WEIGHTED ENDPOINT ESTIMATES FOR MULTILINEAR LITTLEWOOD-PALEY OPERATORS

LIU LANZHE

ABSTRACT. In this paper, we prove weighted endpoint estimates for multilinear Littlewood-Paley operators.

1. Introduction and Results

Let $\psi$ be a fixed function on $\mathbb{R}^n$ which satisfies the following properties:

1. $\int \psi(x)dx = 0$,
2. $|\psi(x)| \leq C(1 + |x|)^{-(n+1)}$,
3. $|\psi(x+y) - \psi(x)| \leq C|y|(1 + |x|)^{-(n+2)}$ when $2|y| < |x|$;

Let $m$ be a positive integer and $A$ be a function on $\mathbb{R}^n$. The multilinear Littlewood-Paley operator is defined by

$$g_{\mu}^A(f)(x) = \left( \int \int_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t+|x-y|} \right)^{n\mu} |F_t^A(f)(x,y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2}, \quad \mu > 1,$$

where

$$F_t^A(f)(x,y) = \int_{\mathbb{R}^n} \frac{R_{m+1}(A; x, z)}{|x-z|^m} \psi_t(y-z)f(z)dz,$$

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha A(y)(x-y)^\alpha,$$

and $\psi_t(x) = t^{-n} \psi(x/t)$ for $t > 0$. We denote by $F_t(f)(y) = f * \psi_t(y)$. We also define

$$g_{\mu}(f)(x) = \left( \int \int_{\mathbb{R}_+^{n+1}} \left( \frac{t}{t+|x-y|} \right)^{n\mu} |F_t(f)(y)|^2 \frac{dydt}{t^{n+1}} \right)^{1/2},$$

which is the Littlewood-Paley operator (see [17]).

Received June 11, 2003.
2000 Mathematics Subject Classification. Primary 42B20, 42B25.
Key words and phrases. Multilinear operator, Littlewood-Paley operator, Hardy space, BMO space.
Supported by the NNSF (Grant: 10271071).
Let $H$ be the Hilbert space $H = \{ h : \| h \| = \left( \int \int_{R^{n+1}} |h(t)|^2 \, dy dt/t^{n+1} \right)^{1/2} < \infty \}$. Then for each fixed $x \in R^n$, $F^A_t(f)(x, y)$ may be viewed as a mapping from $(0, +\infty)$ to $H$, and it is clear that

$$g^A_{\mu}(f)(x) = \left\| \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} F^A_t(f)(x, y) \right\|,$$

$$g_{\mu}(f)(x) = \left\| \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} F_t(f)(y) \right\|.$$ 

We also consider the variant of $g^A_{\mu}$, which is defined by

$$\tilde{g}^A_{\mu}(f)(x) = \left[ \int \int_{R^{n+1}} \left( \frac{t}{t + |x - y|} \right)^{n\mu} \left| \tilde{F}^A_t(f)(x, y) \right|^2 \, dy \, dt/t^{n+1} \right]^{1/2}, \quad \mu > 1,$$

where

$$\tilde{F}^A_t(f)(x, y) = \int_{R^n} \frac{Q_{m+1}(A; x, z)}{|x - z|^m} \psi_r(y - z) f(z) \, dz$$

and

$$Q_{m+1}(A; x, z) = R_m(A; x, z) - \sum_{|\alpha| = m} D^\alpha A(x)(x - z)^\alpha.$$ 

Note that when $m = 0$, $g^A_{\mu}$ is just the commutator of Littlewood-Paley operator (see [1], [14], [15]). It is well known that multilinear operators, as an extension of commutators, are of great interest in harmonic analysis and have been widely studied by many authors (see [4] – [8], [12], [13]). In [11], [16], the endpoint boundedness properties of commutators generated by the Calderon-Zygmund operator and BMO functions are obtained. The main purpose of this paper is to study the weighted endpoint boundedness of the multilinear Littlewood-Paley operators. Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of $f$, $Q$ will denote a cube of $R^n$ with side parallel to the axes. For a cube $Q$ and any locally integral function $f$ on $R^n$, we denote that $f(Q) = \int_Q f(x) \, dx$, $f_Q = |Q|^{-1} \int_Q f(x) \, dx$ and $f^\#(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y) - f_Q| \, dy$.

