COEFFICIENT ESTIMATES IN SUBCLASSES OF THE CARATHÉODORY CLASS RELATED TO CONICAL DOMAINS

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Abstract. We study some properties of subclasses of the Carathéodory class of functions, related to conic sections, and denoted by $\mathcal{P}(p_k)$. Coefficients bounds, estimates of some functionals are given.

1. Introduction

We denote by $\mathcal{P}$ the class of Carathéodory functions analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$, e.g.

\[(1.1) \quad \mathcal{P} = \{p : p \text{ analytic in } \mathcal{U}, \ p(0) = 1, \ \Re p(z) > 0\}.
\]

Some special subclasses of $\mathcal{P}$ play an important role in geometric function theory because of their relations with subclasses of univalent functions. Many such classes have been introduced and studied; some became the well-known, for instance, the class of analytic functions $p$ in the unit disk $\mathcal{U}$ such that $p(0) = 1$ and $p \prec (1 + Az)/(1 + Bz)$, that is the class of functions for which $p(\mathcal{U})$ is a subset of a disk, or a half-plane. The other choice is the class of all $p$ such that $p \prec [(1+z)/(1-z)]^\gamma$. In this case $p(\mathcal{U})$ is a subset of a sector, contained in a right half-plane with a vertex at the origin and symmetric about the real axis. Here the symbol “$\prec$” denotes the subordinations (cf. e.g. [7]).

Let $k \in [0, \infty)$. For arbitrarily chosen $k$ let $\Omega_k$ denote the following domain

\[(1.2) \quad \Omega_k = \{u + iv : u^2 > k^2(u - 1)^2 + k^2v^2\}.
\]

Note that $\Omega_k$ is convex and symmetric in the real axis and $1 \in \Omega_k$ for all $k$. $\Omega_0$ is nothing but the right half-plane and when $0 < k < 1$, $\Omega_k$ is an unbounded domain enclosed by the right branch of the hyperbola

\[(1.3) \quad \left(\frac{(1-k^2)u + k^2}{k}\right)^2 - \left(\frac{(1-k^2)v}{\sqrt{1-k^2}}\right)^2 = 1.
\]
with foci at 1 and $-(1+k^2)/(1-k^2)$. When $k=1$, the domain $\Omega_1$ is still unbounded domain enclosed by the parabola
\[ 2u = v^2 + 1 \]
with the focus at 1. When $k > 1$, the domain $\Omega_k$ becomes bounded domain being the interior of the ellipse
\[ \left( \frac{(k^2-1)u-k^2}{k} \right)^2 - \left( \frac{(k^2-1)v}{\sqrt{k^2-1}} \right)^2 = 1 \]
with foci at 1 and $(k^2+1)/(k^2-1)$. It should be noted that, for no choice of parameter $k$, $\Omega_k$ reduce to a disk. $\{\Omega_k, k \in [0, \infty)\}$ forms the family of domains bounded by conic sections, convergent in the sense of the kernel convergence.

Let $p_k$ denote the conformal mapping of $\mathcal{U}$ onto $\Omega_k$ determined by conditions $p_k(0) = 1$, $p_k'(0) > 0$. The concrete form of $p_k$ was given in [7], [8], [11] and in [5].

**Theorem 1.1.** Let $k \in [0, \infty)$. The conformal mapping of $\mathcal{U}$ onto $\Omega_k$ is of the form
\[ p_k(z) = \begin{cases} 
1 & \text{for } k = 0, \\
1 + \frac{2}{\pi k^2} \sinh^2 \left( A(k) \arctanh \sqrt{z} \right) & \text{for } k \in (0, 1), \\
1 + \frac{2}{\pi} \left( \arctanh \sqrt{z} \right)^2 & \text{for } k = 1, \\
1 + \frac{2}{\pi} \sin^2 \left( \frac{\pi}{2k(t)} F\left( \sqrt{\frac{z}{t^2}} \right) \right) & \text{for } k > 1,
\end{cases} \]
where $A(k) = (2/\pi) \arccos k$, $F(w, t)$ is the Jacobi elliptic integral of the first kind
\[ F(w, t) = \int_0^w \frac{dx}{\sqrt{(1-x^2)(1-t^2t^2)}}, \]
and $k = \cosh \mu(t) = \cosh \left( \frac{\pi K'(t)}{2k(t)} \right), t \in (0, 1)$.

By $\mathcal{P}(p_k)$ we denote the subclass of the Carathéodory class $\mathcal{P}$, consisting of functions $p$, analytic in $\mathcal{U}$, $p(0) = 1$, $\Re p(z) > 0$ in $\mathcal{U}$, and such that $p < p_k$ in $\mathcal{U}$. Observe that when $k$ varies, $\mathcal{P}(p_k)$ generate a number of subclasses of the class $\mathcal{P}$.

