ERROR ESTIMATES FOR FINITE VOLUME SCHEME FOR PERONA-MALIK EQUATION

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Abstract. We present Perona-Malik nonlinear image selective smoothing equation (modified in the sense of Catté, Lions, Morel and Coll) which is investigated especially from numerical point of view. Error estimates in $L_2$ norms for fully discrete numerical finite volume scheme are derived and proved. Some numerical examples are presented.

1. Introduction

1.1. Mathematical model of the problem

We are dealing with Perona-Malik type problem discussed in [4] in the following form

\begin{align*}
\frac{\partial_t u}{\partial} - \nabla \cdot (g(|\nabla G_\sigma \ast u|) \nabla u) &= 0 \quad \text{in } Q_T \equiv I \times \Omega, \\
\partial_\nu u &= 0 \quad \text{on } I \times \partial \Omega, \\
u(0, \cdot) &= u_0 \quad \text{in } \Omega,
\end{align*}

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where \( \Omega \subset \mathbb{R}^d \) is a rectangular domain, \( I = [0, T] \) is a scaling interval, and

\[
(4) \quad g(s) \text{ is a Lipschitz continuous decreasing function,}
\]
with Lipschitz constant \( L_g \)

\[
(5) \quad g(0) = 1, 0 < g(s) \to 0 \text{ for } s \to \infty,
\]

\[
(6) \quad G_\sigma \in C^\infty(\mathbb{R}^d) \text{ is a smoothing kernel with compact support } K_\sigma
\]
with \( \int_{\mathbb{R}^d} G_\sigma(x)dx = 1 \)
and \( G_\sigma(x) \to \delta_x \) for \( \sigma \to 0 \), \( \delta_x \) – Dirac function at point \( x \),

\[
(7) \quad \text{initial condition } u_0 \text{ is such that regularity below is fulfilled.}
\]

We can rewrite the partial differential equation (1) in the form

\[
(8) \quad \partial_t u - \nabla \cdot (g(\|J(u)(x)\|)\nabla u) = 0 \quad \text{in } Q_T \equiv I \times \Omega,
\]

where \( J(u) : L_2(\Omega) \to (C^\infty(\Omega))^d \). In our case we use \( J(u)(t, \cdot) = \nabla G_\sigma \ast u(t, \cdot) \) for \( t \) fixed, but we can choose any smoothing operator with these properties.

Let us define a weak solution to the problem (8),(2),(3). Equation (8) is multiplied by a test function \( \varphi \in \Psi \), where \( \Psi \) is the space of smooth test functions

\[
\Psi = \{ \varphi \in C^{1,2}([0, T] \times \overline{\Omega}), \nabla \varphi . \tilde{n} = 0 \text{ on } (0, T) \times \partial \Omega, \varphi(T, \cdot) = 0 \}.
\]

After integrating over \([0, T]\) and \( \Omega \) and after applying integration by parts and properties of a test function, we come to a definition of the weak solution.
Definition 1.1. The weak solution of the regularized Perona-Malik problem (1)–(3) is a function $u \in L_2(I, W^{1,2}(\Omega))$ satisfying the identity

$$ (9) \quad \int_0^T \int_{\Omega} u \frac{\partial \varphi}{\partial t} (t, x) \, dx \, dt + \int_{\Omega} u_0(x) \varphi(0, x) \, dx - \int_0^T \int_{\Omega} g(|J(u(t, x))|) \nabla u(t, x) \nabla \varphi(t, x) \, dx \, dt = 0 $$

for all $\varphi \in \Psi$.

It is well known from the regularity theory of such a solution [10] that, because of the properties of the operator $J(u)$, the weak solution of our problem is a function $u \in L_2(I, W^{2,2}(\Omega))$ for initial condition $u_0 \in L_\infty(\Omega)$. Moreover from the embedding theory for dimension $d = 2$, or $d = 3$ follows that $u \in C(\bar{Q}_T)$.

For our error estimates we need further regularity of the solution, more precise $u \in L_2(I, W^{2,2}(\Omega)) \cap L_\infty(I, W^{1,2}(\Omega))$ and $u_{tt} \in L_1(I, L_1(\Omega))$.

