NOTE ON THE Ψ-BOUNDEDNESS OF THE SOLUTIONS OF A SYSTEM OF DIFFERENTIAL EQUATIONS

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Abstract. It is proved a necessary and sufficient condition for the existence of Ψ-bounded solutions of a linear nonhomogeneous system of ordinary differential equations.

1. Introduction

The purpose of this note is to give a necessary and sufficient condition so that the nonhomogeneous system

\[ x' = A(t)x + f(t) \]

have at least one Ψ-bounded solution for every continuous and Ψ-bounded function \( f \), in supplementary hypothesis that \( A(t) \) is a Ψ-bounded matrix on \( \mathbb{R}_+ \).

Here, Ψ is a continuous matrix function. The introduction of the matrix function Ψ permits to obtain a mixed asymptotic behavior of the components of the solutions.

The problem of Ψ-boundedness of the solutions for systems of ordinary differential equations has been studied by many authors, as e.g. O. Akinyele [1], A. Constantin [3], C. Avramescu [2], T. Hallam [8], J. Morchalo [10]. In these papers, the function Ψ is a scalar continuous function (and increasing, differentiable and bounded in \([1] \), nondecreasing and such that \( \Psi(t) \geq 1 \) on \( \mathbb{R}_+ \) in \([3] \)).

Let \( \mathbb{R}^d \) be the Euclidean \( d \)-space. For \( x = (x_1, x_2, \ldots, x_d)^T \in \mathbb{R}^d \), let \( \|x\| = \max \{|x_1|, |x_2|, \ldots, |x_d|\} \) be the norm of \( x \). For a \( d \times d \) real matrix \( A \), we define the norm \( |A| \) by \( |A| = \sup_{\|x\| \leq 1} \|Ax\| \). Let \( \Psi_i : \mathbb{R}_+ \to (0, \infty), \ i = 1, 2, \ldots, d \), be continuous functions and

\[ \Psi = \text{diag} [\Psi_1, \Psi_2, \ldots, \Psi_d] . \]

Definition 1.1. A function \( \phi : \mathbb{R}_+ \to \mathbb{R}^d \) is said to be Ψ-bounded on \( \mathbb{R}_+ \) if \( \Psi(t)\phi(t) \) is bounded on \( \mathbb{R}_+ \).

Let \( A \) be a continuous \( d \times d \) real matrix and the associated linear differential system

\[ y' = A(t)y \]
Let $Y$ be the fundamental matrix of (2) for which $Y(0) = I_d$ (identity $d \times d$ matrix).

Let $X_1$ denote the subspace of $\mathbb{R}^d$ consisting of all vectors which are values of $\Psi$-bounded solutions of (2) for $t = 0$ and let $X_2$ an arbitrary fixed subspace of $\mathbb{R}^d$, supplementary to $X_1$.

We suppose that $X_2$ is a closed subspace of $\mathbb{R}^d$. We denote by $P_1$ the projection of $\mathbb{R}^d$ onto $X_1$ (that is $P_1$ is a bounded linear operator $P_1 : \mathbb{R}^d \to \mathbb{R}^d$, $P_1^2 = P_1$, $\text{Ker} P_1 = X_2$) and $P_2 = I - P_1$ the projection onto $X_2$.

In our papers [5] and [6] we have proved the following results (Lemma 1, Lemma 2 and respectively Theorem 2.1):

**Lemma 1.** Let $Y(t)$ be an invertible matrix which is a continuous function of $t$ on $\mathbb{R}_+$ and let $P$ be a projection.