The aim of this paper is to present some properties of the class $\mathcal{P}(p_k)$. In Section 2 we prove the continuity of functions “extremal” in $\mathcal{P}(p_k)$ as regards the parameter $k$. Some coefficients problems are treated in Section 3, in particular we obtain the sharp bound on the coefficient functional $\left| b_2 - \mu b_1^2 \right| (-\infty < \mu < \infty)$.

2. General properties of the family $\mathcal{P}(p_k)$

We recall some notation and properties of Jacobi elliptic functions which will be used in next theorems (cf. e.g. [1], [4]).

The **elliptic integral** (or normal elliptic integral) of the first kind has been defined at (1.5). By $\mathcal{K}(t)$ we denote the **complete elliptic integral of the first kind**
\[ \mathcal{K}(t) = F(1, t), \quad \text{and let} \quad \mathcal{K}'(t) = \mathcal{K}(t'), \quad t' = \sqrt{1-t^2}, \quad t \in (0, 1). \]
Let $E(w, t)$ denote the elliptic integral of the second kind, e.g.

\[ E(w, t) = \int_0^w \sqrt{1 - t^2 x^2} \, dx, \]

and let $E(t) = E(1, t)$ be the complete elliptic integral of the second kind, $t \in (0, 1)$. Also, set $E'(t) = E'(t)$. Changing the variable by $x = \sin \theta$ integrals (1.5) and (2.1) reduce to the Legendre form

\[ F(\varphi, t) = \int_0^{\varphi} (1 - t^2 \sin^2 \theta)^{-1/2} \, d\theta, \]

\[ E(\varphi, t) = \int_0^{\varphi} \sqrt{1 - t^2 \sin^2 \theta} \, d\theta. \]

The equation $z = F(\varphi, t)$, where $z$ is assumed to be real, defines $\varphi$ as a function of $z$ which has been called by Jacobi the amplitude of $z$ and denoted $\varphi = \text{am}(z, t)$. Further Jacobi introduced $\sin(\text{am}z)$, and $\cos(\text{am}z)$ (\textit{sinus and cosinus amplitudinus}) that have several applications in geometry and mechanics. Among numerous interesting properties of elliptic functions, the following will be used in the proof:

\[ \lim_{t \to 0^+} K'(t) = \lim_{t \to 0^+} E(t) = \lim_{t \to 1^-} K'(t) = \frac{\pi}{2}, \]

\[ \lim_{t \to 0^+} K'(t) = \infty, \quad \lim_{t \to 0^+} E'(t) = 1, \]

\[ \lim_{t \to 1^-} K(t) = \infty, \quad \lim_{t \to 1^-} E(t) = 1, \]

\[ \lim_{t \to 1^-} (1 - t^2)K(t) = 0, \quad \lim_{t \to 1^-} \frac{K'(t)}{K(t)} = 0, \]

\[ \lim_{t \to 1^-} \frac{K'(t)}{K(t)} = \infty, \quad \text{so that} \quad \lim_{t \to 1^-} \left( \frac{\pi K'(t)}{4K(t)} \right) = \infty. \]

Functions $K, K', E, E'$ are continuous and differentiable on $(0, 1)$, and

\[ \frac{dK(t)}{dt} = \frac{E(t) - (1 - t^2)K(t)}{t(1 - t^2)}, \quad \frac{dE(t)}{dt} = \frac{E(t) - K(t)}{t}, \]

\[ \frac{dK'(t)}{dt} = \frac{t^2 K'(t) - E(t)}{t(1 - t^2)} \]

from which the Legendre identity can be derived (cf. [1, p. 112])

\[ E(t)K'(t) + E'(t)K(t) - K(t)K'(t) = \frac{\pi}{2}. \]

Further, (2.6) and the above identity yields the result

\[ \frac{dK'(t)/K(t)}{dt} = \frac{K(t)K'(t) - E(t)K'(t) - E'(t)K(t)}{t(1 - t^2)K^2(t)} = \frac{\pi}{2t(1 - t^2)K^2(t)}. \]
Moreover
\[ (2.8) \quad \frac{\pi}{1 + \sqrt{1 - t^2}} \leq K(t) \leq \frac{\pi}{2\sqrt{1 - t^2}}, \]
(c.f. [2], see also [3]).
Finally, by a simple computation we arrive at
\[ (2.9) \quad \lim_{t \to 1^-} F(\sqrt{z/t}, t) = \text{arctanh} \sqrt{z}. \]

We now return to functions “extremal” in the class \( \mathcal{P}(p_k) \).