Moreover, for a weight functions $w \in A_1$ (see [10]), $f$ is said to belong $\text{BMO}(w)$ if $f^\# \in L^\infty(w)$ and define $\| f \|_{\text{BMO}(w)} = \| f^\# \|_{L^\infty(w)}$, if $w = 1$, we denote that $\text{BMO}(R^n) = \text{BMO}(w)$. Also, we give the concepts of atomic and weighted $H^1$ space. A function $a$ is called a $H^1(w)$ atom if there exists a cube $Q$ such that $a$ is supported on $Q$, $\| a \|_{L^\infty(w)} \leq w(Q)^{-1}$ and $\int a(x) \, dx = 0$. It is well known that, for $w \in A_1$, the weighted Hardy space $H^1(w)$ has the atomic decomposition characterization (see [2]).

We shall prove the following theorems in Section 3.

**Theorem 1.** Let $D^\alpha A \in \text{BMO}(R^n)$ for $|\alpha| = m$ and $w \in A_1$. Then $g^A_{\mu}$ is bounded from $L^\infty(w)$ to $\text{BMO}(w)$. 
**Theorem 2.** Let $D^\alpha A \in \text{BMO}(R^n)$ for $|\alpha| = m$ and $w \in A_1$. Then $\tilde{g}_\mu^A$ is bounded from $H^1(w)$ to $L^1(w)$.

**Theorem 3.** Let $D^\alpha A \in \text{BMO}(R^n)$ for $|\alpha| = m$ and $w \in A_1$. Then $g_\mu^A$ is bounded from $H^1(w)$ to $L^1(w)$.

**Theorem 4.** Let $D^\alpha A \in \text{BMO}(R^n)$ for $|\alpha| = m$ and $w \in A_1$.

(i) If for any $H^1(w)$-atom $a$ supported on certain cube $Q$ and $u \in 3Q \setminus 2Q$, there is

\[
\int_{(4Q)^c} \left( \frac{t}{t + |x-y|} \right)^{n\mu/2} \sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{(x-u)^\alpha}{|x-u|^m} \int_Q \psi_t(y-z)D^\alpha A(z)dz \| w(x)dx \leq C,
\]

then $g_\mu^A$ is bounded from $H^1(w)$ to $L^1(w)$;

(ii) If for any cube $Q$ and $u \in 3Q \setminus 2Q$, there is

\[
\frac{1}{w(Q)} \int_Q \left( \frac{t}{t + |x-y|} \right)^{n\mu/2} \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)_Q) \cdot \int_{(4Q)^c} \frac{(u-z)\alpha}{|u-z|^m} \psi_t(y-z)f(z)dz \| w(x)dx 
\leq C \| f \|_{L^\infty(w)},
\]

then $\tilde{g}_\mu^A$ is bounded from $L^\infty(w)$ to $\text{BMO}(w)$.

**Remark.** In general, $g_\mu^A$ is not bounded from $H^1(w)$ to $L^1(w)$.

2. **Some Lemmas**

We begin with two preliminary lemmas.

**Lemma 1.** (see [7],) Let $A$ be a function on $R^n$ and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then

\[
|R_m(A : x, y)| \leq C |x - y|^m \sum_{|\alpha|=m} \left( \frac{1}{(|Q(x,y)|)^{1/q}} \int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q dz \right)^{1/q},
\]

where $\tilde{Q}(x,y)$ is the cube centered at $x$ and having side length $5\sqrt{n}|x - y|$.

**Lemma 2.** Let $w \in A_1$, $1 < p < \infty$ $1 < r \leq \infty$, $1/q = 1/p + 1/r$ and $D^\alpha A \in \text{BMO}(R^n)$ for $|\alpha| = m$. Then $g_\mu^A$ is bounded from $L^p(w)$ to $L^r(w)$, that is