If there exist a continuous function $\varphi : \mathbb{R}_+ \to (0, \infty)$ and a positive constant $M$ such that
\[
\int_0^t \varphi(s) |\Psi(t) P Y^{-1}(s) \Psi^{-1}(s)| \, ds \leq M, \quad \text{for all } t \geq 0,
\]
and
\[
\int_0^\infty \varphi(s) \, ds = +\infty,
\]
then, there exists a constant $N > 0$ such that
\[
|\Psi(t) Y(t) P| \leq Ne^{-M \int_0^t \varphi(s) \, ds}, \quad \text{for all } t \geq 0.
\]
Consequently,
\[
\lim_{t \to \infty} |\Psi(t) Y(t) P| = 0.
\]

**Lemma 2.** Let $Y(t)$ be an invertible matrix which is a continuous function of $t$ on $\mathbb{R}_+$ and let $P$ be a projection.

If there exists a constant $M > 0$ such that
\[
\int_t^\infty |\Psi(t) Y(t) P Y^{-1}(s) \Psi^{-1}(s)| \, ds \leq M, \quad \text{for all } t \geq 0,
\]
then, for any vector $x_0 \in \mathbb{R}^d$ such that $Px_0 \neq 0$,
\[
\limsup_{t \to \infty} \|\Psi(t) Y(t) Px_0\| = +\infty.
\]

**Theorem 2.1.** If $A$ is a continuous $d \times d$ matrix, then the system (1) has at least one $\Psi$-bounded solution on $\mathbb{R}_+$ for every continuous and $\Psi$-bounded function $f$ on
\( \mathbb{R}_+ \) if and only if there is a positive constant \( K \) such that

\[
\int_0^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \, ds + \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \, ds \leq K,
\]

for all \( t \geq 0 \).

2. The main results

In this section we give the main results of this note.

**Theorem 2.1.** Let \( A \) be a continuous \( d \times d \) real matrix such that

\[
|\Psi(t)A(t)\Psi^{-1}(t)| \leq L, \quad \text{for all } t \geq 0.
\]

Let \( \Psi(t) \) such that

\[
|\Psi(t)\Psi^{-1}(s)| \leq M, \quad \text{for } t \geq s \geq 0.
\]

Then, the system (1) has at least one \( \Psi \)-bounded solution on \( \mathbb{R}_+ \) for every continuous and \( \Psi \)-bounded function \( f \) on \( \mathbb{R}_+ \) if and only if there are two positive constants \( K_1 \) and \( \alpha \) such that

\[
|\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \leq K_1 e^{-\alpha(t-s)}, \quad 0 \leq s \leq t,
\]

\[
|\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \leq K_1 e^{-\alpha(s-t)}, \quad 0 \leq t \leq s,
\]

Proof. First, we prove the "only if" part.

From the hypotheses and Theorem 2.1, [6], it follows that there is a positive constant \( K \) such that

\[
\int_0^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \, ds + \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \, ds \leq K,
\]

for all \( t \geq 0 \).

From \( Y'(t) = A(t)Y(t), \ t \geq 0 \), it follows that

\[
Y(t) = Y(s) + \int_s^t A(u)Y(u) \, du, \quad \text{for } t \geq s \geq 0.
\]

Therefore,

\[
\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) = \Psi(t)\Psi^{-1}(s) + \int_s^t \Psi(t)A(u)Y(u)Y^{-1}(s)\Psi^{-1}(s) \, du.
\]
Thereafter, for \( t \geq s \geq 0, \)

\[
|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)|
\]

\[
\leq |\Psi(t)\Psi^{-1}(s)| + \int_{s}^{t} |\Psi(t)\Psi^{-1}(u)\|\Psi(u)A(u)\Psi^{-1}(u)\|\Psi(u)Y(u)Y^{-1}(s)\Psi^{-1}(s)|\,du.
\]

From the hypotheses and Gronwall’s inequality it follows that

\[
(5) \quad |\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)| \leq Me^{LM(t-s)}, \quad t \geq s \geq 0.
\]

Now, we show that (3) and (5) imply (4).

