ON P-EXTENDING MODULES

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Abstract. Let $R$ be a ring. A right $R$-module $M$ is called quasi-principally injective if it is $M$-principally injective. In this paper, we give some characterizations and properties of principally injective modules, which generalize results of Nicholson and Yousif. For a quasi-principally injective module $M$, we show: 1. For isomorphic submodules $H, K$ of $M$, we have $SH = SK$, where $S$ is the endomorphism ring of $M$. 2. $M$ has ($PC_2$), and consequently has ($PC_3$). We characterize when a direct sum of $P$-extending modules is $P$-extending, and when a direct sum of a $P$-extending module and a semi-simple module is $P$-extending. We also characterize when a direct sum of $FP$-extending modules is $FP$-extending. Finally, we discuss when a direct sum of $P$-extending modules with relatively EC-injective is $P$-extending.

1. Introduction

In [7], Nicholson and Yousif have introduced and studied the structure of principally injective rings, and have given some characterizations of such rings in terms of the internal properties of these rings. In fact, they defined principally injective modules in the following sense: A right module $M$ over a ring $R$ is called principally injective (for short $P$-injective) if every $R$-homomorphism from a principal right ideal of $R$ to $M$ can be extended to $R$. In [8], Wongwai extended this notion to modules by making use of $M$-cyclic submodules of $M$.

Here, we adopt the extension of the concept of principally injective rings, which is given in [7], to modules. The fact that, cyclic and $M$-cyclic submodules of a module $M$ are not the same (e.g., as $Z$-modules, the integers $Z$ is cyclic submodule, but not a $Q$-cyclic submodule, of $Q$, and $Q$ is $Q$-cyclic but not cyclic, of $Q$), gives the independence of the concepts of $N$-principally injective by Wongwai and the one we are dealing with.

We also introduce the definitions of principally extending, (for short $P$-extending), and $P$-(quasi-)continuous modules as follows: For a right $R$-module $M$,

1. $M$ is called a $P$-extending module if every cyclic submodule of $M$ is essential in a direct summand of $M$ or, equivalently, every EC-closed submodule of $M$ is a summand. $M$ is called an FP-extending module if every finite uniform dimension EC-closed submodule of $M$ is a summand.

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2. $M$ is called a $P$-quasi-continuous module if it is $P$-extending, and the following condition holds: $(PC_3)$ For each $a, b \in M$, if $aR$ and $bR \leq \oplus M$ with $aR \cap bR = 0$, then $aR \oplus bR \leq \oplus M$.

3. $M$ is called a $P$-continuous module if it is $P$-extending, and the following condition holds: $(PC_2)$ For each $a, b \in M$, if $aR \cong bR$ and $bR \leq \oplus M$, then $aR \leq \oplus M$.

It is known that in regular rings the condition $(C_2)$ is satisfied, and so such rings are continuous if and only if they are extending. Consequently, every regular ring is $P$-continuous as a module over itself. It is also clear that regular rings are $P$-injective rings. This allows us to find $P$-injective modules, which are not injective.

Direct sums of extending modules have been investigated in great detail, in a long series of papers, by Dung and Smith [3], and by Kamal and Muller [4], [5]. The present paper studies direct sums of $P$-extending modules, and we investigate when such direct sums are $P$-extending.

It is known that $M$ is $N$-injective if and only if for every submodule $A$ of $N \oplus M$ with $A \cap M = 0$, there exists a submodule $B$ of $N \oplus M$ such that $A \leq B$, and $N \oplus M = B \oplus M$. In analogue, we introduce the concept of $N$-EC-injectivity, and give a characterization of such modules different from the diagram description. This helps us to build up blocks of $P$-extending modules, which are relatively EC-injective to obtain $P$-extending modules. We prove that, if $M = M_1 \oplus M_2$, then $M_i$ is $P$-extending and is $M_j$-EC-injective ($i \neq j = 1, 2$) if and only if $M = C \oplus M'_i \oplus M_j$, where $M'_i \leq M_i$, for every EC-closed submodule $C$ of $M$ with $C \cap M_j = 0$ ($i \neq j = 1, 2$).