1.2. Formulation of the finite volume scheme

Let $\tau_h$ be a uniform mesh of $\Omega$ with cells $p$ of measure $m(p)$ (we assume rectangular cells here). For every cell $p$ we consider a set of neighbours $N(p)$ consisting of all cells $q \in \tau_h$ for which the common interface of $p$ and $q$, denoted by $e_{pq}$, is of non-zero measure $m(e_{pq})$. We denote the set of all these edges for all volumes $p \in \tau_h$ by $E$ and by $e_{pq}I$ we denote the edge which connects the volumes $p$ and $q$. (Clearly $e_{pq} = e_{qp} = e_{pq}I$). It is assumed that for every $p$, there exists a representative point $x_p \in p$, such that for every pair $p, q, q \in N(p)$, the vector $\frac{x_q - x_p}{|x_q - x_p|}$ is equal to a unit vector $n_{pq}$ which is normal to $e_{pq}$ and oriented from $p$ to $q$. We denote $d_{pq} := |x_p - x_q|$. In a simple case of a uniform grid $x_p$ is just the center of the pixel. Then, let $x_{pq}$ be the point of $e_{pq}$ intersecting the segment $x_p x_q$. We define

$$ (10) \quad T_{pq} := \frac{m(e_{pq})}{d_{pq}}. $$
Discrete approximation of a solution of partial differential equation is considered to be piecewise constant on control volumes [5]. Let \( u_p^n \) be the value of the numerical solution in the n-th scale step on a volume \( p \). The finite volume semi-implicit scheme on a uniform grid is then written as follows:

Let \( 0 = t_0 \leq t_1 \leq \ldots \leq t_n \leq \ldots \leq t_{N_{\text{max}}} \), \( N_{\text{max}} \cdot k = T \) denote the scale discretization steps with \( t_l = t_{l-1} + k \), where \( k \) is the discrete scale step, \( l = 1, 2, \ldots, N_{\text{max}} \).

For \( n = 0, \ldots, N_{\text{max}} \) we look for \( u_p^{n+1} \), \( p \in \tau_h \), satisfying the identities

\[
(u_p^{n+1} - u_p^n) m(p) = k \sum_{q \in N(p)} g_{pq}^{\sigma,n} T_{pq} (u_q^{n+1} - u_p^{n+1}) ,
\]

\[
u^0_p = \frac{1}{m(p)} \int_p u_0(x) dx ,
\]

(12) \( g_{pq}^{\sigma,n} := g(|J(\tilde{u}(t_n,x_{pq}))|) \),

where \( \tilde{u} \) is a periodic extension of the discrete image computed in the n-th scale step. Its \( L_2 \) norm can be estimated with constant \( B \) by \( L_2 \) norm of function \( u \). \( u_p^n \) is a value of the numerical solution on the volume \( p \) in the n-th scale step.

2. Stability and convergence results

We briefly mention results of Mikula and Ramarosy, see [12], obtained for the semi-implicit finite volume scheme concerning the stability and convergence properties. Explicit time discretization are discussed also in [7] and [8]. Stability estimates are of the following type [12]:

**Lemma 2.1** (A priori estimates in \( L_2(Q_T) \)). It holds, that there exist positive constants \( C_1, C_2 \) such that
Let us denote by \( \bar{u}_{h,k} \) the finite volume numerical solution for some fixed space and scale mesh \( h \) and \( k \). This solution is piecewise constant on each finite volume and in each scale step as it is usual for finite volume numerical schemes of a parabolic type. By \( \bar{u}^l \) we denote the function piecewise constant on each finite volume in the \( l \)-th scale step. Then we have:

**Lemma 2.2** (Convergence of \( \bar{u}_{h,k} \)). There exists \( u \in L^2(Q_T) \) which is the weak solution of (9) such that

\[
\bar{u}_{h,k} \rightarrow u \text{ in } L^2(Q_T)
\]

as \( h, k \rightarrow 0 \). Furthermore, the convergence is pointwise.

### 3. Error estimates

#### 3.1. \( L_{\infty} \) stability for a discrete solution

We rewrite the original discrete equation (11) in the following way:

\[
(13) \quad u_p^{n+1} + \frac{k}{m(p)} \sum_{q \in N(p)} g_{pq}^n T_{pq} (u_p^{n+1} - u_q^{n+1}) = u_p^n.
\]
Now let $u_p^{n+1}$ be the maximal value for the fixed scale step $n + 1$ and $p \in \tau_h$. Then the second term on the left hand side of (13) is nonnegative and:

$$u_p^{n+1} \leq u_p^n.$$ 

Recursively we have

$$||u^n||_{L_\infty} \leq |u_p^{n+1}| \leq |u_p^n| \leq ||u^0||_{L_\infty} \leq C.$$ 