For \( v \in \mathbb{R}^{d} \) and \( 0 \leq s \leq t \leq s + 1, \) we have

\[
\|\Psi(t)Y(t)P_1 v\| = \|\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)\Psi(s)Y(s)P_1 v\|
\]

\[
\leq |\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)| \cdot \|\Psi(s)Y(s)P_1 v\|
\]

\[
\leq Me^{LM}\|\Psi(s)Y(s)P_1 v\|
\]

For \( P_1 v \neq 0, \) let

\[
q(t) = \|\Psi(t)Y(t)P_1 v\|^{-1} \quad \text{and} \quad Q(t) = \int_{0}^{t} q(s)\,ds.
\]

We have

\[
q(t) \geq M^{-1}e^{-LM}q(s), \quad \text{for } 0 \leq s \leq t \leq s + 1.
\]

Thus,

\[
Q(s+1) = \int_{0}^{s+1} q(u)\,du \geq \int_{s}^{s+1} q(u)\,du \geq M^{-1}e^{-LM}q(s).
\]

From Lemma 1, [5], it follows that

\[
\|\Psi(t)Y(t)P_1 v\| \leq KQ^{-1}(s+1)e^{-K^{-1}(t-s-1)}, \quad \text{for } t \geq s + 1
\]

and hence

\[
(7) \quad \|\Psi(t)Y(t)P_1 v\| \leq KMe^{LM}q^{-1}(s)e^{-K^{-1}(t-s-1)}
\]

\[
= KMe^{LM}e^{-K^{-1}(t-s-1)}\|\Psi(s)Y(s)P_1 v\|, \quad \text{for } t \geq s + 1.
\]

From (6) and (7) it results that

\[
(8) \quad \|\Psi(t)Y(t)P_1 v\| \leq N_1 e^{-K^{-1}(t-s)}\|\Psi(s)Y(s)P_1 v\|,
\]

for \( t \geq s \) and \( v \in \mathbb{R}^{d}, \) where \( N_1 = Me^{LM+K^{-1}} \max\{1, K\}. \)

Similarly, for \( P_2 v \neq 0, \) let

\[
r(t) = \|\Psi(t)Y(t)P_2 v\|^{-1}.
\]
From (3) and Lemma 2, [5], it follows that the function \( R(t) = \int_t^\infty r(u) \, du \) exists for \( t \geq 0 \) and

\[
(9) \quad r^{-1}(t) \int_t^T r(u) \, du \leq K, \quad \text{for } T \geq t \geq 0.
\]

Hence,

\[
R'(t) = -r(t) \leq -K^{-1} R(t)
\]

and then,

\[
(10) \quad R(t) \leq R(t_0)e^{-K^{-1}(t-t_0)}, \quad t \geq t_0 \geq 0.
\]

On the other hand, for \( t \geq s \geq 0 \), we have

\[
r^{-1}(t) = \| \Psi(t)Y(t)P_2v \| = \| \Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)\Psi(s)Y(s)P_2v \|
\leq |\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s)| \cdot \| \Psi(s)Y(s)P_2v \|
\leq Me^{LM(t-s)}r^{-1}(s).
\]

Consequently,

\[
r(s) \geq M^{-1}e^{-LM(s-t)}r(t), \quad s \geq t \geq 0.
\]

Hence,

\[
R(t) \geq M^{-1}r(t) \int_t^\infty e^{-LM(s-t)} \, ds = L^{-1}M^{-2}r(t).
\]

Combining this with \((9)\) and \((10)\), we obtain, for \( t \geq t_0 \geq 0 \):

\[
\| \Psi(t)Y(t)P_2v \| = r^{-1}(t) \geq L^{-1}M^{-2}R^{-1}(t)
\geq L^{-1}M^{-2}R^{-1}(t_0)e^{K^{-1}(t-t_0)}
= (LM^2)^{-1}e^{K^{-1}(t-t_0)}\| \Psi(t_0)Y(t_0)P_2v \|.
\]

It results that

\[
(11) \quad \| \Psi(t)Y(t)P_2v \| \leq N_2e^{-K^{-1}(s-t)}\| \Psi(s)Y(s)P_2v \|,
\]

for \( s \geq t \geq 0, \ v \in \mathbb{R}^d \), where \( N_2 = LM^2 \).