All modules here are right modules over a ring $R$. The right (respectively, left) annihilator of a subset $X$ of a module is denoted by $\tau_R(X)$ (resp. $l_R(X)$). A submodule $A$ of a module $M$ is called essential in $M$ or $M$ is an essential extension of $A$ (denoted by $A \leq \oplus M$), if every non-zero submodule of $M$ has non-zero intersection with $A$. $X \leq \oplus M$ signifies that $X$ is a direct summand of $M$.

A submodule $A$ of $M$ is called $M$-cyclic submodule of $M$ if it is isomorphic to $M/X$, for some submodule $X$ of $M$. The injective hull and the uniform dimension of a module $M$ will be denoted by $E(M)$ and $U - \dim(M)$ respectively. The endomorphism ring of a module $M$ is denoted by $\text{End}(M)$. A submodule is closed in $M$ if it has no proper essential extensions in $M$. The graph of a homomorphism $f : N \to M$ is the submodule $(f) = \{n - f(n) : n \in N\}$ of $N \oplus M$.

A module $M$ is extending ($n$-extending) if every closed submodule $A$ (with $U - \dim(A) \leq n$) is a direct summand of $M$, or equivalently to the requirement that every submodule $A$ (with $U - \dim(A) \leq n$) is essential in a direct summand of $M$.

A module $M$ is called quasi-continuous if it is extending module, and the following condition holds: $(C_3)$ For all $X$, and $Y \leq \oplus M$, with $X \cap Y = 0$, one has $X \oplus Y \leq \oplus M$. $M$ is called continuous if it is extending module, and the following condition holds: $(C_2)$ If a submodule $A$ of $M$ is isomorphic to a direct summand of $M$, then $A$ is a direct summand of $M$.
2. Principally Injective Modules

Let $R$ be a ring and $M, N$ be $R$-modules. $M$ is called $N$-principally injective (for short $N$-$P$-injective) if every $R$-homomorphism from a cyclic submodule of $N$ to $M$ can be extended to $N$. Equivalently, for each $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$, there exists $f \in \text{Hom}_R(N, M)$ such that $m = f(n)$.

Within the proof of [2, Proposition 1.1], it was observed that $M$ is $N$-injective if and only if $N \oplus M = C \oplus M$, for every complement $C$ of $M$ in $N \oplus M$. The condition 3. in the next Proposition is analogous with such observation.

**Proposition 2.1.** Let $M$ and $N$ be $R$-modules, and $S = \text{End}(M)$. Then the following are equivalent:

1. $M$ is $N$-$P$-injective;
2. For each $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$, we have $Sm \subseteq \text{Hom}_R(N, M)n$;
3. For each $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$, there is a complement $C$ of $M$ in $N \oplus M$ with $n - m \in C$ and $N \oplus M = C \oplus M$;
4. For each $n \in N$, $l_Mr_R(n) = \text{Hom}_R(N, M)n$;
5. For each $n \in N$ and $a \in R$, $l_M[aR \cap r_R(n)] = l_M(a) + \text{Hom}_R(N, M)n$.

**Proof.**

1. $\Rightarrow$ 2.: Let $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$. Since $M$ is $N$-$P$-injective, there then exists a homomorphism $f : N \to M$ such that $m = f(n)$. Let $\phi \in S$, then $\phi(n) \in \text{Hom}_R(N, M)n$. Therefore, $Sm \subseteq \text{Hom}_R(N, M)n$.

2. $\Rightarrow$ 3.: Let $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$, then by 2, there exists a homomorphism $f : N \to M$ such that $m = f(n)$. Hence $N \oplus M = (f) \oplus M$, where $(f)$ is the graph of a homomorphism $f : N \to M$. Therefore, $C = (f)$ is a complement of $M$ in $N \oplus M$ with $N \oplus M = C \oplus M$ and $n - m \in C$.

