ON THE VOLUME OF THE Trajectory SURFACES UNDER THE HOMOTHETIC MOTIONS

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Abstract. The volumes of the surfaces of 3-dimensional Euclidean Space which are traced by a fixed point as a trajectory surface during 3-parametric motions are given by H. R. Müller [3], [4], [5] and W. Blaschke [1].

In this paper, the volumes of the trajectory surfaces of fixed points under 3-parametric homothetic motions are computed. Also, using a certain pseudo-Euclidean metric we generalized the well-known classical Holditch Theorem, [2], to the trajectory surfaces.

1. Introduction

Let $R$ and $R'$ be moving and fixed spaces and $\{O; e_1, e_2, e_3\}$ and $\{O'; e_1', e_2', e_3'\}$ be their orthonormal coordinate systems, respectively. If $e_j = e_j(t_1, t_2, t_3)$ and $u = u(t_1, t_2, t_3)$, then a 3-parameter motion $B_3$ of $R$ with respect to $R'$ is defined, where $u = \overrightarrow{O'O}$ and $t_1, t_2, t_3$ are the real parameters. For the rotation part of $B_3$, we have the anti-symmetric system of differentiation equations (Ableitungsgleichungen)

$$de_i = e_k \omega_j - e_j \omega_k, \quad i, j, k = 1, 2, 3 \text{ (cyclic)}$$

with the conditions of integration (Integrierbarkeitsbedingungen)

$$d \omega_i = \omega_j \wedge \omega_k,$$

where “$d$” is the exterior derivative and “$\wedge$” is the wedge product of the differential forms. For the translation part of $B_3$

$$d \overrightarrow{O'O} = \sigma = \sigma_1 e_1 + \sigma_2 e_2 + \sigma_3 e_3,$$

where the conditions of integration are

$$d \sigma_i = \sigma_j \wedge \omega_k - \sigma_k \wedge \omega_j.$$

During $B_3$, $\omega_i$ and $\sigma_i$ are the linear differential forms with respect to $t_1, t_2, t_3$. We assume that $\omega_i, i = 1, 2, 3$ are linear independent, i.e., $\omega_1 \wedge \omega_2 \wedge \omega_3 \neq 0$.

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2. The volume of the trajectory surface under the homothetic motions

I. Now, let us consider the 3-parametric homothetic motion of the fixed point $X = (x_i)$ with respect to arbitrary moving Euclidean space. We may write

$$x' = u + hx,$$

where $x$ and $x'$ are the position vectors of the point $X$ with respect to the moving and fixed coordinate systems, respectively, and $h = h(t_1, t_2, t_3)$ is the homothetic scale of the motion. Then, we get

$$dx' = \sigma + xdh + hx \times \omega,$$

where $\omega = \sum \omega_i e_i$ is the rotation vector and “×” denotes the vector product.

If we write $dx' = \sum \tau_i e_i$, we get

$$\tau_i = \sigma_i + x_i dh + h(x_j \omega_k - x_k \omega_j). \quad (1)$$

The volume element of the trajectory surface of $X$ is

$$dJ_X = \tau_1 \wedge \tau_2 \wedge \tau_3. \quad (2)$$

Thus, the integration of the volume element over the region $G$ of the parameter space yields the volume of the trajectory surface, i.e., $J_X = \int_G dJ_X$. Let $\Gamma$ be the closed and orientated edge surface of $G$.

If we replace (1) in (2), for the volume of the trajectory surface of $X$ we get

$$J_X = J_O + \sum_{i=1}^3 \tilde{A}_i x_i^2 + \sum_{i\neq j} A_{ij} x_i x_j + \sum_{i=1}^3 B_i x_i + \left(\sum_{i=1}^3 x_i^2\right) \left(\sum_{i=1}^3 C_i x_i\right), \quad (3)$$

where

$$\tilde{A}_i = \int_G (h^2 \sigma_i \wedge \omega_j \wedge \omega_k + hdh \wedge \sigma_j \wedge \omega_j + hdh \wedge \sigma_k \wedge \omega_k),$$

$$A_{ij} = \int_G (hdh \wedge \omega_i \wedge \sigma_j + hdh \wedge \omega_j \wedge \sigma_i + h^2 \sigma_j \wedge \omega_j \wedge \omega_k + h^2 \sigma_i \wedge \omega_k \wedge \omega_i),$$

$$B_i = \int_G (h\sigma_i \wedge \sigma_k \wedge \omega_k + dh \wedge \sigma_j \wedge \sigma_k + h\sigma_i \wedge \sigma_j \wedge \omega_j) = \int_{\Gamma} h\sigma_j \wedge \sigma_k,$$