### 3.2. Error estimates

Let now $t_n < t \leq t_{n+1}$. Multiplying PDE (8) by $v_p^{n+1}$ and then integrating over volume $p$ and using integration by parts, we have:

$$\int_p \partial_t u(t, x) v_p^{n+1} \, dx - \int_{\partial p} g(|J(u)|) \nabla u \cdot n_p v_p^{n+1} \, dx = 0,$$

where $\partial p$ is the boundary of the volume $p$ and $n_p$ is the outward unit normal vector to the boundary of volume $p$ and further analogously $n_{pq}$ will be the outward unit normal vector to the edge $e_{pq}$. We can write

$$\partial p = \bigcup_{q \in N(p)} e_{pq}.$$ 

For the discrete scheme we have

$$\left(\frac{u_p^{n+1} - u_p^n}{k}\right) v_p^{n+1} m(p) - \sum_{q \in N(p)} g_{pq}^{\sigma,n} T_{pq} (u_q^{n+1} - u_p^n) v_p^{n+1} = 0.$$ 

Now we denote

$$e_p^n = u(t_n, x_p) - u_p^n,$$
where $x_p$ is a representative point of a volume $p$, $p \in \tau_h$.

Then posing $v^n_p = e^n_p$ and subtracting (16) from (15) we obtain:

$$
\int_p \left( \frac{e_{p}^{n+1} - e_{p}^{n}}{k} \right) e_{p}^{n+1} + \sum_{q \in N(p)e_{pq}} \int \left( g_{\sigma,n} \frac{u_{q}^{n+1} - u_{p}^{n+1}}{d_{pq}} - g(|J(u)|) \nabla u \cdot n_{pq} \right) e_{p}^{n+1} \\
= \int_p \left( \frac{u(t_{n+1},x_p) - u(t_n,x_p)}{k} - \partial_t u \right) e_{p}^{n+1}.
$$

Now after summation over all volumes $p \in \tau_h$ and integration over $I_n = \langle t_n, t_{n+1} \rangle$ for all $n = 0, 1, \ldots, m - 1$ and rearranging the equation we obtain:

$$
\int_{\Omega} |\varepsilon^n|^2 + \sum_{n=0}^{m-1} \int_{\Omega} |\varepsilon^{n+1} - \varepsilon^n|^2 + 2 \sum_{n=0}^{m-1} \int_{I_n} \sum_{p \in \tau_h} \sum_{q \in N(p)e_{pq}} \int \left( g_{\sigma,n} \frac{u_{q}^{n+1} - u_{p}^{n+1}}{d_{pq}} - g(|J(u)|) \nabla u \cdot n_{pq} \right) e_{p}^{n+1} \\
= \int_{\Omega} |\varepsilon^0|^2 + 2 \sum_{n=0}^{m-1} \int_{I_n} \sum_{p \in \tau_h} \int \left( \frac{u(t_{n+1},x_p) - u(t_n,x_p)}{k} - \partial_t u \right) e_{p}^{n+1}.
$$

(17)
The third term on the left hand side of the last equation can be expressed as it is usual in the finite volume theory, see [5]:

\[
"\text{Third"} = 2 \sum_{n=0}^{m-1} \int_{I_n} \sum_{p \in \tau_n} \sum_{q \in N(p)_{e_{pq}}} \int \left( g_{\sigma,n} \frac{u_{q,n+1}^n - u_{p,n+1}^n}{d_{pq}} - g(|J(u)|) \nabla u \cdot n_{pq} \right) e_{n+1}^p e_{n+1}^q
\]

\[
= 2 \sum_{n=0}^{m-1} \int_{I_n} \sum_{\mathcal{E}_{e_{pq}I}} \int \left( g_{\sigma,n} \frac{u_{q,n+1}^n - u_{p,n+1}^n}{d_{pq}} - g(|J(u)|) \nabla u \cdot n_{pq} \right) (e_{n+1}^p - e_{n+1}^q).
\]