3. $\Rightarrow$ 4.: Let $n \in N$ and $x \in l_Mr_R(n)$, then $r_R(n) \subseteq r_R(x)$. By 3, there is a complement $C$ of $M$ in $N \oplus M$ with $n - x \in C$ and $N \oplus M = C \oplus M$. So, there exists a homomorphism $f : N \to M$ such that $C = (f)$. Since $n - x = n' - f(n')$, for some $n' \in N$. So, $n = n'$ and $x = f(n') = f(n)$. Hence $x \in \text{Hom}_R(N, M)n$, and $l_Mr_R(n) \subseteq \text{Hom}_R(N, M)n$. The other conclusion is obvious.

4. $\Rightarrow$ 5.: Let $n \in N$, $a \in R$, and $x \in l_M[aR \cap r_R(n)]$, then $x(aR \cap r_R(n)) = 0$ and so $r_R(na) \subseteq r_R(xa)$. Hence $l_Mr_R(xa) \subseteq l_Mr_R(na) = \text{Hom}_R(N, M)na$, by 4. Therefore, $xa = f(na) = f(n)a$, for some $f \in \text{Hom}_R(N, M)$. So $(x - f(n))a = 0$ and $x - f(n) \in l_M(a)$. Thus $x \in l_M(a) + \text{Hom}_R(N, M)n$, and so $l_M[aR \cap r_R(n)] \subseteq l_M(a) + \text{Hom}_R(N, M)n$. On the other hand, let $x \in l_M(a) + \text{Hom}_R(N, M)n$, then $x = m + f(n)$ for some $m \in l_M(a)$ and $f \in \text{Hom}_R(N, M)$. So $xa = ma + f(n)a = f(n)a$. Let $ar \in aR \cap r_R(n)$, then $x(ar) = f(na)r = f(nar) = 0$, and so $x \in l_M[aR \cap r_R(n)]$. Thus $l_M(a) + \text{Hom}_R(N, M)n \subseteq l_M[aR \cap r_R(n)]$.

5. $\Rightarrow$ 1.: Let $m \in M$ and $n \in N$ with $r_R(n) \subseteq r_R(m)$, then $l_Mr_R(m) \subseteq l_Mr_R(n)$. By 5, we get $l_Mr_R(n) = \text{Hom}_R(N, M)n$, and so there is a homomorphism $f : N \to M$ such that $f(n) = m$. Thus $M$ is $N$-$P$-injective. $\square$
Proposition 2.2. Let $M$ be $N$-$P$-injective, then $M$ is $X$-$P$-injective, for every submodule $X$ of $N$. If, in addition, $X$ is a direct summand of $N$, then $M$ is $N/X$-$P$-injective.

Proof. It is clear. □

Lemma 2.3. Let $M$ be $N$-$P$-injective and $K \subseteq M$, then $K$ is $N$-$P$-injective.

Proof. It is obvious. □

Lemma 2.4. Let $\{M_i\}_{i \in I}$ be a family of modules. Then the direct product $\prod_{i \in I} M_i$ is $N$-$P$-injective if and only if $M_i$ is $N$-$P$-injective, for every $i \in I$.

Proof. It is clear. □

Proposition 2.5. If $M$ is a quasi-principally injective module, and $S = \text{End}(M)$, then $SH = SK$, for any isomorphic $R$-submodules $H$, $K$ of $M$.

Proof. Since $H \cong K$, then there is a right $R$-isomorphism $\sigma : H \rightarrow K$. For each $k \in K$, $k = \sigma(h)$ for some $h \in H$ and $r_R(h) = r_R(k)$. Since $M$ is quasi-principally injective, then $Sh = Sk$ by Proposition 2.1, and so $Sk \subseteq SH$, for each $k \in K$. Then $SK \subseteq SH$. Similarly, we get $SH \subseteq SK$, and so the result. □

Corollary 2.6. Let $R$ be a $P$-injective ring and $H$, $K$ be two-sided ideals of $R$. If $H \cong K$, as right ideals of $R$, then $H = K$.

Remark. In Corollary 2.6, the condition $P$-injective for the ring $R$ is not avoided. In fact, there are rings which do not satisfy the result in 2.6, for example, the ring $\mathbb{Z}$ of integers.

Theorem 2.7. Let $M$ be a quasi-principally injective module, then $M$ has $(PC_2)$.