Now, we show that

\[
p_i(t) = |\Psi(t)Y(t)P_iY^{-1}(t)\Psi^{-1}(t)|, \quad i = 1, 2,
\]

are bounded for \( t \geq 0 \). Let \( \sigma > 0 \) be such that

\[
p = N_2^{-1}e^{K^{-1}\sigma} - N_1e^{-K^{-1}\sigma} > 0.
\]

From \((8)\) and \((11)\) we deduce that

\[
|\Psi(t+\sigma)Y(t+\sigma)P_1Y^{-1}(t)\Psi^{-1}(t)| \leq N_1e^{-K^{-1}\sigma}p_1(t),
\]

\[
|\Psi(t+\sigma)Y(t+\sigma)P_2Y^{-1}(t)\Psi^{-1}(t)| \geq N_2^{-1}e^{K^{-1}\sigma}p_2(t).
\]
Hence,
\[|p_1^{-1}(t)\Psi(t + \sigma)Y(t + \sigma)P_1Y^{-1}(t)\Psi^{-1}(t)| + p_2^{-1}(t)\Psi(t + \sigma)Y(t + \sigma)P_2Y^{-1}(t)\Psi^{-1}(t)| \geq p.\]

It follows that
\[|\Psi(t + \sigma)Y(t + \sigma)Y^{-1}(t)\Psi^{-1}(t)(p_1^{-1}(t)\Psi(t)Y(t)P_1Y^{-1}(t)\Psi^{-1}(t)| + p_2^{-1}(t)\Psi(t)Y(t)P_2Y^{-1}(t)\Psi^{-1}(t)| \geq p,\]

or
\[p \leq |p_1^{-1}(t)\Psi(t)Y(t)P_1Y^{-1}(t)\Psi^{-1}(t)| + p_2^{-1}(t)\Psi(t)Y(t)P_2Y^{-1}(t)\Psi^{-1}(t)|Me^{LM\sigma}.\]

Therefore,
\[pM^{-1}e^{-LM\sigma} \leq |p_1^{-1}(t)I_d + (p_2^{-1}(t) - p_1^{-1}(t))\Psi(t)Y(t)P_2Y^{-1}(t)\Psi^{-1}(t)| \leq p_1^{-1}(t) + |p_2^{-1}(t) - p_1^{-1}(t)|p_2(t) = p_1^{-1}(t)(1 + |p_1(t) - p_2(t)|)
\]
\[= p_1^{-1}(t)(1 + \|\Psi(t)Y(t)P_1Y^{-1}(t)\Psi^{-1}(t)| - |\Psi(t)Y(t)P_2Y^{-1}(t)\Psi^{-1}(t)|)
\]
\[\leq p_1^{-1}(t)(1 + |\Psi(t)Y(t)P_1Y^{-1}(t)\Psi^{-1}(t) + \Psi(t)Y(t)P_2Y^{-1}(t)\Psi^{-1}(t)|)
\]
\[= 2p_1^{-1}(t).\]

It follows that
\[(12)\quad p_1(t) \leq 2Mp^{-1}e^{LM\sigma} = \overline{M}, \quad t \geq 0.\]

Similarly,
\[(13)\quad p_2(t) \leq \overline{M}, \quad t \geq 0.\]

Finally, by (8), (11), (12) and (13) we deduce that
\[|\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \leq K_1e^{-K^{-1}(t-s)}, \quad 0 \leq s \leq t\]
\[|\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \leq K_1e^{-K^{-1}(s-t)}, \quad 0 \leq t \leq s,\]

where \(K_1 = \overline{M}\max\{N_1, N_2\}\).

Now, we prove the "if" part.