After rearranging we get:

\[
"\text{Third"} = 2 \sum_{n=0}^{m-1} \int_{I_n} \sum_{\mathcal{E}_{e_{pq}I}} \int g(|J(u)|) \left( \frac{(e_{n+1}^p - e_{n+1}^q)^2}{d_{pq}} \right)
\]

\[
+ 2 \sum_{n=0}^{m-1} \int_{I_n} \sum_{\mathcal{E}_{e_{pq}I}} \int \left( g_{\sigma,n} - g(|J(u)|) \right) \frac{u_{q,n+1}^n - u_{p,n+1}^n}{d_{pq}} (e_{n+1}^p - e_{n+1}^q)
\]

\[
+ 2 \sum_{n=0}^{m-1} \int_{I_n} \sum_{\mathcal{E}_{e_{pq}I}} \int g(|J(u)|) \left( \frac{u(t_{n+1,x_q}) - u(t_{n+1,x_p})}{d_{pq}} - \nabla u \cdot n_{pq} \right) (e_{n+1}^p - e_{n+1}^q).
\]
Involving these terms to the (17) equation we obtain:

$$
\int_{\Omega} |\bar{e}^m|^2 + \sum_{n=0}^{m-1} \int_{\Omega} |\bar{e}^{n+1} - \bar{e}^n|^2 + 2 \sum_{n=0}^{m-1} \int_{I_n} \int_{\mathcal{E}} g(|J(u)|) \frac{(e_{p+1} - e_{q+1})^2}{d_{pq}}
$$

$$
= \int_{\Omega} |\bar{e}^0|^2 + 2 \sum_{n=0}^{m-1} \int_{I_n} \int_{\mathcal{E}} \left( \frac{u(t_{n+1}, x_p) - u(t_n, x_p)}{k} - \partial_t u \right) e_{p+1}^n
$$

$$
+ 2 \sum_{n=0}^{m-1} \int_{I_n} \int_{\mathcal{E}} g_{pq} \left( g(|J(u)|) \frac{u_{q+1}^n - u_p^n}{d_{pq}} (e_{p+1}^n - e_{q+1}^n) \right)
$$

$$
+ 2 \sum_{n=0}^{m-1} \int_{I_n} \int_{\mathcal{E}} g(|J(u)|) \left( \frac{u(t_{n+1}, x_q) - u(t_{n+1}, x_p)}{d_{pq}} - \nabla u \cdot n_{pq} \right) (e_{p+1}^n - e_{q+1}^n),
$$

or briefly

$$
\int_{\Omega} |\bar{e}^m|^2 + \sum_{n=0}^{m-1} \int_{\Omega} |\bar{e}^{n+1} - \bar{e}^n|^2 + 2 \sum_{n=0}^{m-1} \int_{I_n} \int_{\mathcal{E}} g(|J(u)|) \frac{(e_{p+1}^n - e_{q+1}^n)^2}{d_{pq}}
$$

$$
= \int_{\Omega} |\bar{e}^0|^2 + I + II + III.
$$

**Remark.** To obtain an appropriate error estimate we must take into account the regularity of the solution which plays an important role in error analysis. Error estimates could be done also better, but further regularity for time derivative is needed. If we suppose $u_0 \in L_\infty(\Omega)$ only, no further regularities are available.
Now we must estimate each of the last three terms on the right hand side.

\[
I = 2 \sum_{n=0}^{m-1} \int \sum_{p \in \tau_h} \left( \frac{u(t_{n+1}, x_p) - u(t_n, x_p)}{k} - \partial_t u \right) e_p^{n+1}
\]

\[
= 2 \sum_{n=0}^{m-1} \sum_{p \in \tau_h} \int (u(t_{n+1}, x_p) - u(t_n, x) + u(t_n, x) - u(t_n, x_p) \right) e_p^{n+1}
\]

\[
= 2 \sum_{p \in \tau_h} \int \left( \sum_{n=0}^{m-1} (u(t_n, x_p) - u(t_n, x)) \left( e_p^n - e_p^{n+1} \right) \right)
\]

\[
+ 2 \sum_{p \in \tau_h} \int (u(t_m, x_p) - u(t_m, x)) e_p^m - (u(0, x_p) - u(0, x)) e_p^0\]

\[
\leq \sqrt{2} \sum_{n=0}^{m-1} \int h \lvert \nabla u(t_n, \cdot) \rvert \lvert e^{n+1} - e^n \rvert + \sqrt{2} \int h \lvert \nabla u(t_m, \cdot) \rvert \lvert e^m \rvert + \sqrt{2} \int h \lvert \nabla u(0, \cdot) \rvert \lvert e^0 \rvert.
\]

