the generator of the action \( \tau : X \to X \) lifts to an action \( \tilde{\tau} : P_{\text{Spin}} \to P_{\text{Spin}} \) of the Spin bundle \( P_{\text{Spin}} \). Such an action is of even type if \( \tilde{\tau} \) has order \( 2^p \) and is of odd type if \( \tilde{\tau} \) has order \( 2^p + 1 \).

In [2], Bryan (see also [6]) used Furuta’s technique of “finite dimensional approximation” and the equivariant \( K \)-theory to improve the above bound by \( p \) under the assumption that \( X \) has a spin odd type \( \mathbb{Z}/2^p \)-action satisfying some non-degeneracy conditions analogous to the condition \( m \neq 0 \). More precisely, he proved

**Theorem 1.3** (Bryan [2]). Let \( X \) be a smooth, closed, connected spin 4-manifold with \( b_1(X) = 0 \). Assume that \( \tau : X \to X \) generates a spin smooth \( \mathbb{Z}/2^p \)-action of odd type. Let \( X_i \) denote the quotient of \( X \) by \( \mathbb{Z}/2^i \subset \mathbb{Z}/2^p \). Then

\[ 2k + 1 + p \leq m \]

if \( m \neq 2k + b_2^+(X_1) \) and \( b_2^+(X_i) \neq b_2^+(X_j) > 0 \) for \( i \neq j \).
In the paper [9], Kim gave the same bound for smooth, spin, even type $\mathbb{Z}/2\mathbb{Z}$-action on $X$ satisfying some non-degeneracy conditions analogous to Bryan’s.

In the paper [10], Liu gave the bound for even type spin $S_3$ action on 4-manifolds, that is

**Theorem 1.4.** Let $X$ be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. Let $k = -\sigma(X)/16$ and $m = b^+_2(X)$. Then,

$$2k + 2 \leq m$$

if $b^+_2(X/ < x_1 >) > 0$, $b^+_2(X/ < x_2 >) > 0$ and $b^+_2(X/ < x_1 >)$.

The purpose of this paper is to study the spin symmetric group $S_3$ actions of odd type on spin 4-manifolds, we prove that $b^+_2(X) \geq |\sigma(X)|/8 + 3$ under some non-degeneracy conditions. We also obtain some results about $\text{Ind}_{S_3}^G D$, where $\tilde{S}_3$ is the extension of $S_3$ by $Z_2$.

We organize the remainder of this paper as follows. In Section 2, we give some preliminaries to prove the main theorem. In Section 3, we use equivariant $K$-theory and representation theory to study the $G$-equivariant properties of the moduli space. In the last section we give our main results.

## 2. Notations and preliminaries

We assume that we have completed every Banach spaces with suitable Sobolev norms. Let $S = S^+ \oplus S^-$ denote the decomposition of spinor bundles into positive and negative spinor bundles. Let $D : \Gamma(S^+) \to \Gamma(S^-)$ be the Dirac operator, and $\rho : \Lambda^*_C \to \text{End}_C(S)$ be the Clifford multiplication. The Seiberg-Witten equations are for a pair $(a, \phi) \in \Omega^1(X, \sqrt{-1}T) \times \Gamma(S^+)$ and they are

$$D\phi + \rho(a)\phi = 0, \quad \rho(d^*a) - \phi \otimes \phi^* + \frac{1}{2} |\phi|^2 \text{id} = 0, \quad d^*a = 0.$$  

Let

$$V = \Gamma(\sqrt{-1}\Lambda^1 \oplus S^+),$$

$$W' = (S^- \oplus \sqrt{-1}\text{su}(S^+) \oplus \sqrt{-1}\Lambda^0).$$

We can think of the equation as the zero set of a map

$$\mathcal{D} + \mathcal{Q} : V \to W,$$

where $\mathcal{D}(a, \phi) = (D\phi, \rho(d^*a), d^*a))$, $\mathcal{Q}(a, \phi) = (\rho(a)\phi, \phi \otimes \phi^* - \frac{1}{4} |\phi|^2 \text{id}, 0)$, and $W$ is defined to be the orthogonal complement to the constant functions in $W'$.