$$C_i = \int_G h^2 dh \wedge \omega_j \wedge \omega_k = \frac{1}{3} \int_{\Gamma} h^3 \omega_j \wedge \omega_k$$

and $J_O = \int_G \sigma_1 \wedge \sigma_2 \wedge \sigma_3$ is the volume of the trajectory surface of the origin point $O$. 
Let us suppose that $\sigma_i \wedge \omega_i$, $i = 1, 2, 3$, have the same sign when integrated over any consistently orientated simplex from $\Gamma$. Then, using the mean-value theorem for double integrals, we obtain

$$
\int_\Gamma h^2 \sigma_i \wedge \omega_i = h^2(u_i, v_i) \int_\Gamma \sigma_i \wedge \omega_i, \quad i = 1, 2, 3,
$$

where $u_i$ and $v_i$ are the parameters. If we assume that $h^2(u_1, v_1) = h^2(u_2, v_2) = h^2(u_3, v_3)$, then using (4) and (5) we can find the parameters $u_0$ and $v_0$ such that

$$
J_X = J_O + h^2(u_0, v_0) \sum_{i=1}^3 A_i x_i^2 + \sum_{i \neq j} A_{ij} x_i x_j + \sum_{i=1}^3 B_i x_i
$$

(6)

$$
+ \left( \sum_{i=1}^3 x_i^2 \right) \left( \sum_{i=1}^3 C_i x_i \right),
$$

where

$$
A_i = \frac{1}{2} \int_\Gamma (\sigma_j \wedge \omega_j + \sigma_k \wedge \omega_k).
$$

Now, let us consider the plane $P : C_1 x + C_2 y + C_3 z = 0$. The volumes of the trajectory surfaces of points on $P$ are quadratic polynomial with respect to $x_i$. If we choose the moving coordinate system such that the coefficients of the mixture quadratic terms vanish, i.e. $A_{ij} = 0$, then we get for a point $X \in P$

$$
J_X = J_O + h^2(u_0, v_0) \sum_{i=1}^3 A_i x_i^2 + \sum_{i=1}^3 B_i x_i.
$$

(7)

Hence, we may give the following theorem:

**Theorem 1.** All the fixed points of $P$ whose trajectory surfaces have equal volume during the homothetic motion lie on the same quadric.

II.

Let $X$ and $Y$ be two fixed points on $P$ and $Z$ be another point on the line segment $XY$, that is, $z_i = \lambda x_i + \mu y_i$, $\lambda + \mu = 1$.

Using (7), we get

$$
J_Z = \lambda^2 J_X + 2\lambda\mu J_{XY} + \mu^2 J_Y,
$$

(8)

where

$$
J_{XY} = J_{YX} = J_O + h^2(u_0, v_0) \sum_{i=1}^3 A_i y_i x_i + \frac{1}{2} \sum_{i=1}^3 B_i (x_i + y_i) + \left( \sum_{i=1}^3 x_i^2 \right) \left( \sum_{i=1}^3 C_i y_i \right).
$$
is called the mixture trajectory surface volume. It is clearly seen that $J_{XX} = J_X$.
Since
\begin{equation}
J_X - 2J_{XY} + J_Y = h^2(u_0, v_0) \sum_{i=1}^{3} A_i(x_i - y_i)^2,
\end{equation}
we can rewrite (8) as follows:
\begin{equation}
J_Z = \lambda J_X + \mu J_Y - \varepsilon h^2(u_0, v_0) \lambda \mu \sum_{i=1}^{3} A_i(x_i - y_i)^2.
\end{equation}

We will define the distance $D(X, Y)$ between the points $X, Y$ of $P$ by
\begin{equation}
D^2(X, Y) = \sum_{i=1}^{3} A_i(x_i - y_i)^2, \quad \varepsilon = \pm 1, \quad [4].
\end{equation}
By the orientation of the line $XY$ we will distinguish $D(X, Y) = -D(Y, X)$. Therefore, from (10) we have
\begin{equation}
J_Z = \lambda J_X + \mu J_Y - \varepsilon h^2(u_0, v_0) \lambda \mu D^2(X, Y).
\end{equation}
Since $X, Y$ and $Z$ are collinear, we may write
\[D(X, Z) + D(Z, Y) = D(X, Y).\]
Thus, if we denote
\[\lambda = \frac{D(Z, Y)}{D(X, Y)}, \quad \mu = \frac{D(X, Z)}{D(X, Y)},\]
from (12) we get
\begin{equation}
J_Z = \frac{1}{D(X, Y)} [D(Z, Y)J_X + D(X, Z)J_Y]
- \varepsilon h^2(u_0, v_0)D(X, Z)D(Z, Y).
\end{equation}
Now, we consider that the points $X$ and $Y$ trace the same trajectory surface. In this case, we get $J_X = J_Y$. Then, from (13) we obtain
\begin{equation}
J_X - J_Z = \varepsilon h^2(u_0, v_0)D(X, Z)D(Z, Y)
\end{equation}
which is the generalization of Holditch’s result, [2], for trajectory surfaces during the homothetic motions. (14) is also equivalent to the result given by [6]. We may give the following theorem:

**Theorem 2.** Let $XY$ be a line segment with the constant length on $P$ and the endpoints of this line segment have the same trajectory surface. Then, the point $Z$ on this line segment traces another trajectory surface. The volume between these trajectory surfaces depends on the distances (in the sense of (11)) of $Z$ from the endpoints and the homothetic scale $h$.