\[
\|g_\mu^A(f)\|_{L^r(w)} \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} \| f \|_{L^p(w)}.
\]
Proof. By Minkowski inequality and the condition of \( \psi \), we have
\[
g_{\mu}^{A}(f)(x) \\
\leq \int_{\mathbb{R}^{n}} \frac{|f(z)||R_{n+1}(A; x, z)|}{|x - z|^{m}} \left( \int_{R_{n}^{+1}} |\psi_{t}(y - z)|^{2} \left( \frac{t}{t + |x - y|} \right)^{n\mu} d\gamma dt \right)^{1/2} d\eta
\]
\[
\leq C \int_{\mathbb{R}^{n}} \frac{|f(z)||R_{n+1}(A; x, z)|}{|x - z|^{m}} \left( \int_{0}^{\infty} \int_{R_{n}} \left( \frac{t}{t + |x - y|} \right)^{n\mu} d\eta dt \right)^{1/2} d\eta
\]
noting that
\[
t^{-n} \int_{R_{n}} \left( \frac{t}{t + |x - y|} \right)^{n\mu} d\eta \\
\leq CM \left( \frac{1}{(t + |x - z|)^{2n+2}} \right)(x) \leq C \frac{1}{(t + |x - z|)^{2n+2}}
\]
and
\[
\int_{0}^{\infty} \frac{tdt}{(t + |x - z|)^{2n+2}} = C|x - z|^{-2n},
\]
we obtain
\[
g_{\mu}^{A}(f)(x) \\
\leq C \int_{\mathbb{R}^{n}} \frac{|f(z)||R_{n+1}(A; x, z)|}{|x - z|^{m+n}} \left( \int_{0}^{\infty} \frac{tdt}{(t + |x - z|)^{2n+2}} \right)^{1/2} d\eta
\]
\[
= C \int_{\mathbb{R}^{n}} \frac{|f(z)||R_{n+1}(A; x, z)|}{|x - z|^{m+n}} d\eta,
\]
thus, the lemma follows from [8] [9].

3. Proof of Theorems

Proof of Theorem 1. It is only to prove that there exists a constant \( C_{Q} \) such that
\[
\frac{1}{w(Q)} \int_{Q} |g_{\mu}^{A}(f)(x) - C_{Q}|w(x)dx \leq C ||f||_{L^{\infty}(w)}
\]
holds for any cube \( Q \). Fix a cube \( Q = Q(x_{0}, l) \). Let \( \tilde{Q} = 5\sqrt{n}Q \) and \( \tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^{\alpha}A)\tilde{Q}x^{\alpha} \), then \( R_{m}(A; x, y) = R_{m}(\tilde{A}; x, y) \) and \( D^{\alpha}A = D^{\alpha}A - (D^{\alpha}A)\tilde{Q} \) for \( |\alpha| = m \). We write \( F_{x}^{A}(f) = F_{x}^{A}(f_{1}) + F_{x}^{A}(f_{2}) \) for \( f_{1} = f\chi_{\tilde{Q}} \)
To estimate II, we write

\[
\frac{1}{w(Q)} \int_Q |g^A_\mu(f)(x) - g^A_\mu(f_2)(x_0)|w(x)dx
\]

= \frac{1}{w(Q)} \int_Q \left| \left( \frac{t}{t + |x - y|} \right)^{n/2} F^A_t(f)(x, y) \right|
\left| \left( \frac{t}{t + |x_0 - y|} \right)^{n/2} F^A_t(f_2)(x_0, y) \right| w(x)dx

\leq \frac{1}{w(Q)} \int_Q g^A_\mu(f)(x)w(x)dx

+ \frac{1}{w(Q)} \int_Q \left| \left( \frac{t}{t + |x - y|} \right)^{n/2} F^A_t(f_2)(x, y) \right|
\left| \left( \frac{t}{t + |x_0 - y|} \right)^{n/2} F^A_t(f_2)(x_0, y) \right| w(x)dx

:= I(x) + II(x).

Now, let us estimate I and II. First, by the \(L^\infty\) boundedness of \(g^A_\mu\) (Lemma 2), we gain

\[
I \leq \|g^A_\mu(f_1)\|_{L^\infty} \leq C \|f\|_{L^\infty(w)}.
\]

To estimate II, we write

\[
\left( \frac{t}{t + |x - y|} \right)^{n/2} F^A_t(f_2)(x, y) - \left( \frac{t}{t + |x_0 - y|} \right)^{n/2} F^A_t(f_2)(x_0, y)
\]

= \left( \frac{t}{t + |x - y|} \right)^{n/2} \int \left[ \frac{1}{|x - z|^m} - \frac{1}{|x_0 - z|^m} \right] \psi_t(y - z) R_m(\tilde{A}; x, z) f_2(z)dz

+ \left( \frac{t}{t + |x - y|} \right)^{n/2} \int \frac{\psi_t(y - z) f_2(z)}{|x_0 - z|^m} \left[ R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z) \right]dz

+ \int \left[ \left( \frac{t}{t + |x - y|} \right)^{n/2} - \left( \frac{t}{t + |x_0 - y|} \right)^{n/2} \right] \psi_t(y - z) R_m(\tilde{A}; x_0, z) f_2(z)dz