Now it is time to describe the group of symmetries of the equations. Define $\text{Pin}(2) \subset SU(2)$ to be the normalizer of $S^1 \subset SU(2)$. Regarding $SU(2)$ as the group of unit quaternions and taking $S^1$ to be elements of the form $e^{\sqrt{-1}\theta}$, then $\text{Pin}(2)$ consists of the form $e^{\sqrt{-1}\theta}$ or $e^{\sqrt{-1}\theta} J$. We define the action of $\text{Pin}(2)$ on $V$ and $W$ as follows: since $S^+$ and $S^-$ are $SU(2)$ bundles, $\text{Pin}(2)$ naturally acts on $\Gamma(S^+)$ by multiplication on the left. $Z_2$ acts on $\Gamma(\Lambda^*_C)$ by multiplication by $\pm 1$ and this pullback also describes the action of $\text{Pin}(2)$ on $\sqrt{-1}\text{su}(S^+)$. Both $\mathcal{D}$ and $\mathcal{Q}$ are seen to be $\text{Pin}(2)$ equivariant maps.
Let $X$ be a smooth closed spin 4-manifold and suppose that $X$ admits a spin structure preserving action by a compact Lie group (or finite group) $G$. We may assume a Riemannian metric on $X$ so that $G$ acts by isometries. If the action is of even type, both $D$ and $Q$ are $\tilde{G} = \text{Pin}(2) \times G$ equivariant maps.

Now we define $V_\lambda$ to be the subspace of $V$ spanned by the eigenspaces $D^*D$ with eigenvalues less than or equal to $\lambda \in \mathbb{R}$. Similarly, we define $W_\lambda$ using $DD^*$. The virtual $G$-representation $[V_\lambda \otimes C] - [W_\lambda \otimes C] \in \mathbb{R}(\tilde{G})$ is the $\tilde{G}$-index of $D$ and can be determined by the $\tilde{G}$-index and is independent of $\lambda \in \mathbb{R}$, where $\mathbb{R}(\tilde{G})$ is the complex representation of $\tilde{G}$. In particular, since $V_0 = \ker D$ and $W_0 = \text{Coker } D \oplus \text{Coker } d^+$, we have

$$[V_\lambda \otimes C] - [W_\lambda \otimes C] = [V_0 \otimes C] - [W_0 \otimes C] \in \mathbb{R}(\tilde{G}).$$

Note that $\text{Coker } d^+ = H^2_2(X, R)$.

The $G$-action on $X$ can always be lifted to $\hat{G}$-actions on spinor bundles, where $\hat{G}$ is the following extension

$$1 \to \mathbb{Z}_2 \to \hat{G} \to G \to 1.$$

Recall that the $G$-action is of even type if $\hat{G}$ contains a subgroup isomorphic to $G$, otherwise it is of odd type.

For $S_3$ action of odd type, it is easy to know that the extension of $S_3$ by $\mathbb{Z}_2$ is isomorphic to the group

$$\tilde{S}_3 = \langle a, b \mid a^6 = 1, b^2 = a^3, ba = a^{-1}b \rangle.$$

The group $\tilde{S}_3$ has 12 elements and can be partitioned into 6 conjugacy classes: the identity element 1, $\{b, a^2b\}$, $\{a^2, a^4\}$, $\{a, a^5, a^4b\}$, $\{a^3\}$, and $\{ab, a^3b, a^5b\}$.