Proof. Let $a, b \in M$ with $aR \cong bR$ and $bR \leq^{\otimes} M$. Then $bR = eM$ for some idempotent $e \in \text{End}(M)$. Since $aR \cong bR$, then there is an isomorphism $\sigma : bR \rightarrow aR$. Let $\sigma e = h$, then $aR = hM$ and $\sigma^{-1} h = e$. Since $bR \leq^{\otimes} M$, then by Lemma 2.3, $bR$ is $M$-$P$-injective, and so there exists a homomorphism $\phi : M \rightarrow bR$ such that $\phi(a) = \sigma^{-1}(a)$. Then $\phi$ is an epimorphism, $\phi h = e$, and so $f = h\phi$ is an idempotent endomorphism of $M$. Hence $fM = h\phi M = h(bR) = heM = hM$, and so $aR \leq^{\otimes} M$. □

Remark. It is known that every summand right ideal of a ring $R$ is generated by an idempotent element in $R$. Then every summand right ideal of $R$ is cyclic and so, $R$ has $(PC_i)$ if and only if $R$ has $(C_i)$, $i = 2, 3$. Therefore by [6, Proposition 2.2], if $R$ has $(PC_2)$, then $R$ has $(C_3)$.

Corollary 2.8 ([7], Theorem 2.3.). If $R$ is a $P$-injective ring, then $R$ has $(C_2)$.

Lemma 2.9. Let $M$ be an $R$-module. If $M$ has $(PC_2)$, then $M$ has $(PC_3)$. 


Proof. Let $aR \leq_{e} M$ and $bR \leq_{e} M$ with $aR \cap bR = 0$, then $aR = eM = \text{Im } e$, for some $e^{2} = e \in \text{End}(M)$, and so $aR \oplus bR = eM \oplus (1-e)bR$. Since $(1-e)bR \leq_{e} M$ and $M$ has $(PC_{2})$, then $(1-e)bR = fM$ for some $f^{2} = f \in \text{End}(M)$. Then $ef = 0$, and $h = e + f - fe$ is an idempotent in $\text{End}(M)$. Therefore, $aR \oplus bR = eM \oplus fM = (e + f - fe)M = hM \leq_{e} M$. \hfill $\Box$

Corollary 2.10. If $M$ is a quasi-principally injective module, then $M$ has $(PC_{3})$.

Definition 2.1. By an EC-(closed) submodule $C$ of a module $M$, we mean a (closed) submodule $C$ which contains essentially a cyclic submodule; i.e. there exists $c \in C$ such that $cR \leq e C$.

Lemma 2.11. Every summand of an EC- submodule of $M$ is EC-submodule.

Proof. Let $cR \leq e C$ be an EC-submodule of $M$, and $C_{1} \leq_{e} C$, then $C = C_{1} \oplus C_{2}$, for some submodule $C_{2}$ in $C$. Let $c = c_{1} + c_{2}$, where $c_{1} \in C_{1}$ and $c_{2} \in C_{2}$. It is easy to see that $c_{1}R \leq e C_{1}$. Therefore, $C_{1}$ is an EC-submodule of $M$. \hfill $\Box$

Corollary 2.12. Every summand of an EC-closed submodule of $M$ is EC-closed.

Lemma 2.13. Every summand of a $P$-(quasi-)continuous module is $P$-(quasi-) continuous.

Proof. It is obvious by Corollary 2.12 \hfill $\Box$

Lemma 2.14. For an indecomposable module $M$, the following are equivalent:
1. $M$ is extending;
2. $M$ is $P$-extending;
3. $M$ is uniform.

Lemma 2.15. A module $M$ over a right noetherian ring $R$, is 1-extending if and only if it is $P$-extending.

Proof. Let $M$ be a 1-extending module, and $cR \leq e C$ be an EC-closed submodule of $M$. Since $R$ is a noetherian ring, then $C$ has a finite uniform dimension. Since $M$ is 1-extending, then by Proposition (4) in [4], $M$ is n-extending. Hence $C$ is a summand, and so $M$ is $P$-extending. For the converse, it is obvious. \hfill $\Box$

Corollary 2.16. Let $M$ be a module with finite uniform dimension, then the following are equivalent:
1. $M$ is extending;
2. $M$ is 1-extending;
3. $M$ is $P$-extending.