**Theorem 2.1.** Functions \( p_k \) is continuous as regards the parameter \( k \in [0, \infty) \).

**Proof.** First we observe that
\[
\lim_{k \to 1^-} p_k(z) = \lim_{k \to 0^+} \left( 1 + \frac{2}{1 - k^2} \sinh^2 \left( A(k) \arctanh \sqrt{z} \right) \right) = 1 + \frac{2}{\pi^2} \pi = p_1(z).
\]

Simultaneously, by (1.4) and setting \( t = 2 \arctanh \sqrt{z} \) one gets
\[
\lim_{k \to 1^+} p_k(z) = \lim_{k \to 1^-} \left( 1 + \frac{2}{1 - k^2} \sinh^2 \left( A(k) \arctanh \sqrt{z} \right) \right) = 1 + \frac{2}{\pi^2} \pi = p_1(z).
\]

Here, \( A(k) = (2/\pi) \arccos k \to 0^+ \) as \( k \to 1^- \).

Finally, we will prove the right-hand continuity of \( p_k \) at \( k = 1 \) if we show that
\[ (2.10) \quad \lim_{k \to 1^+} \frac{\sin \left( \frac{\pi}{2K(t)} F(\sqrt{z/t}, t) \right)}{\sqrt{k^2 - 1}} = \frac{2}{\pi} \arctanh \sqrt{z}. \]

Note that \( k = \cosh \left( \frac{\pi K'(t)}{2K(t)} \right) \), so that \( \sqrt{k^2 - 1} = \sinh \left( \frac{\pi K'(t)}{2K(t)} \right) \) and if \( k \to 1^+ \) then \( t \to 1^- \), thus (2.10) is equivalent to
\[ (2.11) \quad \lim_{t \to 1^-} \frac{\sin \left( \frac{\pi}{2K(t)} F(\sqrt{z/t}, t) \right)}{\sinh \left( \frac{\pi K'(t)}{2K(t)} \right)} = \frac{2}{\pi} \arctanh \sqrt{z}. \]
Since by (2.4) and (2.9) both, the numerator and the denominator tend to 0 we need to prove, by the l’Hospital rule, that there exists the limit of the quotient of derivatives of (2.11), or equivalently

\[
\lim_{t \to 1^-} \left( \cos \left( \frac{\pi}{2K(t)} F(\sqrt{z/t}, t) \right) \cdot \frac{\pi}{2} \left[ \frac{E(t)-(1-t^2)K(t)}{t(1-t^2)K(t)} F(\sqrt{z/t}, t) - \frac{1}{K(t)} \frac{d[F(\sqrt{z/t}, t)]}{dt} \right] \right) = \cosh \left( \frac{\pi K'(t)}{2K(t)} \right) \frac{\pi}{2} \frac{1}{t(1-t^2)K^2(t)}.
\]

(2.12)

Set

\[ D(z, t) = \left( E(t) - (1-t^2)K(t) \right) F(\sqrt{z/t}, t) - t(1-t^2)K(t) \frac{d[F(\sqrt{z/t}, t)]}{dt}. \]

Then (2.12) reduces to

\[
\frac{2}{\pi} \lim_{t \to 1^-} \frac{\cos \left( \frac{\pi}{2K(t)} F(\sqrt{z/t}, t) \right)}{\cosh \left( \frac{\pi K'(t)}{2K(t)} \right)} D(z, t).
\]

Differentiating with respect to \( t \) we obtain from (1.5)

\[
\frac{d[F(\sqrt{z/t}, t)]}{dt} = \int_0^{\sqrt{z/t}} \frac{tx^2}{\sqrt{1-x^2}(1-t^2x^2)^3} \, dx = -\frac{\sqrt{z}}{2t\sqrt{1-z}\sqrt{1-\sqrt{z}}}
\]

so that

\[
\lim_{t \to 1^-} \frac{d[F(\sqrt{z/t}, t)]}{dt} = \frac{1}{4} \log \frac{1 - \sqrt{z}}{1 + \sqrt{z}} - \frac{1}{2(1 - \sqrt{z})}.
\]

Since, by (2.4)

\[
\lim_{t \to 1^-} t(1-t^2)K(t) = 0, \quad \text{and} \quad \lim_{t \to 1^-} E(t) = 1
\]

then using (2.9) we obtain

\[
\lim_{t \to 1^-} D(z, t) = \arctanh \sqrt{z}.
\]

Therefore, the above and first and fourth relation from (2.4) finally yield

\[
\frac{2}{\pi} \lim_{t \to 1^-} \frac{\cos \left( \frac{\pi}{2K(t)} F(\sqrt{z/t}, t) \right)}{\cosh \left( \frac{\pi K'(t)}{2K(t)} \right)} D(z, t) = \frac{2}{\pi} \arctanh \sqrt{z},
\]

that is equivalent to (2.11). It completes the proof. \( \square \)

**Theorem 2.2.** Let \( k \in [0, \infty) \) be fixed. The function \( p_k(z) \) has the positive Taylor coefficients around the origin.
The proof of Theorem 2.2 appeared complicated for the case $k \in (1, \infty)$ and has been proved in [11] using the theory of continued fractions. We quote it here for the sake of completeness. Applying Theorem 2.2 estimates of the modulus and the real part of $p \in \mathcal{P}(p_k)$ were derived [11].