The character table for $\tilde{S}_3$ is as following

<table>
<thead>
<tr>
<th></th>
<th>$1$</th>
<th>$a^3$</th>
<th>$a^2$</th>
<th>$b$</th>
<th>$a$</th>
<th>$ab$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\eta_0$</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\eta_1$</td>
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<td>-1</td>
<td>1</td>
<td>-1</td>
<td>i</td>
<td>-i</td>
</tr>
<tr>
<td>$\eta_2$</td>
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<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$\eta_3$</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>i</td>
<td>i</td>
</tr>
<tr>
<td>$\eta_4$</td>
<td>2</td>
<td>2</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\eta_5$</td>
<td>2</td>
<td>-2</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

3. The index of $D$ and the character formula for the $K$-theory degree

The virtual representation $[V_{\lambda,c}]-[W_{\lambda,c}] \in R(\tilde{G})$ is the same as $\text{Ind}(D) = [\ker D] - [\text{Coker } D]$. Furuta determines $\text{Ind}(D)$ as a $\text{Pin}(2)$ representation; denoting the restriction map $r : R(\tilde{G}) \to R(\text{Pin}(2))$, Furuta shows

$$r(\text{Ind}(D)) = 2kh - m\tilde{1}$$
where $k = -\sigma(X)/16$ and $m = b_2^+(X)$. Thus $\text{Ind}(\mathcal{D}) = sh - t\tilde{I}$ where $s$ and $t$ are polynomials such that $s(1) = 2k$ and $t(1) = m$. For a spin odd $S_3$ action, $\tilde{G} = \text{Pin}(2) \times \tilde{S}_3$, we can write

$$s(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) = a_0 + b_0\eta_1 + c_0\eta_2 + d_0\eta_3 + e_0\eta_4 + f_0\eta_5,$$

and

$$t(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5) = a_1 + b_1\eta_1 + c_1\eta_2 + d_1\eta_3 + e_1\eta_4 + f_1\eta_5,$$

such that $a_0 + b_0 + c_0 + d_0 + 2e_0 + 2f_0 = 2k$ and $a_1 + b_1 + c_1 + d_1 + 2e_1 + 2f_1 = m = b_2^+(X)$.

For any element $g \in \tilde{S}_3$, denote by $\langle g \rangle$ the subgroup of $\tilde{S}_3$ generated by $g$. Then we have

$$\dim(H^+(X)\tilde{S}_3) = d_1 = b_2^+(X/\tilde{S}_3),$$

$$\dim(H^+(X)(\alpha)) = a_1 + c_1 + 2e_1 = b_2^+(X/(\alpha^2)),$$

$$\dim(H^+(X)(\beta)) = a_1 + d_1 = b_2^+(X/(\beta^2)),$$

$$\dim(H^+(X)(\alpha\beta)) = a_1 + e_1 + f_1 = b_2^+(X/(\alpha\beta)),$$

The Thom isomorphism theory in equivariant $K$-theory for a general compact Lie group is a deep theory proved using elliptic operator [1]. The subsequent character formula of this section uses only elementary properties of the Bott class.

Let $V$ and $W$ be complex $\Gamma$ representations for some compact Lie group $\Gamma$. Let $BV$ and $BW$ denote balls in $V$ and $W$ and let $f : BV \rightarrow BW$ be a $\Gamma$-map preserving the boundaries $SV$ and $SW$. $K_\Gamma(V)$ is by definition $K_\Gamma(BV, SV)$, and by the equivariant Thom isomorphism theorem, $K_\Gamma(V)$ is a free $R(\Gamma)$ module with generator the Bott class $\lambda(V)$. Applying the $K$-theory functor to $f$ we get a map

$$f^* : K_\Gamma(W) \rightarrow K_\Gamma(V)$$

which defines a unique element $\alpha_f \in R(\Gamma)$ by the equation $f^*(\lambda(W)) = \alpha_f \cdot \lambda(V)$.

The element $\alpha_f$ is called the $K$-theory degree of $f$.