Proposition 2.17. Let $M = M_{1} \oplus M_{2}$, and let $C \cap M_{1}$ be an EC-submodule of $M$, for every EC-closed submodule $C$ of $M$. Then $M$ is $P$-extending if and only if every EC-closed submodule $C$, with $C \cap M_{1} = 0$, or $C \cap M_{2} = 0$, is a summand.
Proof. The necessary condition is obvious. For the sufficient condition, let \( cR \leq C \) be an EC-closed submodule of \( M \). If \( C \cap M_1 = 0 \), then we are done. Otherwise, \( C \cap M_1 \) is an EC-submodule of \( M \), by assumption. Let \( C_1 \) be a maximal essential extension of \( C \cap M_1 \) in \( C \), then \( C_1 \) is an EC-closed submodule of \( M \), with \( C_1 \cap M_2 = 0 \). Hence by the assumption, \( C_1 \) is a summand of \( M \). Write \( M = C_1 \oplus C_2 \), by the modular law, \( C = C_1 \oplus (C \cap C_2) \). By Corollary 2.12, \( C \cap C_2 \) is an EC-closed submodule of \( M \) with \( (C \cap C_2) \cap M_1 = 0 \), and therefore, \( C \cap C_2 \) is a summand of \( M \). Thus \( C \) is a summand of \( M \), and therefore, \( M \) is \( P \)-extending. \( \square \)

Proposition 2.18. Let \( M = M_1 \oplus M_2 \), where \( M_1 \) is of finite uniform dimension. Then \( M \) is \( P \)-extending if and only if every EC-closed submodule \( C \) of \( M \), with \( C \cap M_1 = 0 \), or \( C \) is of finite uniform dimension, is a summand.

Proof. The necessary condition is obvious. For the sufficient condition, let \( mR \leq^* C \) be an EC-closed submodule of \( M \). If \( C \cap M_1 = 0 \), then we are done. Now let \( 0 \neq c \in C \cap M_1 \), and \( C_1 \) be a maximal essential extension of \( cR \) in \( C \). Since \( M_1 \) is of finite uniform dimension, so is \( C_1 \). By the given assumption, \( C_1 \) is a summand of \( M \). Write \( M = C_1 \oplus K \). Hence \( C = C_1 \oplus C^* \), where \( C^* := K \cap C \) is closed in \( M \). Let \( m = c_1 + c^* \), where \( c_1 \in C_1 \) and \( c^* \in C^* \). Since \( C^* \) is a summand of an EC-closed submodule \( C \), then by Corollary 2.12, \( C^* \) is EC-closed. If \( C^* \cap M_1 = 0 \), then by assumption \( C^* \) is a summand, and hence \( C \) is a summand of \( M \). On the other hand, if \( C^* \cap M_1 \neq 0 \), then by repeating the previous steps, we have \( C^* \supseteq C_2 \oplus C_3 \), where \( C_2 \) is a summand and has a nonzero intersection with \( M_1 \). Continuing in this manner, we should stop after a finite steps (due to \( M_1 \) a finite uniform dimensional module) and end with \( C = C_1 \oplus C_2 \oplus \ldots \oplus C_n \), where \( C_i \) is a summand of \( M \) \( (i = 1, 2, \ldots, n - 1) \), and \( C_n \) contains an essential cyclic submodule with \( C_n \cap M_1 = 0 \). Hence \( C_n \) is a summand of \( M \), by assumption, and therefore \( C \) is a summand of \( M \). \( \square \)

Corollary 2.19. Let \( M = M_1 \oplus M_2 \), where \( M_1 \) is of finite uniform dimension. Then \( M \) is \( P \)-extending if and only if every EC-closed submodule \( C \) of \( M \), with \( C \cap M_1 = 0 \), or \( C \) is of finite uniform dimension, is a summand.

Proposition 2.20. Let \( M = M_1 \oplus M_2 \). Then \( M \) is \( FP \)-extending if and only if every EC-closed submodule \( C \) of \( M \) with finite uniform dimensional such that \( C \cap M_1 = 0 \), or \( C \cap M_2 = 0 \), is a summand.