3. Coefficient bounds

Now, we find some bounds in the family $\mathcal{P}(p_k)$. The first problem, we discuss, is the Fekete-Szegő-Goluzin’s problem in the class $\mathcal{P}(p_k)$. We begin by proving the theorem that is itself interesting, since it improves the Livingston result in $\mathcal{P}(p_k)$ [12]. For fixed $k$, set

$$p_k(z) = 1 + p_1(k)z + p_2(k)z^2 + \cdots, \quad z \in \mathcal{U}.$$  

**Theorem 3.1.** Let $0 \leq k < \infty$ be fixed. Then

$$|P_2^2(k) - P_2(k)| \leq P_1(k). \tag{3.1}$$

**Proof.** The inequality (3.1) is obvious for $k = 0$, by Livingston result [12], therefore we assume $k > 0$. We consider separately cases:

1. $k \in (0, 1)$,
2. $k = 1$,
3. $k > 1$.

**Case 1.** By virtue of (1.4) a precise form of coefficients of $p_k$ were derived (cf. [5], [9])

$$P_1(k) = \frac{2A^2(k)}{1-k^2}, \quad P_2(k) = \frac{2A^2(k)(A^2(k) + 2)}{3(1-k^2)} = P_1(k) \frac{A^2(k) + 2}{3},$$

$$A(k) = \frac{2}{\pi} \arccos k.$$

Since $P_1(k)$ is positive for $k \in (0, 1)$, the inequality (3.1) reduces to proving $|P_1(k) - (A^2(k) + 2)/3| \leq 1$. Note that

$$P_1(k) - \frac{A^2(k) + 2}{3} = \frac{A^2(k)(5 + k^2) + 1 - k^2}{3(1-k^2)} > 0$$

for $k \in (0, 1)$, then it suffices to show the inequality $P_1(k) - (A^2(k) + 2)/3 \leq 1$, or equivalently

$$\frac{5(1-k^2)}{k^2 + 5} - \frac{4}{\pi^2} \arccos^2 k \geq 0.$$

Set

$$h(k) := \sqrt{\frac{5(1-k^2)}{k^2 + 5}} - \frac{2}{\pi} \arccos k.$$  

Then functions $h$ is well defined on a closed interval $[0, 1]$ and

$$h'(k) = \frac{2}{\pi \sqrt{1-k^2}} \left[ 1 - \frac{3\sqrt{5}\pi k}{(k^2 + 5)^{3/2}} \right].$$
Note that $h'(0) > 0, h'(1^-) < 0$. Since $g(k) = \frac{3\sqrt{5}k}{k^2 + 5/2}$ is monotone then there exists the only point $k_0 \in (0,1)$ such that $h'(k_0) = 0$. Then $h'(k) > 0$ in $0 < k < k_0$ and $h'(k) < 0$ in $0 < k_0 < k < 1$. Therefore $h(k) \geq \min\{h(0), h(1)\} = 0$ in $0 \leq k < 1$, so that the proof of the case 1. is complete.

2. In this instance $P_1(k) = 8/\pi^2$ and $P_2(k) = 16/(3\pi^2) = 2P_1(k)/3$ (cf. [14], [15]). The inequality (3.1) now follows immediately by means of the relation $-1/3 < P_1(k) < 5/3$.

Case 3. To this purpose we use the following form of $p_k$ for $k > 1$

$$p_k(z) = \frac{1}{k^2 - 1} \sin \left( \frac{\pi}{2K(t)} \right) \left( u(z)/\sqrt{t} + \frac{k^2}{k^2 - 1} \right),$$

(cf. [5, p. 20]), where $u(z) = (z - \sqrt{t})/(1 - \sqrt{t})$ and $k = \cosh(\mu(t)/2)$ for $t \in (0,1)$.

In view of [5]

$$P_1(k) = \frac{\pi^2}{2(k^2 - 1)K^2(t)\sqrt{t}(1 + t)},$$

$$P_2(k) = P_1(k) \frac{4K^2(t)(t^2 + 6t + 1) - \pi^2}{24K^2(t)\sqrt{t}(1 + t)} =: P_1(k)D(k),$$

see also [9]. Since $P_1(k)$ is positive for all $t \in (0,1)$, the inequality (3.1) will hold if $|P_1(k) - D(k)| \leq 1$, equivalently $P_1(k) \leq D(k) + 1$ and $D(k) \leq P_1(k) + 1$. Now, we will show that the inequality $D(k) \leq P_1(k) + 1$ holds. We rewrite the inequality $D(k) \leq P_1(k) + 1$ into the form

$$\frac{4K^2(t)(t^2 + 6t + 1) - \pi^2}{4K^2(t)} \leq \frac{3\pi^2}{2(k^2 - 1)K^2(t)} + 6\sqrt{t}(1 + t).$$

Observing that $k^2 - 1 = \sinh^2(\pi K'(t)/(4K(t)))$, the relation (3.3) becomes

$$t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t} + 1 - \frac{\pi^2}{4K^2(t)} \leq \frac{3\pi^2}{2K^2(t)\sinh^2 \left( \frac{\pi K(t)}{4K(t)} \right)}.$$ 

Observe next, that if $k \to 1^+$ then $t \to 1^-$ and the case $k \to \infty$ corresponds to the case $t \to 0^+$. Thus, we may study the inequality (3.4) as regards $t \in (0,1)$.