Let $V_g$ and $W_g$ denote the subspaces of $V$ and $W$ fixed by an element $g \in \Gamma$ and let $V_g^\perp$ and $W_g^\perp$ be the orthogonal complements. Let $f^g : V_g \rightarrow W_g$ be the restriction of $f$ and let $d(f^g)$ denote the ordinary topological degree of $f^g$ (by definition, $d(f^g) = 0$ if $\dim V_g \neq \dim W_g$). For any $\beta \in R(\Gamma)$, let $\lambda_{-1} \beta$ denote the alternating sum $\Sigma(-1)^i \lambda_i \beta$ of exterior powers.

T. tom Dieck proves the following character formula for the degree $\alpha_f$:

**Theorem (3).** Let $f : BV \rightarrow BW$ be a $\Gamma$-map preserving boundaries and let $\alpha_f \in R(\Gamma)$ be the $K$-theory degree. Then

$$\text{tr}_g(\alpha_f) = d(f^g) \text{tr}_g(\lambda_{-1}(W_g^\perp - V_g^\perp))$$

where $\text{tr}_g$ is the trace of the action of an element $g \in \Gamma$. 

This formula is very useful in the case where \( \dim V_g \neq \dim W_g \) so that \( d(f^g) = 0 \).

Recall that \( \lambda_{-1}(\Sigma a_i r_i) = \prod_i (\lambda_{-1} r_i)^{a_i} \) and that for a one dimensional representation \( r \), we have \( \lambda_{-1} r = (1 - r) \). A two dimensional representation such as \( h \) has \( \lambda_{-1} h = (1 - h + \Lambda^2 h) \). In this case, since \( h \) comes from an \( SU(2) \) representation, \( \Lambda^2 h = \det h = 1 \) so \( \lambda_{-1} h = (2 - h) \).

In the following by using the character formula to examine the K-theory degree \( \alpha_{f_2} \) of the map \( f_2 : BV_{\lambda,C} \to BW_{\lambda,C} \) coming from the Seiberg-Witten equations. We will abbreviate \( \alpha_{f_2} \) as \( \alpha \) and \( V_{\lambda,C} \) and \( W_{\lambda,C} \) as just \( V \) and \( W \). Let \( \phi \in S^1 \subset Pin(2) \subset G \) be the element generating a dense subgroup of \( S^1 \), and recall that there is the element \( J \in Pin(2) \) coming from the quaternion. Note that the action of \( J \) on \( h \) has two invariant subspaces on which \( J \) acts by multiplication with \( \sqrt{-1} \) and \( -\sqrt{-1} \).

**4. THE MAIN RESULTS**

Consider \( \alpha = \alpha_{f_2} \in R(Pin(2) \times S_3) \), it has the following form

\[
\alpha = \alpha_0 + \hat{\alpha}_0 \hat{1} + \sum_{i=1}^{\infty} \alpha_i h_i.
\]

where \( \alpha_0 = l_1 n_1 + n_1 + n_2 + p_1 n_3 + q_1 n_4 + r_1 n_5 \), \( i \geq 0 \) and \( \hat{\alpha}_0 = \tilde{l}_0 + \tilde{n}_0 n_1 + \tilde{r}_0 n_2 + \tilde{p}_0 n_3 + \tilde{q}_0 n_4 + \tilde{r}_0 n_5 \).

Since \( \phi \) acts non-trivially on \( h \) and trivially on \( \hat{1} \), then we have

\[
\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h)_{\phi} - \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)\hat{1})_{\phi} = -(a_1 + b_1 + c_1 + d_1 + 2c_1 + 2f_1) = -b_2^+(X).
\]

So if \( b_2^+(X) \neq 0 \), \( \text{tr}_\phi \alpha = 0 \).

\( \phi^a \) acts non-trivially on \( V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h \) but trivially on \( a_1 \hat{1} \). Besides, the action of \( a \) on \( e_1 \eta_4 \) and \( f_1 \eta_5 \) both have a one-dimensional invariant subspace, then we have

\[
\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5))_{\phi^a} - \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)\hat{1})_{\phi^a} = -(a_1 + b_1 + f_1) = -b_2^+(X/(a)\). \]

So if \( a_1 + b_1 + f_1 = b_2^+(X/(a)) \neq 0 \), \( \text{tr}_{\phi^a} \alpha = 0 \).