After using Young’s inequality we get

\[
I \leq h^2 \sum_{n=0}^{m-1} \int \lvert \nabla u(t_n, \cdot) \rvert^2 + \frac{1}{2} \sum_{n=0}^{m-1} \int \lvert e^{n+1} - e^n \rvert^2 + h^2 \int \lvert \nabla u(t_m, \cdot) \rvert^2 + \frac{1}{2} \int \lvert e^m \rvert^2 + h^2 \int \lvert \nabla u(0, \cdot) \rvert^2 + \frac{1}{2} \int \lvert e^0 \rvert^2.
\]
Finally

\[ I \leq \frac{1}{2} \sum_{n=0}^{m-1} \int_{\Omega} |\tilde{e}^{n+1} - \tilde{e}^n|^2 + \frac{1}{2} \int_{\Omega} |\tilde{e}^n|^2 + \frac{1}{2} \int_{\Omega} |\tilde{e}^0|^2 + \left( \frac{h^2 T}{k} + 2h^2 \right) \|\nabla u\|_{L_\infty(I, L_2(\Omega))} \]

We estimate the second term in the following way:

\[ \Pi = 2 \sum_{n=0}^{m-1} \int_{I_n} \sum_{\varepsilon \in \mathcal{E}_{pq}} \int_{\epsilon_{pq}} (g^{\sigma,n}_{pq} - g(|J(u(x))|)) \frac{u^{n+1}_{q} - u^{n+1}_{p}}{d_{pq}} (e^{n+1}_{p} - e^{n+1}_{q}) dx. \]

First we estimate

\[ |g^{\sigma,n}_{pq} - g(|J(u)|)| = |g(|\nabla G^{\sigma} \ast \tilde{u}(t_n, x_{pq})|) - g(|\nabla G^{\sigma} \ast \tilde{u}(t, x)|)| \]

\[ \leq L_g \left| \int_{\mathbb{R}^d} \nabla G^{\sigma}(x_{pq} - \eta) \tilde{u}_{h,k}(t_n, \eta) d\eta - \int_{\mathbb{R}^d} \nabla G^{\sigma}(s - \eta) \tilde{u}(t, \eta) d\eta \right| \]

\[ \leq L_g \int_{\mathbb{R}^d} |\nabla G^{\sigma}(x_{pq} - \eta) - \nabla G^{\sigma}(s - \eta)||\tilde{u}_{h,k}(t_n, \eta)| d\eta \]

\[ + L_g \int_{\mathbb{R}^d} |\nabla G^{\sigma}(s - \eta)||\tilde{u}_{h,k}(t_n, \eta) - \tilde{u}(t, \eta)| d\eta. \]

We obtain

\[ |g^{\sigma,n}_{pq} - g(|J(u)|)| \]

\[ \leq \frac{L_g B}{\sqrt{2}} \cdot h \|D^2 G^{\sigma}\|_{L_\infty(\Omega)} \|u_{h,k}\|_{L_\infty(Q_T)} m(K^{\sigma}) + L_g B \|\nabla G^{\sigma}\|_{L_\infty(\Omega)} \]

\[ \cdot \left( \left( \int_{\Omega} |\tilde{e}^n|^2 dx \right)^\frac{1}{2} + \int_{t_n}^T \int_{\Omega} |\partial_t u(s, x)| ds dx + \sum_{p \in \mathcal{E}_{h}} \int_{x_p}^x \left| \frac{\partial u(t, y)}{\partial n} \right| dy dx \right), \]
where \( m(K_{\sigma}) \) is measure of the compact support \( K_{\sigma} \), \( \sigma \) is fixed, \( B \) is the estimation for mirror reflexion function. We denote

\[
\begin{align*}
C_3 &= 2L_g B \| D^2 G_{\sigma} \|_{L_\infty(\Omega)} \| u_{h,k} \|_{L_\infty(Q_T)} m(K_{\sigma}), \\
C_4 &= 2L_g B \| \nabla G_{\sigma} \|_{L_\infty(\Omega)}, \\
C_g &\text{ is such that } g(|J(u)|) \geq C_g.
\end{align*}
\]