- \sum_{|\alpha|=m} \frac{1}{\alpha!} \int \left[ \left( \frac{t}{t + |x - y|} \right)^{n/2} \frac{(x - z)^\alpha}{|x - z|^m} \right]
\left[ \left( \frac{t}{t + |x_0 - y|} \right)^{n/2} \frac{(x_0 - z)^\alpha}{|x_0 - z|^m} \right] \psi_t(y - z) D^\alpha \tilde{A}(z) f_2(z)dz

:= II_1^1(x) + II_2^1(x) + II_3^1(x) + II_4^1(x),
\]
Note that $|x - z| \sim |x_0 - z|$ for $x \in Q$ and $z \in R^n \setminus \tilde{Q}$, similar to the proof of Lemma 2 and by Lemma 1, we have

$$
\frac{1}{w(Q)} \int_{Q} ||I_1^2(x)|| w(x) dx 
\leq \frac{C}{w(Q)} \int_{Q} \left( \int_{R^n \setminus \tilde{Q}} \frac{|x - x_0||f(z)|}{|x - z|^{n+m+1}} |R_m(\tilde{A}; x, z)| dz \right) w(x) dx
$$

$$
\leq \frac{C}{w(Q)} \int_{Q} \left( \sum_{k=0}^{\infty} \int_{2^{k+1}Q \setminus 2^k \tilde{Q}} \frac{|x - x_0||f(z)|}{|x - z|^{n+m+1}} |R_m(\tilde{A}; x, z)| dz \right) w(x) dx
$$

$$
\leq C \sum_{k=0}^{\infty} \frac{k|l(2^k l)|^m}{(2^k l)^{n+m+1}} \sum_{|\alpha| = m} ||D^\alpha A||_{\text{BMO}} \left( \int_{2^{k+1} \tilde{Q}} |f(z)| dz \right)
$$

$$
\leq C \sum_{|\alpha| = m} ||D^\alpha A||_{\text{BMO}} ||f||_{L^\infty(w)} \sum_{k=0}^{\infty} k2^{-k}
$$

Thus, for $x \in Q$,

$$
||I_1^2(x)|| 
\leq C \int_{R^n} \frac{|f_2(z)|}{|x - z|^{n+m}} |R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z)| dz
$$

$$
\leq C \sum_{|\alpha| = m} ||D^\alpha A||_{\text{BMO}} \int_{R^n} \frac{|x - x_0|^m + \sum_{0 < |\beta| < m} |x_0 - z|^{m-|\beta|}|x - x_0||^{|\beta|}}{|x_0 - z|^{m+n}} |f_2(z)| dz
$$

$$
\leq C \sum_{|\alpha| = m} ||D^\alpha A||_{\text{BMO}} \sum_{k=0}^{\infty} \frac{k|l|^m}{(2^k l)^{m+n}} \int_{2^{k+1} \tilde{Q}} |f(z)| dz
$$

$$
\leq C \sum_{|\alpha| = m} ||D^\alpha A||_{\text{BMO}} ||f||_{L^\infty(w)} \sum_{k=1}^{\infty} k2^{-km}
$$

$$
\leq C \sum_{|\alpha| = m} ||D^\alpha A||_{\text{BMO}} ||f||_{L^\infty(w)} ;
$$
For $II_2^3(x)$, by the inequality: $a^{1/2} - b^{1/2} \leq (a - b)^{1/2}$ for $a \geq b > 0$, we obtain, similar to the estimate of Lemma 2 and $II_1^3$,

$$||II_2^3(x)|| \leq C \int_{R^n} \left( \int_{R_2^{+1}} \left[ \frac{11^{\mu/2} |x - x_0|^{1/2} |\psi_t(y - z)||f_2(z)||R_m(\tilde{A}; x_0, z)|}{(t + |x - y|)^{n+\mu/2}} \right] |R_m(\tilde{A}; x_0, z)| \right)^{2 \frac{dydt}{m+1}} dz $$

Combining these estimates, we complete the proof of Theorem 1.