The left-hand side function satisfies

$$t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t} + 1 - \frac{\pi^2}{4K^2(t)} \leq t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t} + 1 - (1 - t^2).$$

by means of the right-hand estimation in (2.8). Set

$$w(t) := t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t} + 1 - (1 - t^2) = 2t^2 - 6t\sqrt{t} + 6t - 6\sqrt{t}.$$ 

The function $w(t)$ is defined on the closed interval $[0,1]$ and it decreases continuously from $w(0) = 0$ to $w(1) = -4$. Indeed $w'(t) = 4t - 9\sqrt{t} + 6 - 3/\sqrt{t} = (t - 3) + 3(\sqrt{t} - 1)/\sqrt{t} < 0$ on $(0,1)$. Therefore $w(t) < 0$ on $(0,1)$.
Now we show that the right-hand function of (3.4) is positive in (0, 1) and increases from 0 to $96/\pi^2$. Let

$$W(t) := \frac{3\pi^2}{2K^2(t) \sinh^2 \left( \frac{\pi K'(t)}{4K(t)} \right)}.$$  

Making use the first relation in (2.2) and the last relation in (2.5) we find that

$$\lim_{t \to 0^+} W(t) = 0.$$  

Also, after necessary transformations, we obtain

$$\lim_{t \to 1^-} W(t) = \lim_{t \to 1^-} \frac{24 \left( \frac{\pi K'(t)}{4K(t)} \right)^2}{(K')^2(t) \sinh^2 \left( \frac{\pi K'(t)}{4K(t)} \right)} = \frac{96}{\pi^2},$$

because of the last formula in (2.4) and the fact that $\lim_{x \to 0^+} \sinh x/x = 1$. Now, we will show that $W(t)$ is increasing. Differentiating $W(t)$ and using (2.6) and (2.7) one gets

$$W'(t) = -\frac{3\pi^2}{t(1 - t^2)K^3(t) \sinh^3 \left( \frac{\pi K'(t)}{4K(t)} \right)} \left( \frac{\pi K'(t)}{4K(t)} \right) \text{cosh} \left( \frac{\pi K'(t)}{4K(t)} \right)$$

In order to show that $W'(t) > 0$ it suffices to prove that the expression in the square brackets of $W'(t)$ is negative for $t \in (0, 1)$. Such relation may be rewritten in the form

$$(\mathcal{E}(t) - (1 - t^2)K(t)) \sinh \left( \frac{\pi K'(t)}{4K(t)} \right) < \frac{\pi^2}{8K(t)} \cosh \left( \frac{\pi K'(t)}{4K(t)} \right)$$

or

(3.6)  

$$\frac{8}{\pi^2} \left[ \frac{\mathcal{E}(t) - (1 - t^2)K(t)}{K(t)} \right] < \coth \left( \frac{\pi K'(t)}{4K(t)} \right).$$

Set

$$\phi(t) = \mathcal{E}(t) - (1 - t^2)K(t).$$

Then, in view of (2.6) we have

$$\frac{d\phi(t)}{dt} = \frac{\mathcal{E}(t) - K(t)}{t} - \frac{-2tK(t) + \mathcal{E}(t) - K(t) + t^2K(t)}{t} = tK(t) > 0$$

in (0, 1). Moreover $\phi(0^+) = 0$ and $\phi(1^-) = 1$ by the second and third relation in (2.4). Thus $0 < \phi(t) < 1$ in (0, 1). Note also that $K(t)$ is increasing from $\pi/2$ to $\infty$, when $t \in (0, 1)$. Therefore

$$\frac{8}{\pi^2} \left[ \frac{\mathcal{E}(t) - (1 - t^2)K(t)}{K(t)} \right] < \frac{8}{\pi^2} \frac{1}{\pi/2} = \frac{16}{\pi^3} < 1,$$

whereas the right-hand side of (3.6) is greater than 1 since $\left( \frac{\pi K'(t)}{4K(t)} \right) > 0$. Then the inequality (3.6) holds, equivalently $W'(t) > 0$ on (0, 1) so that $W$ increases
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on $(0, 1)$. Thus, having in view properties of $w(t)$ and $W(t)$ we conclude that $w(t) \leq W(t)$ for all $t \in (0, 1)$, so that (3.4) is satisfied.