The last estimate can be established due to the properties of the solution \( u \). Hence the whole term II can be estimated as follows:

\[
\begin{align*}
II &\leq C_3 h \cdot \left( \sum_{n=0}^{m-1} k \sum_{e_{pq} I} |u_{n+1}^q - u_{n+1}^p|^2 d_{pq} \right)^{\frac{1}{2}} \left( \sum_{n=0}^{m-1} k \sum_{e_{pq} I} |e_{n+1}^p - e_{n+1}^q|^2 d_{pq} \right)^{\frac{1}{2}} \\
&\quad + C_4 \cdot \left( \sum_{n=0}^{m-1} \left( \sum_{e_{pq} I} |u_{n+1}^q - u_{n+1}^p|^2 d_{pq} \right) \right)^{\frac{1}{2}} \left( \sum_{e_{pq} I} \frac{|e_{n+1}^p - e_{n+1}^q|^2}{d_{pq}} \right)^{\frac{1}{2}} \left( \int_{\Omega} |e|^2 \right)^{\frac{1}{2}} \\
&\quad + C_4 \cdot \sum_{n=0}^{m-1} \int_{I_n} \left( \sum_{e_{pq} I} |u_{n+1}^q - u_{n+1}^p|^2 d_{pq} \right)^{\frac{1}{2}} \left( \sum_{e_{pq} I} \frac{|e_{n+1}^p - e_{n+1}^q|^2}{d_{pq}} \right)^{\frac{1}{2}} \left( \int_{\Omega} \int_{I_n} \left( \int_{\partial t u(s,x)} dsdx + \sum_{p \in \tau_n} \int_{x_p}^{x} \left| \frac{\partial u(t,y)}{\partial n} \right| dydx \right) \right)^{\frac{1}{2}}.
\end{align*}
\]
\[
\Pi \leq \frac{4C_2C_3^2h^2}{C_g^2} + \frac{1}{2} \sum_{n=0}^{m-1} \int \int_{\epsilon_{pq}I} g(|J(u)|) \frac{(e_p^{n+1} - e_q^{n+1})^2}{d_{pq}} \\
+ \frac{4C_4^2}{C_g^2} \cdot \sum_{n=0}^{m-1} k \sum_{\epsilon_{pq}I} \int \int_{\epsilon_{pq}I} \frac{|u_q^{n+1} - u_p^{n+1}|^2}{d_{pq}} \int_{\Omega} |e^n|^2 \\
+ \frac{4C_4^2}{C_g^2} \cdot \sum_{n=0}^{m-1} \int_{I_n} \int_{\epsilon_{pq}I} \frac{|u_q^{n+1} - u_p^{n+1}|^2}{d_{pq}} \\
\cdot \left( \int_{\Omega} \int_{t_n} \frac{\nabla \cdot (g(|J(u)|\nabla u))}{|\epsilon_{pq}I|} ds dx + \sum_{p \in \tau_h} \int_{x_p}^{x} \frac{\partial u(t, y)}{\partial n} |dy dx| \right) \\
= \frac{4C_2C_3^2h^2}{C_g^2} + \Pi_1 + \Pi_2 + \Pi_3,
\]

where the inequalities (14), (ii) and the equation (1) has been used. The last term can be estimated using the properties of the exact solution:

\[
\Pi_3 \leq \left( \frac{8C_4^2LgC_2}{C_g^2} ||D^2G\sigma||_{L_{\infty}(\Omega)} ||\nabla u||_{L_{\infty}(I,L_2(\Omega))} + \frac{4C_4^2C_2}{C_g^2} ||\Delta u||_{L_2(I,L_2(\Omega))} \right) \cdot k \\
+ \left( \frac{8C_4^2LgC_2}{C_g^2} ||DG\sigma||_{L_{\infty}(\Omega)} ||\nabla u||_{L_{\infty}(I,L_2(\Omega))} \right) \cdot h.
\]
Finally the third term can be estimated:

\[
III = 2 \sum_{n=0}^{m-1} \int_{I_n} \sum_{\mathcal{E}} \int_{e_{pq} I} g(|J(u)|) \left( \frac{u(t_{n+1}, x_q) - u(t_{n+1}, x_p)}{d_{pq}} - \nabla u \cdot n_{pq} \right) \left( e_{n+1}^p - e_{n+1}^q \right)
\]

We denote

\[
R_{pq} = \frac{1}{m(e_{pq})} \left( - \int_{13454568 e_{pq} I} \nabla u \cdot n_{pq} + \frac{u(t_{n+1}, x_q) - u(t_{n+1}, x_p)}{d_{pq}} m(e_{pq}) \right).
\]

Applying the properties of function \( g \), this term can be estimated as

\[
|III| \leq 2 \sum_{n=0}^{m-1} \int_{I_n} \sum_{\mathcal{E}} \int_{e_{pq} I} |R_{pq}| |e_{n+1}^p - e_{n+1}^q|.
\]