Proof. Is similar to the proof of Proposition 2.18. \( \square \)

Proposition 2.21. Let \( M = M_1 \oplus M_2 \), where \( M_1 \) is a semisimple module. Then \( M \) is \( P \)-extending if and only if every EC-closed submodule \( C \) of \( M \) with \( C \cap M_1 = 0 \), is a summand.

Proof. The necessary condition is obvious. For the sufficient condition, let \( C \) be an EC-closed submodule of \( M \). If \( C \cap M_1 = 0 \), then we are done. On the other hand, since \( M_1 \) is a semisimple, we get \( C \cap M_1 \leq^\oplus M_1 \) and so \( C = C \cap M_1 \oplus C^* \). Since \( C^* \) is an EC-closed submodule of \( M \) and \( C^* \cap M_1 = 0 \), then \( C^* \) is a summand of \( M \). Therefore, \( C \) is a summand of \( M \). \( \square \)
Proposition 2.22. Let $M = M_1 \oplus M_2$, where $M_1$ is $P$-extending and $M_2$ is $M_1$-$P$-injective. If $M_2$ is nonsingular, then every EC-closed submodule $C$ of $M$, with $C \cap M_2 = 0$, is a summand of $M$.

Proof. Let $cR \leq C$ be an EC-closed submodule of $M$ with $C \cap M_2 = 0$, and write $c = c_1 + c_2$, where $c_1 \in M_1$ and $c_2 \in M_2$. Since $M_2$ is $M_1$-$P$-injective, then by Lemma 5 in [4],

$$cR = (c_1R)^* = \{c_1 r + \phi(c_1) r : r \in R\} \subseteq (M_1)^* := \{m_1 + \phi(m_1) : m_1 \in M_1\} \cong M_1$$

and that $M = (M_2)^* \oplus M_2$, where $\phi \in \text{Hom}_R(M_1, M_2)$. Let $x \in C$ and write $x = y + m_2$, where $y \in (M_1)^*$ and $m_2 \in M_2$. Since $cR \leq C$, then there exists an essential right ideal $I$ of $R$ such that $m_2I = 0$. Since $M_2$ is nonsingular, then $m_2 = 0$. It follows that $C \subseteq (M_1)^*$. Since $(M_1)^*$ is $P$-extending, we have $C \leq (M_1)^* \leq M$.

Definition 2.2. Let $M = M_1 \oplus M_2$ be a module. The module $M_2$ is called $M_1$-EC-injective, if for every EC-(closed) submodule $N$ of $M_1$, and every homomorphism from $N$ to $M_2$ can be extended to $M_1$.

This is equivalent to for every EC-(closed) submodule $N$ of $M$ such that $N \cap M_2 = 0$, there exists $N' \leq M$ such that $N \leq N'$, and $M = N' \oplus M_2$.

Observe that every module over a regular ring $R$ is $R$-EC-injective.

Lemma 2.23. Let $M = M_1 \oplus M_2$ and $M_2$ be $M_1$-EC-injective. Then:

1. $M_2$ is $K$-EC-injective, for all $K \leq M_1$.
2. $H$ is $M_1$-EC-injective, for all $H \leq M_2$.
3. $H$ is $K$-EC-injective, for all $K \leq M_1$, and $H \leq M_2$.

Proof. Let $K$ be a submodule of $M_1$, and $N$ be an EC-submodule of $K \oplus M_2$ with $N \cap M_2 = 0$. Then $N$ is an EC-submodule of $M$. Since $M_2$ is $M_1$-EC-injective, then there is $N' \leq M$ such that $N \leq N'$, and $M = N' \oplus M_2$. Then $K \oplus M_2 = (K \oplus M_2) \cap (N' \oplus M_2) = (N' \cap (K \oplus M_2)) \oplus M_2$ and $N \leq N' \cap (K \oplus M_2)$. Hence $M_2$ is $K$-EC-injective.