From (4), for \(t \geq 0\) we have
\[\int_0^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| ds + \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| ds \leq K_1 \int_0^s e^{-\alpha(t-s)} ds + K_1 \int_t^\infty e^{-\alpha(s-t)} ds < \frac{2K_1}{\alpha}.\]

From this and Theorem 2.1, [6], it follows the conclusion of theorem. The proof is now complete. \(\square\)
Remark 2.1. If $\Psi(t)$ and fundamental matrix $Y(t)$ do not fulfil the condition (5), then the conditions (4) may not be true.

This is shown by the

**Example 2.1.** Consider the linear system (2) with $A(t) = \begin{pmatrix} -2 & e^t \\ 0 & 2 \end{pmatrix}$.

A fundamental matrix for the system (2) is

$$Y(t) = \begin{pmatrix} e^{-2t} & \frac{1}{2}(e^{3t} - e^{-2t}) \\ 0 & e^{2t} \end{pmatrix}.$$

Consider

$$\Psi(t) = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{pmatrix}.$$

We have

$$\Psi(t)Y(t)Y^{-1}(s)\Psi^{-1}(s) = \begin{pmatrix} e^{-3(t-s)} & \frac{1}{5}e^{2t}(1 - e^{-5(s-t)}) \\ 0 & 1 \end{pmatrix}.$$

This shows that (5) is not satisfied.

Instead,

$$\Psi(t)\Psi^{-1}(s) = \begin{pmatrix} e^{-(t-s)} & 0 \\ 0 & e^{-2(t-s)} \end{pmatrix},$$

is bounded for $0 \leq s \leq t$.

But then, in this case, we have

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thereafter,

$$\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s) = \begin{pmatrix} e^{-3(t-s)} & \frac{1}{5}e^{-3t}(1 - e^{5s}) \\ 0 & 0 \end{pmatrix},$$

which is unbounded for $0 \leq s \leq t$.

Thus, the conditions (4) is not true.

**Remark 2.2.** If in Theorem 2.1 we put $\Psi(t) = I_d$, then the conclusion of the Theorem 3, Chapter V, [4], follows.

We prove finally a theorem in which we will see that the asymptotic behavior of solutions of (1) is determined completely by the asymptotic behavior of $f(t)$ as $t \to \infty$.

**Theorem 2.2.** Suppose that:

1. the fundamental matrix $Y(t)$ of (2) satisfies the conditions

$$|\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \leq K e^{-\alpha(t-s)}, \quad 0 \leq s \leq t,$$

$$|\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)| \leq K e^{-\alpha(s-t)}, \quad 0 \leq t \leq s,$$

where $K$ and $\alpha$ are positive constants and $P_1, P_2$ are supplementary projections, $P_1 \neq 0$;
2. the continuous and $\Psi$-bounded function $f : \mathbb{R}_+ \to \mathbb{R}^d$ satisfies one of the following conditions:
   a) $\lim_{t \to \infty} \|\Psi(t)f(t)\| = 0$,
   b) $\int_{0}^{t} \|\Psi(t)f(t)\| \, dt$ is convergent,
   c) $\lim_{t \to \infty} \int_{t}^{t+1} \|\Psi(s)f(s)\| \, ds = 0$.

Then, every $\Psi$-bounded solution $x(t)$ of (1) is such that
   $$\lim_{t \to \infty} \|\Psi(t)x(t)\| = 0.$$  

Proof. a) It follows from the Theorem 2.1, [6].
   b) It is similar to the proof of Theorem 2.1, [6].
   c) By the hypothesis 2, it follows that there exists a positive constant $C$ such that
   $$\int_{t}^{t+1} \|\Psi(s)f(s)\| \, ds \leq C, \quad \text{for all } t \geq 0.$$

Let $x(t)$ be a $\Psi$-bounded solution of (1). There is a positive constant $M$ such that $\|\Psi(t)x(t)\| \leq M$, for all $t \geq 0$.