Now using the regularity of a weak solution and the estimates well known in the finite volume method see, [5, Chapter 3.1.6], we get

\[
|III| \leq \frac{C}{C_g} h^2 \sum_{n=0}^{m-1} \int_{I_n} \int_{\Omega} (H(u)(z))^2 + \frac{1}{4} \sum_{n=0}^{m-1} \int_{I_n} \sum_{\mathcal{E}} \int_{e_{pq} I} g(|J(u)|) \frac{(e_{n+1}^p - e_{n+1}^q)^2}{d_{pq}}.
\]
Here $|H(u)(z)|^2 = \sum_{i,j=1}^{d} |D_iD_j u(z)|^2$ and $D_i$ denote the weak derivatives with respect to the component $z_i$. Since $u \in L_2(I, W^{2,2}(\Omega))$ we can denote

$$C_5 = \frac{C}{C_g} \|H(u)\|_{L_2(Q_T)}^2$$

and we have

$$\text{III} \leq C_5 h^2 + \frac{1}{4} \sum_{n=0}^{m-1} \int_{I_n} \sum_{\varepsilon} \int_{e_{pq} I} g(|J(u)|) \left(\frac{e_{p}^{n+1} - e_{q}^{n+1}}{d_{pq}}\right)^2.$$

Putting all these estimates together, we obtain:

$$\int_{\Omega} |\mathcal{e}|^2 + \sum_{n=0}^{m-1} \int_{\Omega} |\mathcal{e}^{n+1} - \mathcal{e}^n|^2 + \sum_{n=0}^{m-1} \int_{I_n} \sum_{\varepsilon} \int_{e_{pq} I} g(|J(u)|) \left(\frac{e_{p}^{n+1} - e_{q}^{n+1}}{d_{pq}}\right)^2$$

$$\leq 4 \int_{\Omega} |\mathcal{e}^0|^2 + 2\left(\frac{h^2 T}{k} + 2h^2\right) \|\nabla u\|_{L_\infty(I,L_2(\Omega))} + \left(\frac{4C_2 C_3}{C_g} + 2C_5\right) h^2$$

$$+ \left(\frac{8C_2^2 L g}{C_g^2} \|D^2 G_\sigma\|_{L_\infty(\Omega)} \|\nabla u\|_{L_\infty(I,L_2(\Omega))} + \frac{4C_2^2 C_2}{C_g^2} \|\Delta u\|_{L_2(I,L_2(\Omega))}\right) k$$

$$+ \left(\frac{8C_2^2 L g C_2}{C_g^2} \|D G_\sigma\|_{L_\infty(\Omega)} \|\nabla u\|_{L_\infty(I,L_2(\Omega))}\right) \cdot h$$

$$+ \frac{4C_4}{C_g} \cdot \left(\sum_{n=0}^{m-1} k \varepsilon \int_{e_{pq} I} \frac{|u_{q}^{n+1} - u_{p}^{n+1}|^2}{d_{pq}} \int_{\Omega} |\mathcal{e}^n|^2\right).$$
If we realize that the first term on the right hand side with $e_0$ is less than $Ch$ because of the properties of the exact solution and the definition of $u_p^0$, we are prepared to use Gronwall’s lemma in the form:

**Lemma 3.1.** If $u(t)$ and $v(t)$ are non-negative measurable functions for $t \leq 0$ and $u_0$ is a non-negative constant, then the inequality $u(t) \leq u_0 + \int_0^t v(s) u(s) ds$ implies that $u(t) \leq u_0 \exp \left( \int_0^t v(s) ds \right)$.

To estimate the last term of the previous inequality let us denote for $t \in I_n = (t_{n-1}, t_n)$

$$v(t) = \sum_{p,q} \int_{e_{pq}I} \frac{|u_{p+1}^{n+1} - u_p^n|^2}{d_{pq}}, \quad u(t) = \int_{\Omega} |\varepsilon^m|^2 dx.$$  