2. Let $H$ be a summand of $M_2$, and $N$ be an EC-submodule of $M_1 \oplus H$ with $N \cap H = 0$. Then $N$ is an EC-submodule of $M$, and $N \cap M_2 = 0$. Since $M_2$ is $M_1$-EC-injective, then there is $N' \leq M$ such that $N \leq N'$, and $M = N' \oplus M_2$. Since $H \leq M_2$, then $M_2 = H \oplus H'$, and so $M_1 \oplus H = (M_1 \oplus H) \cap (N' \oplus H \oplus H') = H \oplus (M_1 \oplus H) \cap (N' \oplus H')$. Since $N \leq N'$, then $N \leq (M_1 \oplus H) \cap (N' \oplus H')$. Therefore, $H$ is $M_1$-EC-injective.

3. Follows from 1. and 2. □

Proposition 2.24. Let $M = M_1 \oplus M_2$, where $M_1$ is $P$-extending and $M_2$ is $M_1$-EC-injective. Then $M = C \oplus M_1 \oplus M_2$, where $M_1' \leq M_1$, for every EC-closed submodule $C$ of $M$, with $C \cap M_2 = 0$.

Proof. Let $cR \leq C$ be an EC-closed submodule of $M$ with $C \cap M_2 = 0$. Define $X := M_1 \cap (C \oplus M_2)$. Then $c_1 R \leq C$, where $c = c_1 + c_2$, where $c_1 \in M_1$ and $c_2 \in M_2$. Let $N_1$ be a maximal essential extension of $X$ in $M_1$. Then $N_1$ is an EC-closed submodule of $M_1$. Since $M_1$ is $P$-extending, we have $N_1 \leq M_1$. □
Write $M_1 = N_1 \oplus M'_1$, where $M'_1 \leq M_1$. Now $C \oplus M_2 = X \oplus M_2 \leq N_1 \oplus M_2$; i.e. $C \leq N_1 \oplus M_2$, and $C \subseteq N_1 \oplus M_2$. Then $C$ is a complement of $M_2$ in $N_1 \oplus M_2$. Since $M_2$ is $M_1$-EC-injective, and $N_1$ is a summand of $M_1$, then by Lemma 2.23 1., $M_2$ is $N_1$-EC-injective, and so there exists $N' \leq N_1 \oplus M_2$ such that $C \leq N'$, and $N_1 \oplus M_2 = N' \oplus M_2$. Hence $N'$ is a complement of $M_2$ in $N_1 \oplus M_2$, but $C$ is a complement of $M_2$ in $N_1 \oplus M_2$. Therefore, $N' = C$ and $M = M_1 \oplus M_2 = N_1 \oplus M'_1 \oplus M_2 = C \oplus M'_1 \oplus M_2$.

**Corollary 2.25.** Let $M = M_1 \oplus M_2$, where $M_i$ is $P$-extending and is $M_j$-EC-injective ($i \neq j = 1, 2$) if and only if $M = C \oplus M'_i \oplus M_j$, where $M'_i \leq M_i$, for every $EC$-closed submodule $C$ of $M$, with $C \cap M_j = 0$ ($i \neq j = 1, 2$).

**Proposition 2.26.** Let $M = M_1 \oplus M_2$, where $M_1$ and $M_2$ are relatively $EC$-injective, and either $M_1$ or $M_2$ is of finite uniform dimension. Then $M$ is $P$-extending if and only if $M_1$ and $M_2$ are $P$-extending.

**Proof.** It is follows by Corollaries 2.25, and 2.19.

**Proposition 2.27.** Let $M = \bigoplus_{i \in I} M_i$ be an $R$-module, where $M(F)$ is $P$-extending and $M(I \setminus F) = M(F)$-$EC$-injective, for all finite subset $F$ of $I$. Then $M$ is $P$-extending.

**Proof.** Let Let $c \in M$ and $C$ be a maximal essential extension of $cR$ in $M$. Then $cR \subseteq M(F)$ and $cR \cap M(I \setminus F) = 0$, for a finite subset $F$ of $I$. Since $cR \leq C$, then $C \cap M(I \setminus F) = 0$. Since $M(I \setminus F)$ is $M(F)$-$EC$-injective and $C$ is EC-closed submodule of $M$, then by Proposition 2.24, $C$ is a summand of $M$. Hence $M$ is $P$-extending.

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