Consider the function
   $$y(t) = x(t) - Y(t)P_1x(0) - \int_{0}^{t} Y(t)P_1Y^{-1}(s)f(s) \, ds + \int_{t}^{\infty} Y(t)P_2Y^{-1}(s)f(s) \, ds,$$
   for all $t \geq 0$.

For $v \geq t \geq 0$ we have
   $$\int_{t}^{v} P_2Y^{-1}(s)f(s) \, ds \leq \int_{t}^{v} \|P_2Y^{-1}(s)f(s)\| \, ds$$
   $$\leq |Y^{-1}(t)\Psi^{-1}(t)| \int_{t}^{v} \|\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)\| \cdot \|\Psi(s)f(s)\| \, ds$$
   $$\leq K|Y^{-1}(t)\Psi^{-1}(t)| \int_{t}^{v} e^{-\alpha(s-t)} \|\Psi(s)f(s)\| \, ds$$
   $$\leq KC(1-e^{-\alpha})^{-1}|Y^{-1}(t)\Psi^{-1}(t)|,$$

by using a Lemma of J. L. Massera and J. J. Schäffer, [9].

It follows that the integral
   $$\int_{t}^{\infty} Y(t)P_2Y^{-1}(s)f(s) \, ds$$
   is convergent.
Clearly, the function $y(t)$ is continuously differentiable on $\mathbb{R}_+$. For $t \geq 0$, we have

$$y'(t) = x'(t) - Y'(t)P_1x(0) - Y'(t) \int_0^t P_1Y^{-1}(s)f(s)\,ds - Y(t)P_1Y^{-1}(t)f(t)$$

$$+ Y'(t) \int_t^\infty P_2Y^{-1}(s)f(s)\,ds - Y(t)P_2Y^{-1}(t)f(t)$$

$$= A(t)x(t) + f(t) - A(t)Y(t)P_1x(0) - A(t)Y(t) \int_0^t P_1Y^{-1}(s)f(s)\,ds$$

$$+ A(t)Y(t) \int_t^\infty P_2Y^{-1}(s)f(s)\,ds - Y(t)(P_1 + P_2)Y^{-1}(t)f(t)$$

$$= A(t)y(t).$$

Thus, the function $y(t)$ is a solution of the linear system (2).

Since the hypothesis 1. implies that $\lim_{t \to \infty} \Psi(t)Y(t)P_1 = 0$ (see Lemma 1, [5]), there exists a positive constant $N$ such that $|\Psi(t)Y(t)P_1| \leq N$ for all $t \geq 0$.

It follows that

$$\|\Psi(t)y(t)\| \leq \|\Psi(t)x(t)\| + |\Psi(t)Y(t)P_1| \cdot \|x(0)\|$$

$$+ \int_0^t |\Psi(t)Y(t)P_1Y^{-1}(s)\Psi^{-1}(s)| \cdot \|\Psi(s)f(s)\|\,ds$$

$$+ \int_t^\infty |\Psi(t)Y(t)P_2Y^{-1}(s)\Psi^{-1}(s)|\|\Psi(s)f(s)\|\,ds$$

$$\leq M + N\|x(0)\| + K \int_0^t e^{-\alpha(t-s)}\|\Psi(s)f(s)\|\,ds$$

$$+ K \int_t^\infty e^{-\alpha(s-t)}\|\Psi(s)f(s)\|\,ds$$

$$\leq M + N\|x(0)\| + 2KC(1 - e^{-\alpha})^{-1},$$

for all $t \geq 0$,

by using of above Lemma of Massera and Schäffer.

Thus, the function $y(t)$ is a $\Psi$-bounded solution of the linear system (2).