Since \( \phi^{a^2} \) acts non-trivially on \( V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h \), and trivially on \( a_1 \hat{1}, b_1 \eta_1 \hat{1} \) and \( d_1 \eta_3 \hat{1} \), then we have

\[
\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5))_{\phi^{a^2}} - \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5))_{\phi^{a^2}} = -(a_1 + b_1 + c_1 + d_1) = -b_2^+(X/(a^2)) \]

So if \( a_1 + b_1 + c_1 + d_1 = b_2^+(X/(a^2)) \neq 0 \), \( \text{tr}_{\phi^{a^2}} \alpha = 0 \).

\( \phi^{a^2} \) acts non-trivially on \( V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h \) but trivially on \( a_1 \hat{1} \) and \( c_1 \eta_4 \hat{1} \). Besides, the action of \( a^3 \) on \( e_1 \eta_4 \) has a two-dimensional invariant subspaces, so we
have
\[
\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5))_{\phi^3} - \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5))_{\phi^3} = -(a_1 + c_1 + 2e_1) = -b^2_2(X/\langle a^3 \rangle).
\]

So if \(a_1 + c_1 + 2e_1 = b^2_2(X/\langle a^3 \rangle) \neq 0\), \(\text{tr}_{\phi^3} \alpha = 0\).

Since \(\phi \phi b\) acts non-trivially on \(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h\) and trivially on \(a_1 \tilde{I}\) and \(c_1 \eta_2 \tilde{I}\), then we have
\[
\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h)_{\phi b} - \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)\tilde{\phi} h) = -(a_1 + c_1) = -b^2_2(X/\langle b \rangle).
\]

So if \(a_1 + c_1 = b^2_2(X/\langle b \rangle) \neq 0\), \(\text{tr}_{\phi b} \alpha = 0\).

\(\phi \phi b\) acts non-trivially on \(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h\) but trivially on \(a_1 \tilde{I}\). Besides, the action of \(ab\) on \(c_1 \eta_4\) and \(f_1 \eta_5\) both have a one-dimensional invariant subspace, then we have
\[
\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h)_{\phi \phi b} - \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)\tilde{\phi} b) = -(a_1 + c_1 + f_1) = -b^2_2(X/\langle ab \rangle).
\]

So if \(a_1 + c_1 + f_1 = b^2_2(X/\langle ab \rangle) \neq 0\), \(\text{tr}_{\phi \phi b} \alpha = 0\).