Next, we will show that $P_1(k) \leq D(k) + 1$ holds for $t \in (0, 1)$, or equivalently

$$\frac{3\pi^2}{2K^2(t)\sinh^2\left(\frac{\pi K'(t)}{4K(t)}\right)} \leq t^2 + 6t\sqrt{t} + 6t + 6\sqrt{t} + 1 - \frac{\pi^2}{2K^2(t)},$$

by reversing the inequality in (3.4). Since $K(t) > \frac{\pi}{2}$ we have $-\frac{\pi^2}{4K^2(t)} > -1$ so that it suffices to show that

$$\frac{3\pi^2}{2K^2(t)\sinh^2\left(\frac{\pi K'(t)}{4K(t)}\right)} \leq t^2 + 6t\sqrt{t} + 6t + 6\sqrt{t}. \tag{3.7}$$

Set

$$r(t) := t^2 + 6t\sqrt{t} + 6t + 6\sqrt{t}.$$

Then $r'(t) = 2t + 9\sqrt{t} + 6 + 3\sqrt{t} > 0$ on $(0, 1)$ and $r(0) = 0, r(1) = 19$. Moreover $r(1/\sqrt{2}) \approx 14.158379 > \frac{9\pi}{2}$ whereas the value $\frac{9\pi}{2}$ is the supremum of the left hand side of (3.7) as was shown in the first part of the proof of that case. Thus, it suffices to show (3.7) for $t \in (0, 1/\sqrt{2})$. Since $K(t) > \frac{\pi}{2}$ then

$$\frac{3\pi^2}{2K^2(t)\sinh^2\left(\frac{\pi K'(t)}{4K(t)}\right)} < \frac{6}{\sinh^2\left(\frac{\pi K'(t)}{4K(t)}\right)}.$$

Now, we will show that

$$\frac{6}{\sinh^2\left(\frac{\pi K'(t)}{4K(t)}\right)} \leq 6(t + \sqrt{t}) \leq r(t)$$

for $t \in (0, 1/\sqrt{2}]$, which concludes the desired result. The last inequality is obvious therefore it suffices to show that

$$(t + \sqrt{t})\sinh^2\left(\frac{\pi K'(t)}{4K(t)}\right) - 1 \geq 0. \tag{3.8}$$

Let

$$s(t) := (t + \sqrt{t})\sinh^2\left(\frac{\pi K'(t)}{4K(t)}\right) - 1.$$

Now we prove that $s(t)$ decreases in $(0, 1/\sqrt{2})$ to $s(1/\sqrt{2}) > 0$. Differentiating, we obtain

$$s'(t) = \frac{1}{2} \sinh\left(\frac{\pi K'(t)}{2K(t)}\right) \left[ 1 + \frac{1}{2\sqrt{t}} \tanh\left(\frac{\pi K'(t)}{4K(t)}\right) - \frac{\pi^2(\sqrt{t} + 1)}{4\sqrt{t}(1 - t^2)K^2(t)} \right].$$

Since $\sinh\left(\frac{\pi K'(t)}{2K(t)}\right)$ is positive for $t \in (0, 1/\sqrt{2}]$ then $s'(t) < 0$ if and only if the expression in square brackets of $s'(t)$ is negative, or equivalently

$$\left(1 + \frac{1}{2\sqrt{t}} \tanh\left(\frac{\pi K'(t)}{4K(t)}\right) - \frac{\pi^2(\sqrt{t} + 1)}{4\sqrt{t}(1 - t^2)K^2(t)} \right) < 0.$$
The above will be fulfilled if
\[
\frac{2\sqrt{t} + 1}{2(\sqrt{t} + 1)} \tanh \left( \frac{\pi K'(t)}{4K(t)} \right) - \frac{\pi^2}{4(1 - t^2)K^2(t)} < 0,
\]
or, by means of the relation \( \tanh x < 1 \), when the inequality
\[
\frac{2\sqrt{t} + 1}{2(\sqrt{t} + 1)} - \frac{\pi^2}{4(1 - t^2)K^2(t)} < 0
\]
holds. It is easy to see that \( b(t) = \frac{2\sqrt{t} + 1}{2(\sqrt{t} + 1)} \) is increasing on \((0, 1/\sqrt{2})\) with the maximal value \( b(1/\sqrt{2}) \approx 0.73 \). Let
\[
c(t) := \frac{\pi^2}{4(1 - t^2)K^2(t)}.
\]
Since, by (2.6),
\[
c'(t) = \frac{\pi^2}{2} \frac{K(t) - E(t)}{t(1 - t^2)K^3(t)}
\]
and \( K(t) > E(t) \) on \((0, 1)\), then \( c'(t) > 0 \) on \((0, 1)\) so does on \((0, 1/\sqrt{2})\) and therefore \( c(t) > c(0^+) = 1 \) for all \( t \in (0, 1/\sqrt{2}) \). Thus \( b(t) - c(t) < 0 \) for all \( t \in (0, 1/\sqrt{2}) \) and hence \( s(t) \) decreases on \((0, 1/\sqrt{2})\). Thus
\[
b(t) - c(t) < 0 \quad \text{for all } t \in (0, 1/\sqrt{2}).
\]