**Special case:** In the case of $h \equiv 1$, we have the result given by H. R. Müller, [3].
After eliminating the mixture trajectory surface volumes by using (9), we get

\[ J = \lambda_1 x_i + \lambda_2 y_i + \lambda_3 z_i, \quad \lambda_1 + \lambda_2 + \lambda_3 = 1. \]

Let \( P \) and \( Q \) be another point on \( P \) (Fig. 1). Then, we may write

\[ q_i = \lambda_1 x_i + \lambda_2 y_i + \lambda_3 z_i, \quad \lambda_1 + \lambda_2 + \lambda_3 = 1. \]

If we use (7), we obtain

\[ J_Q = \lambda_1^2 J_{X_1} + \lambda_2^2 J_{X_2} + \lambda_3^2 J_{X_3} + 2\lambda_1\lambda_2 J_{X_1X_2} + 2\lambda_1\lambda_3 J_{X_1X_3} + 2\lambda_2\lambda_3 J_{X_2X_3}. \]

After eliminating the mixture trajectory surface volumes by using (9), we get

\[ J_Q = \lambda_1 J_{X_1} + \lambda_2 J_{X_2} + \lambda_3 J_{X_3} - h^2(u_0, v_0) \tag{15} \]

\[ \{ \varepsilon_{ij} \lambda_1 \lambda_2 h^2(X_1, X_2) + \varepsilon_{ik} \lambda_1 \lambda_3 h^2(X_1, X_3) + \varepsilon_{kj} \lambda_2 \lambda_3 h^2(X_2, X_3) \}. \]

On the other hand, if we consider the point \( Q_1 = (a_i) \), we may write

\[ a_i = \mu_1 x_i + \mu_2 y_i, \quad q_i = \mu_3 x_i + \mu_4 a_i, \quad \mu_1 + \mu_2 = \mu_3 + \mu_4 = 1. \]

Thus, we have \( \lambda_1 = \mu_3, \ \lambda_2 = \mu_1 \mu_4, \ \lambda_3 = \mu_2 \mu_4 \) i.e.

\[ \lambda_1 = \frac{D(Q, Q_1)}{D(X_1, Q_1)}, \quad \lambda_2 = \frac{D(X_1, Q)D(Q_1, X_3)}{D(X_1, Q_1)D(X_2, X_3)}, \quad \lambda_3 = \frac{D(X_1, Q)D(X_2, Q_1)}{D(X_1, Q_1)D(X_2, X_3)}. \]

Similarly, considering the points \( Q_2 \) and \( Q_3 \), respectively, we find

\[ \lambda_i = \frac{D(Q, Q_i)}{D(X, Q_i)}, \quad \lambda_i = \frac{D(X, Q)D(Q_k, X_j)}{D(X, Q_i)D(X, X_j)} \]

\[ = \frac{D(X, Q)D(Q_k, X_j)}{D(X, Q_k)D(X, X_j)}, \quad i, j, k = 1, 2, 3 \ (cyclic). \]

Then, from (15) the generalization of (12) is found as

\[ J_Q = \sum_i \frac{D(Q, Q_i)}{D(X, Q_i)} J_{X_i} - h^2(u_0, v_0) \sum \varepsilon_{ij} \left( \frac{D(X, Q)}{D(X, Q_k)} \right)^2 D(Q_k, X_j)D(X_i, Q_k). \]

If \( X_1, X_2, X_3 \) trace the same trajectory surface, then the difference between the volumes is

\[ J_{X_1} - J_Q = h^2(u_0, v_0) \sum \varepsilon_{ij} \left( \frac{D(X, Q)}{D(X, Q_k)} \right)^2 D(Q_k, X_j)D(X_i, Q_k). \]
Then, we can give the following theorem:

**Theorem 3.** Let us consider a triangle on the plane $P$. If the vertices of this triangle trace the same trajectory surface, then a different point on $P$ traces another surface. The volume between these trajectory surfaces depends on the distances (in the sense of (11)) of the moving triangle and the homothetic scale $h$.

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**References**


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