On the other hand, $P_1y(0) = 0$. Therefore, $y(t) = Y(t)g(0) = Y(t)P_2y(0)$. If $P_2y(0) \neq 0$, from the Lemma 2, [5], it follows that $\lim_{t \to \infty} \|\Psi(t)y(t)\| = +\infty$, which is contradictory. Thus, $P_2y(0) = 0$ and then $y(t) = 0$ for $t \geq 0$. 

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Thus, for \( t \geq 0 \) we have
\[
x(t) = Y(t)P_1 x(0) + \int_0^t Y(t)P_1 Y^{-1}(s)f(s) \, ds - \int_t^\infty Y(t)P_2 Y^{-1}(s)f(s) \, ds.
\]

Now, for a given \( \varepsilon > 0 \), there exists \( t_1 \geq 0 \) such that
\[
\int_0^{t+1} \| \Psi(s)f(s) \| \, ds < \varepsilon (4K)^{-1}(1 - e^{-\alpha}), \quad \text{for all } t \geq t_1.
\]

Moreover, there exists \( t_2 > t_1 \) such that, for \( t \geq t_2 \),
\[
\| \Psi(t)Y(t)P_1 \| \leq \frac{\varepsilon}{2} \left( \| x(0) \| + \int_0^{t_1} \| Y^{-1}(s)f(s) \| \, ds \right)^{-1}.
\]

Then, for \( t \geq t_2 \) we have, by using of above Lemma of Massera and Schäffer,
\[
\| \Psi(t)x(t) \| \leq \| \Psi(t)Y(t)P_1 \| \| x(0) \| + \int_0^{t_1} \| \Psi(t)Y(t)P_2 Y^{-1}(s)\Psi^{-1}(s)f(s) \| \, ds
\]
\[
+ \int_0^{t} \| \Psi(t)Y(t)P_1 Y^{-1}(s)\Psi^{-1}(s)f(s) \| \, ds
\]
\[
\leq \| \Psi(t)Y(t)P_1 \| \| x(0) \| + \int_0^{t_1} \| Y^{-1}(s)f(s) \| \, ds
\]
\[
+ \int_{t_1}^{t} e^{-\alpha(t-s)} \| \Psi(s)f(s) \| \, ds + K \int_t^\infty e^{-\alpha(s-t)} \| \Psi(s)f(s) \| \, ds
\]
\[
< \varepsilon.
\]

This shows that \( \lim_{t \to \infty} \| \Psi(t)x(t) \| = 0 \).

The proof is now complete. \( \square \)

**Remark 2.3.** If in Theorem we put \( A(t) = A, \Psi(t) = \varphi^k(t)I_d \), then the conclusion of the Theorem 3.1, [3], follows.

**Remark 2.4.** If the function \( f \) does not fulfill the condition 2 of the theorem, then \( \Psi(t)x(t) \) may be such that
\[
\lim_{t \to \infty} \| \Psi(t)x(t) \| \neq 0.
\]

This can be seen from
**Example 2.2.** Consider the linear system (1) with
\[
A(t) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad f(t) = \begin{pmatrix} e^{(a+1)t} \\ e^{(b-2)t} \end{pmatrix},
\]
where \(a, b \in \mathbb{R}\).

A fundamental matrix for the homogeneous system (2) is
\[
Y(t) = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix}.
\]
Consider
\[
\Psi(t) = \begin{pmatrix} e^{-(a+1)t} & 0 \\ 0 & e^{(1-b)t} \end{pmatrix}.
\]
The first condition of the Theorem 2.2. is satisfied with
\[
P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha = 1, \quad K = 1.
\]
Then, we have \(\|\Psi(t)f(t)\| = 1\) for all \(t \geq 0\) and
\[
\Psi(t)x(t) = \begin{pmatrix} c_1 e^{-t} + \frac{1}{2} e^{-t} \\ c_2 e^t \end{pmatrix} \to 0 \quad \text{as} \quad t \to \infty.
\]

**Remark 2.5.** This Example shows that the components of the solution \(x(t)\) have a mixed asymptotic behavior.

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**References**


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