From the above analysis, we know if \(b^2_2(X/\langle a \rangle) \neq 0\) and \(b^2_2(X/\langle b \rangle) \neq 0\), we have \(\text{tr}_b \alpha = \text{tr}_{\phi \phi b} \alpha = \text{tr}_{\phi \phi b} \alpha = \text{tr}_{\phi \phi b} \alpha = 0\) which implies that
\[
0 = \text{tr}_b \alpha = \text{tr}_b (\alpha_0 + \alpha_0 \tilde{I} + \sum_{i=1}^{\infty} \alpha_i h_i) = \text{tr}_b (\alpha_0 + \tilde{\alpha_0} \tilde{I} + \sum_{i=1}^{\infty} \text{tr}_b \alpha_i (\phi^i + \phi^{-i}))
\]
\[
= (\bar{\alpha}_0 + \alpha_0 + p_0 + q_0 + r_0) + (\bar{\tilde{\alpha}}_0 + \tilde{\alpha}_0 + \tilde{p}_0 + \tilde{q}_0 + \tilde{r}_0)
\]
\[
+ \sum_{i=1}^{\infty} \text{tr}_b \alpha_i (\phi^i + \phi^{-i}),
\]
\[
0 = \text{tr}_{\phi \alpha} \alpha = \text{tr}_{\phi \alpha} (\alpha_0 + \alpha_0 \tilde{I} + \sum_{i=1}^{\infty} \alpha_i h_i) = \text{tr}_a \alpha_0 + \tilde{\alpha}_0 \tilde{I} + \sum_{i=1}^{\infty} \text{tr}_a \alpha_i (\phi^i + \phi^{-i})
\]
\[
= (\bar{\alpha}_0 + \alpha_0 + p_0 + q_0) + (\bar{\tilde{\alpha}}_0 + \tilde{\alpha}_0 + \tilde{p}_0 + \tilde{q}_0 + \tilde{r}_0)
\]
\[
+ \sum_{i=1}^{\infty} \text{tr}_a \alpha_i (\phi^i + \phi^{-i}),
\]
\[
0 = \text{tr}_{\phi \alpha^2} \alpha = \text{tr}_{\phi \alpha^2} (\alpha_0 + \alpha_0 \tilde{I} + \sum_{i=1}^{\infty} \alpha_i h_i) = \text{tr}_{\alpha^2} \alpha_0 + \tilde{\alpha}_0 \tilde{I} + \sum_{i=1}^{\infty} \text{tr}_{\alpha^2} \alpha_i (\phi^i + \phi^{-i})
\]
\[
= (\bar{\alpha}_0 + \alpha_0 + p_0 + q_0 + r_0) + (\bar{\tilde{\alpha}}_0 + \tilde{\alpha}_0 + \tilde{p}_0 + \tilde{q}_0 + \tilde{r}_0)
\]
\[
+ \sum_{i=1}^{\infty} \text{tr}_{\alpha^2} \alpha_i (\phi^i + \phi^{-i}),
\]
0 = \text{tr}_{\phi a^2} \alpha = \text{tr}_{\phi a^2} (\alpha_0 + \tilde{\alpha}_0 \tilde{1}) + \sum_{i=1}^{\infty} \alpha_i h_i = \text{tr}_{a^2} \alpha_0 + \text{tr}_{a^2} \tilde{\alpha}_0 + \sum_{i=1}^{\infty} \text{tr}_{a^2} \alpha_i (\phi^i + \phi^{-i})
\quad = (l_0 - m_0 + n_0 - p_0 + 2q_0 - 2r_0) + (\tilde{l}_0 - \tilde{m}_0 + \tilde{n}_0 - \tilde{p}_0 + 2\tilde{q}_0 - 2\tilde{r}_0) + \sum_{i=1}^{\infty} \text{tr}_{a^2} \alpha_i (\phi^i + \phi^{-i}),

0 = \text{tr}_{\phi b} \alpha = \text{tr}_{\phi b} (\alpha_0 + \tilde{\alpha}_0 \tilde{1}) + \sum_{i=1}^{\infty} \alpha_i h_i = \text{tr}_b \alpha_0 + \text{tr}_b \tilde{\alpha}_0 + \sum_{i=1}^{\infty} \text{tr}_b \alpha_i (\phi^i + \phi^{-i})
\quad = (l_0 - m_0 + n_0 - p_0 - q_0 + r_0) + (\tilde{l}_0 - \tilde{m}_0 + \tilde{n}_0 - \tilde{p}_0 - \tilde{q}_0 + \tilde{r}_0) + \sum_{i=1}^{\infty} \text{tr}_b \alpha_i (\phi^i + \phi^{-i}),

0 = \text{tr}_{\phi ab} \alpha = \text{tr}_{\phi ab} (\alpha_0 + \tilde{\alpha}_0 \tilde{1}) + \sum_{i=1}^{\infty} \alpha_i h_i = \text{tr}_{ab} \alpha_0 + \text{tr}_{ab} \tilde{\alpha}_0 + \sum_{i=1}^{\infty} \text{tr}_{ab} \alpha_i (\phi^i + \phi^{-i})
\quad = (l_0 - im_0 - n_0 + ip_0) + (\tilde{l}_0 - i\tilde{m}_0 - \tilde{n}_0 + i\tilde{p}_0) + \sum_{i=1}^{\infty} \text{tr}_{ab} \alpha_i (\phi^i + \phi^{-i}),

and so on. From these equations, we have \(\alpha_0 = -\tilde{\alpha}_0\) and \(\alpha_i = 0, i > 0\), that is \(\alpha = \alpha_0 (1 - \tilde{1})\).