Next, we show that \( s(0^+) = \infty \). Note that
\[
\sinh^2 \left( \frac{\pi K'(t)}{4K(t)} \right) = \left( \frac{\pi K'(t)}{4K(t)} \right)^2 \left[ 1 + \frac{1}{3!} \left( \frac{\pi K'(t)}{4K(t)} \right)^2 \cdots \right]^2.
\]
Then
\[
\lim_{t \to 0^+} (t + \sqrt{t}) \sinh^2 \left( \frac{\pi K'(t)}{2K(t)} \right) = \lim_{t \to 0^+} \frac{\pi^2(1 + \sqrt{t})}{4K^2(t)} \sqrt{t} (K'(t))^2 \left[ 1 + \frac{1}{3!} \left( \frac{\pi K'(t)}{2K(t)} \right)^2 \cdots \right]^2.
\]
By properties of \( K(t) \) at 0\(^+\) (the relation (2.2)) we have
\[
\lim_{t \to 0^+} \frac{\pi^2(1 + \sqrt{t})}{4K^2(t)} = 1,
\]
so that we need to calculate the limit
\[
\lim_{t \to 0^+} \sqrt{t} (K'(t))^2.
\]
Applying (2.8) to the value \( \sqrt{1 - t^2} \) we obtain
\[
\frac{\pi}{1 + t} \leq K(\sqrt{1 - t^2}) = K'(t) \leq \frac{\pi}{2t},
\]
and since \( \sqrt{t}/(1 + t)^2 \) and \( \sqrt{t}/t^2 \) tend to \( \infty \), as \( t \to 0^+ \), we conclude that
\[
\lim_{t \to 0^+} \sqrt{t} (K'(t))^2 = \infty.
\]
Thus also \( \lim_{t \to 0^+} s(t) = \infty \). Moreover \( s(1/\sqrt{2}) \approx 0.9614 > 0 \) so that we obtain the desired result. Hence the proof of the case 3. is complete. \( \Box \)

**Theorem 3.2.** Let \( 0 \leq k < \infty \) be fixed, and let a function \( p \in \mathcal{P}(p_k) \) be such that \( p(z) = 1 + b_1 z + b_2 z^2 + \cdots \). Then

\[
|b_1^2 - b_2| \leq P_1(k).
\]

The equality holds if \( p(z) = p_k(z^2) \) or one of its rotation.

**Proof.** Since \( p \prec p_k \) then, in view of a definition of the subordination, there exists a function \( \omega(z) = \alpha_1 z + \alpha_2 z^2 + \cdots \), \( |\omega(z)| < 1 \) such that \( p(z) = p_k(\omega(z)) \), therefore

\[
1 + b_1 z + b_2 z^2 + \cdots = 1 + P_1(k)\alpha_1 z + z^2(P_1(k)\alpha_2 + P_2(k)\alpha_1^2) + \cdots.
\]

Comparing the coefficients of \( z \) and \( z^2 \) we have \( b_1 = P_1(k)\alpha_1 \) and \( b_2 = P_1(k)\alpha_2 + P_2(k)\alpha_1^2 \); thus

\[
|b_1^2 - b_2| = |P_1^2(k)\alpha_2^2 - P_1(k)\alpha_2 - P_2(k)\alpha_1^2|
\]

\[
= |\alpha_2^2(P_1^2(k) - P_2(k)) - P_1(k)\alpha_2|
\]

\[
\leq |\alpha_1|^2|P_1^2(k) - P_2(k)| + |P_1(k)||\alpha_2|.
\]

For the Schwarz' function \( \omega \) the classical inequality \( |\alpha_2| \leq 1 - |\alpha_1|^2 \) holds then, on account (3.1), we conclude

\[
|b_1^2 - b_2| \leq |\alpha_1|^2|P_1^2(k) - P_2(k)| + |P_1(k)|(1 - |\alpha_1|^2)
\]

\[
= |\alpha_1|^2 [P_1^2(k) - P_2(k)] - P_1(k)] + P_1(k)
\]

\[
\leq P_1(k),
\]

and the proof of the inequality of (3.11) is complete.