If we use the properties of function $v$ then we can obtain

$$\int_{\Omega} |\varepsilon^m|^2 \leq C (h^2 + h + \frac{h^2}{k} + k) \cdot \exp \left( \sum_{n=0}^{m-1} k \sum_{p,q} \int_{e_{pq}I} \frac{|u_{p+1}^{n+1} - u_p^n|^2}{d_{pq}} \right)$$

$$\leq C \cdot \exp(C_2) (h^2 + k + h + \frac{h^2}{k}),$$

where $C$ is a generic constant depending only on domain $\Omega$, time $T$ and some norms of the exact solution. To obtain convenient error estimate result we can choose

$$(18) \quad k = Ch,$$
Theorem 3.1. Let the relation between scale and space discretization be chosen as in (18). Then for the error estimates for Perona-Malik weak solution and numerical solution obtained via finite volume method it holds

\[ \sum_{n=0}^{N_{\text{max}}} \int_{I_n} \int_{\Omega} |u(t_{n+1}, x) - \bar{u}_{h,k}(t_{n+1}, x)|^2 \leq C h \]

and

\[ \sum_{n=0}^{m-1} \int_{I_n} \int_{e_{pq} I} m(e_{pq}) d_{pq} \left( \frac{u^{n+1}_q - u^{n+1}_p}{d_{pq}} - \frac{1}{m(e_{pq})} \int_{e_{pq}} \nabla u \cdot n_{pq} \right)^2 \leq C h. \]

Proof. It is easy to see that

\[ \sum_{n=0}^{N_{\text{max}}} \int_{I_n} \int_{\Omega} |u(t_{n+1}, x) - \bar{u}_{h,k}(t_{n+1}, x)|^2 \leq 2 h^2 \| \nabla u \|_{L_2(I, L_2(\Omega))} + 2 \sum_{n=0}^{N_{\text{max}}} \int_{I_n} \int_{\Omega} |e^{n+1}|^2 \leq C h \]

and the first inequality is proved. Now

\[ \sum_{n=0}^{m-1} \int_{I_n} \int_{e_{pq} I} m(e_{pq}) d_{pq} \left( \frac{u^{n+1}_q - u^{n+1}_p}{d_{pq}} - \frac{1}{m(e_{pq})} \int_{e_{pq}} \nabla u \cdot n_{pq} \right)^2 \]

\[ \leq C \sum_{n=0}^{m-1} \int_{I_n} \int_{e_{pq} I} \int_{e_{pq}} g(J(u)) \left( \frac{e^{n+1}_q - e^{n+1}_p}{d_{pq}} \right)^2 + C \sum_{n=0}^{m-1} \int_{I_n} \int_{e_{pq} I} \int_{e_{pq}} \left( \frac{u^{n+1}_q - u^{n+1}_p}{d_{pq}} - \frac{1}{m(e_{pq})} \int_{e_{pq}} \nabla u \cdot n_{pq} \right)^2 \]

\[ \leq C h, \]
where we have again used the estimate of the finite volume method for the second term.

4. Numerical experiments

In this section we present experiments with some artificial images perturbed by various types of noise. We want to confirm the relation between scale step, mesh size and the data coefficients obtained in the previous theorem. In simulations, we use the function

\[ g(s) = \frac{1}{1 + Ks^2} \]

and the convolution is realized with the kernel

\[ G_{\sigma}(x) = \frac{1}{Z} e^{\frac{|x|^2}{2\sigma^2}} \]

where the constant \( Z \) is chosen so that \( G_{\sigma} \) has unit mass.

**Example 1.** To every position of the initial image we apply a 10% salt and pepper noise.

**Example 2.** We have chosen another type of picture with a noise function \( f \) defined as follows: if \( \psi(x) \) is a function generating random values in \([0, 2C]\), then for every position \( x \)

\[ f(u_0(x)) = \text{MIN}(255, \text{MAX}(0, u_0(x) - C + \psi)) \]

\( C = 100 \) and the difference in intensity between the two values of the initial image is 200.

In both examples the size of one finite volume corresponds to the size of one pixel. We computed the same example for different scale steps. In both figures we choose the best visual result for every parameter \( K \) in function \( g \) which plays an important role in smoothing effect. For the best cases it seems that the relation between grid mesh, scale step and parameter \( K \) remains is constant.
**Figure 1.** The first column shows the work for parameter $K = 10$ for different scale steps $k = 0.1, 0.5, 1, 2.5$, the second column shows the work for parameter $K = 100$ for scale steps $k = 0.5, 1, 4, 10$. for Example 1.
Figure 2. Pictures shows the work for parameter $K = 10$ for different scale steps $k = 0.2, 1, 5, 10$ (from the top to bottom, from left to right) for Example 2.
Figure 3. Pictures shows the work for parameter $K = 100$ (top) for different scale steps $k = 1, 2, 5, 10$, and for parameter $K = 1000$ for scale parameter $k = 10, 20$ for Example 2.
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