Next we calculate \(\text{tr}_J \alpha\). Since \(J\) acts non-trivially on both \(h\) and \(\tilde{1}\), \(\dim V_J = \dim W_J = 0\), so \(d(f^J) = 1\). Using \(\text{tr}_J h = 0\) and \(\text{tr}_J \tilde{1} = -1\), by the character formula we have

\[
\text{tr}_J(\alpha) = \text{tr}_J(\lambda_- (m \tilde{1} - 2k h)) = \text{tr}_J((1 - \tilde{1})^m (2 - h)^{-2k}) = 2^{m-2k}.
\]

Now we calculate \(\text{tr}_{Ja} \alpha\). \(Ja\) acts non-trivially on both \(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h\), but trivially on \(c_1 \eta_4\tilde{1}\). Besides, the action of \(a\) on \(\eta_1 \eta_4 \tilde{1}\) both have a one-dimensional invariant subspace. So we have

\[
\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h)_{Ja} - \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)\tilde{1})_{Ja} = -(c_1 + e_1 + f_1).
\]

Then, if \(c_1 + e_1 + f_1 \neq 0\), \(\text{tr}_{Ja} \alpha = 0\)

Since \(Ja^2\) acts non-trivially on both \(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h\) and \(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)\tilde{1}\), then \(d(f^Ja^2) = 1\). By tom Dieck formula, we have

\[
\text{tr}_{Ja^2} \alpha = \text{tr}_{Ja^2} [\lambda_- (a_1 + b_1 \eta_1 + c_1 \eta_2 + d_1 \eta_3 + e_1 \eta_4 + f_1 \eta_5) \tilde{1} - \lambda_- (a_0 + b_0 \eta_1 + c_0 \eta_2 + d_0 \eta_3 + e_0 \eta_4 + f_0 \eta_5) h]
= 2^{(a_1 + b_1 + c_1 + d_1) - (a_0 + b_0 + c_0 + d_0)}.
\]

\(Ja^3\) acts non-trivially on \(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h\), but trivially on \(b_1 \eta_1\tilde{1}\) and \(d_1 \eta_3\tilde{1}\). Besides, the action of \(Ja^3\) on \(f_1 \eta_5\tilde{1}\) has two invariant subspaces. So

\[
\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h)_{Ja^3} - \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)\tilde{1})_{Ja^3} = -(b_1 + d_1 + 2f_1).
\]
Then, if $b_1 + d_1 + 2f_1 \neq 0$, $\text{tr}_{J_2} \alpha = 0$.

Since $Jb$ acts non-trivially on $V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h$ but trivially on $b_1 \eta_1 \bar{1}$ and $d_1 \eta_3 \bar{1}$, then
\[
\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h)_{Jb} - \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)\bar{1})_{Jb} = -(b_1 + d_1) = b_2^+(X/(a^2)) - b_2^+(X/(b)).
\]

Then, if $b_1 + d_1 \neq 0$, $\text{tr}_{J_2} \alpha = 0$

$Jab$ acts non-trivially on $V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h$ but trivially on $c_1 \eta_4 \bar{1}$. Besides, the action of $ab$ on $c_1 \eta_4 \bar{1}$ and $f_1 \eta_5 \bar{1}$ both have a one-dimensional invariant subspace. Then we have
\[
\dim(V(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)h)_{Jab} - \dim(W(\eta_1, \eta_2, \eta_3, \eta_4, \eta_5)\bar{1})_{Jab} = -(c_1 + e_1 + f_1).
\]

Then by assuming $b_2^+(X/(a^2)) - b_2^+(X/(b)) \neq 0$ and $b_2^+(X/(a^3)) - b_2^+(X/(b)) \neq 0$, we have $\text{tr}_{J_2} \alpha = 0$, $\text{tr}_{J_2} \alpha = 0$, $\text{tr}_{J_2} \alpha = 0$, $\text{tr}_{J_2} \alpha = 0$.