The equality in (3.11) holds if \( |b_1| = 0 \) and \( |b_2| = P_1(k) \), or equivalently, \( p(z) \) is \( p_k(z^2) \) or one of its rotations. \( \Box \)

**Remark.** Observe that the bound like (3.11) can be used in those subclasses of Carathéodory class for which the inequality similar to (3.1) holds. Let

\[
q(z) = 1 + c_1 z + c_2 z^2 + \cdots,
\]

be such that

\[
|c_1^2 - c_2| \leq c_1, \quad \text{with} \quad c_1 \geq 0,
\]

it means \( q \) satisfies the bounds similar to (3.1). Then, reasoning along the same line as in Theorem 3.2 we may prove that for \( p \in \mathcal{P}(q) \), \( p(z) = 1 + b_1 z + b_2 z^2 + \cdots \) the inequality

\[
|b_1^2 - b_2| \leq c_1,
\]

is satisfied.

For instance, if \( 0 < \gamma \leq 1 \) then the function \( \varphi(z) = [(1 + z)/(1 - z)]^\gamma \) maps the unit disk onto an angle, symmetric with respect to real axis, of width \( \gamma \pi \) and contained in the right half-plane. Moreover, \( \varphi(z) = 1 + 2\gamma z + 2\gamma^2 z^2 + \cdots \). Then, \( |c_1^2 - c_2| = 2\gamma^2 \leq 2\gamma = c_1 \) therefore (3.13) is satisfied, so that (3.14) applies.
Concluding, if \( p \in \mathcal{P}(\varphi) = \{ q : q \prec \varphi \} \) and \( p(z) = 1 + b_1z + b_2z^2 + \cdots \), then \( |b_1^2 - b_2| \leq 2\gamma \). This inequality remarkable improves the result \( |\beta_1^2 - \beta_2| \leq 2 \), due to Ma and Minda [13]. Similarly, the family \( \mathcal{P}((1 + (1 - 2\beta)z)/(1 - z)) \) can be treated. In this instance we obtain the inequality \( |b_1^2 - b_2| \leq 2(1 - \beta) \) for \( p \in \mathcal{P}((1 + (1 - 2\beta)z)/(1 - z)) \).

**Theorem 3.3.** Let \( 0 \leq k < \infty \) be fixed, and let a function \( p \in \mathcal{P}(p_k) \) be of the form \( p(z) = 1 + b_1z + b_2z^2 + \cdots \). Then

\[
|b_2 - \mu b_1^2| \leq \begin{cases} 
P_1(k) - \mu P_1^2(k) & \mu \leq 0, \\
P_1(k) & \mu \in [0, 1], \\
P_1(k) + (\mu - 1)P_1^2(k) & \mu \geq 1.
\end{cases}
\]

When \( \mu < 0 \) or \( \mu > 1 \), the equality holds if \( p(z) = p_k(z) \) or one of its rotations. If \( 0 < \mu < 1 \) then the equality holds if \( p(z) = p_k(z^2) \) or one of its rotations.

**Proof.** Since \( p \prec p_k \) then by Rogosinski Subordination Theorem we have \( |b_n| \leq P_1(k) \) for \( n \geq 1 \) and each fixed \( k \in [0, \infty) \) First assume that \( \mu \geq 1 \). In view of Theorem 3.2, we have \( |b_2 - b_1^2| \leq P_1(k) \) therefore we obtain

\[
|b_2 - \mu b_1^2| \leq |b_1^2 - b_2| + (\mu - 1)|b_1|^2 \leq P_1(k) + (\mu - 1)P_1^2(k).
\]

Next, suppose that \( \mu \leq 0 \). Then

\[
|b_2 - \mu b_1^2| \leq |b_2| + (\mu)|b_1|^2 \leq P_1(k) - \mu P_1^2(k).
\]

Finally, if \( 0 < \mu < 1 \) then \( \mu = 1/t \) with \( t \geq 1 \). Hence one gets

\[
|b_2 - \mu b_1^2| = |b_2 - \frac{1}{t}b_1^2| = \frac{2}{t}|b_2 - b_1^2| = \frac{1}{t}|(t - 1)b_2 + b_2 - b_1^2| \leq \frac{1}{t}|(t - 1)|b_2| + |b_2 - b_1^2| | \leq \frac{1}{t}|(t - 1)P_1(k) + P_1(k)| = P_1(k),
\]

and the proof of all cases of (3.15) is complete.

When \( \mu < 0 \) or \( \mu > 1 \), the equality holds if and only if \( |b_1| = P_1(k) \), that is, \( p(z) = p_k(z) \) or one of its rotation. If \( 0 < \mu < 1 \) then the equality holds if \( |b_1| = 0 \) and \( |b_2| = P_1(k) \), or equivalently, \( p(z) \) is \( p_k(z^2) \) or one of its rotations. \( \square \)

**Remark.** In the paper [13] Ma and Minda proved similar bounds in the class \( \mathcal{P} \). For instance, when \( \mu \leq 0 \) authors obtained the estimate \( |b_2 - \mu b_1^2| \leq 2 - 4\mu \). Observe that in the case of \( \mathcal{P}(p_k) \) the result is far better; the same it holds in the remaining range of the constant \( \mu \).

**References**


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