By direct calculation, we have
\[
\text{tr}_J \alpha_0 = l_0 + m_0 + n_0 + p_0 + 2q_0 + 2r_0 = 2^{m-2k-1},
\]
\[
\text{tr}_{a^2} \alpha_0 = l_0 + m_0 + n_0 + p_0 - q_0 = 2(a_1 + b_1 + c_1 + d_1 - (a_0 + b_0 + c_0 + d_0) - 1),
\]
\[
\text{tr}_a \alpha_0 = l_0 + im_0 - n_0 - ip_0 = 0,
\]
\[
\text{tr}_b \alpha_0 = l_0 - m_0 + n_0 - p_0 + 2q_0 - 2r_0 = 0,
\]
\[
\text{tr}_{ab} \alpha_0 = l_0 + im_0 - n_0 + ip_0 = 0.
\]

Here we use $\text{tr}_{J_2} \alpha = \text{tr}_g(2 \cdot \alpha_0) = 2 \cdot \text{tr}_g \alpha_0$ where $g$ is any element of $\tilde{S}_4$.

From (3), (5), (6) and (8), we get $l_0 + q_0 = 2^{m-2k-3}$. So we have the following main result.

**Theorem 4.1.** Let $X$ be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. Let $k = -\alpha(X)/16$ and $m = b_2^+(X)$. If $X$ admits a spin odd type $S_3$ action, then $2k + 3 \leq m$, if $b_2^+(X/(a)) \neq 0$, $0 \leq b_2^+(X/(a^2)) - b_2^+(X/(b)) \neq 0$ and $b_2^+(X/(a^3)) - b_2^+(X/(b)) \neq 0$.

Besides, from the above six equations, we get
\[
q_0 = r_0 = \frac{2^{m-2k-2} - (a_1 + b_1 + c_1 + d_1 - (a_0 + b_0 + c_0 + d_0) - 2)/3}{3}
\]
\[
l_0 + m_0 + n_0 + p_0 = \frac{2^{m-2k-3} - (a_1 + b_1 + c_1 + d_1 - (a_0 + b_0 + c_0 + d_0) - 2)/3}{3}
\]
Since $q_0 \in \mathbb{Z}$, then $2^{m-2k-2} - (a_1 + b_1 + c_1 + d_1 - (a_0 + b_0 + c_0 + d_0) - 2) \in \mathbb{Z} \subset \mathbb{Z}$. From Theorem 4.1, we know $2^{m-2k-2} \in \mathbb{Z}$. So $2^{m-2k-2} \in \mathbb{Z}$, i.e., $(a_1 + b_1 + c_1 + d_1) \geq (a_0 + b_0 + c_0 + d_0) + 2$. Hence, we have

**Theorem 4.2.** Let $X$ be a smooth spin 4-manifold with $b_1(X) = 0$ and non-positive signature. If $X$ admits a spin odd type $S_3$ action, then
\[
b_2^+(X/(a^2)) \geq \dim((\text{Ind}_{\tilde{S}_3} D)(a^2)) + 2,
\]
if $b_2^+(X/\langle a \rangle) \neq 0$, $b_2^+(X/\langle b \rangle) \neq 0$, $b_2^+(X/\langle a^2 \rangle) - b_2^+(X/\langle b \rangle) \neq 0$ and $b_2^+(X/\langle a^3 \rangle) - b_2^+(X/\langle b \rangle) \neq 0$. Moreover, under this condition, the K-theory degree $\alpha = \alpha_0(1 - \bar{\eta})$ for some $\alpha_0 = l_0(1 + \eta_1 + \eta_2 + \eta_3) + q_0(\eta_4 + \eta_5)$.

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