Proof of Theorem 2. It suffices to show that there exists a constant $C > 0$ such that for every $H^1(w)$-atom $a$ (that is that $a$ satisfies: $\text{supp} a \subset Q = Q(x_0, r)$, $|a|_{L^\infty(w)} \leq w(Q)^{-1}$ and $\int a(y) dy = 0$ (see [8])), we have

$$||\gamma_A^w(a)||_{L^1(w)} \leq C.$$
We write
\[
\int_{R^n} \tilde{g}_\mu^A(a)(x) w(x) dx = \left[ \int_{|x-x_0| \leq 2r} + \int_{|x-x_0| > 2r} \right] \tilde{g}_\mu^A(a)(x) w(x) dx := J + JJ.
\]

For \( J \), by the following equality
\[
Q_{m+1}(A; x, y) = R_{m+1}(A; x, y) - \sum_{|\alpha| = m} \frac{1}{\alpha!} (x - y)^\alpha (D^\alpha A(x) - D^\alpha A(y)),
\]
we have, similar to the proof of Lemma 2,
\[
\tilde{g}_\mu^A(a)(x) \leq g_\mu^A(a)(x) + C \sum_{|\alpha| = m} \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x-y|^n} |a(y)| dy,
\]
thus, \( \tilde{g}_\mu^A \) is \( L^\infty \)-bounded by Lemma 2 and \cite{3}. We see that
\[
J \leq C \| \tilde{g}_\mu^A(a) \|_{L^\infty(w(2Q))} w(2Q) \leq C \| a \|_{L^\infty(w(Q))} w(Q) \leq C.
\]

To obtain the estimate of \( JJ \), we denote that \( \tilde{A}(x) = A(x) - \sum_{|\alpha| = m} \frac{1}{\alpha!} (D^\alpha A)_{2B} x^\alpha \).

Then \( Q_m(A; x, y) = Q_m(\tilde{A}; x, y) \). We write, by the vanishing moment of \( a \) and \( Q_{m+1}(A; x, y) = R_m(\tilde{A}; x, y) - \sum_{|\alpha| = m} \frac{1}{\alpha!} (x - y)^\alpha D^\alpha A(x) \), for \( x \in (2Q) \),
\[
\tilde{F}_t^A(a)(x, y) = \int \frac{\psi_t(y-z)R_m(\tilde{A}; x, z)}{|x-z|^m} a(z) dz - \sum_{|\alpha| = m} \frac{1}{\alpha!} \int \frac{\psi_t(y-z)D^\alpha \tilde{A}(x-z)^\alpha}{|x-z|^m} a(z) dz
\]
\[
= \int \left[ \frac{\psi_t(y-z)R_m(\tilde{A}; x, z)}{|x-z|^m} - \frac{\psi_t(y-x_0)R_m(\tilde{A}; x, x_0)}{|x-x_0|^m} \right] a(z) dz
\]
\[
- \sum_{|\alpha| = m} \frac{1}{\alpha!} \int \left[ \frac{\psi_t(y-z)(x-z)^\alpha}{|x-z|^m} - \frac{\psi_t(y-x_0)(x-x_0)^\alpha}{|x-x_0|^m} \right] D^\alpha \tilde{A}(x) a(z) dz,
\]
thus, similar to the proof of \( II \) in Theorem 1, we obtain
\[
\| \tilde{F}_t^A(a)(x, y) \| \leq C \frac{Q_1^{1+1/n}}{w(Q)} \left( \sum_{|\alpha| = m} ||D^\alpha A||_{\text{BMO}} |x-x_0|^{-n-1} + |x-x_0|^{-n-1} |D^\alpha \tilde{A}(x)| \right),
\]
note that if \( w \in A_1 \), then \( \frac{w(Q_2)}{|Q_2|} \frac{|Q_1|}{w(Q_1)} \leq C \) for all cubes \( Q_1, Q_2 \) with \( Q_1 \subset Q_2 \).

Thus, by Holder’ inequality and the reverse of Holder’ inequality for \( w \in A_1 \),
taking $p > 1$ and $1/p + 1/p' = 1$, we obtain

$$JJ \leq C \sum_{|\alpha| = m} ||D^\alpha A||_{BMO} \sum_{k=1}^{\infty} 2^{-k} \left( \frac{|Q|}{w(Q)} \frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \right)^{1/p} \cdot \left( \frac{1}{2^{k+1}Q} \int_{2^{k+1}Q} |D^\alpha \tilde{A}(x)|^p dx \right)^{1/p'}$$

$$+ C \sum_{|\alpha| = m} \sum_{k=1}^{\infty} 2^{-k} \frac{|Q|}{w(Q)} \left( \frac{1}{2^{k+1}Q} \int_{2^{k+1}Q} |D^\alpha \tilde{A}(x)|^p dx \right)^{1/p}$$

$$\leq C \sum_{|\alpha| = m} ||D^\alpha A||_{BMO} \sum_{k=1}^{\infty} 2^{-k} \left( \frac{w(2^{k+1}Q)}{|2^{k+1}Q|} \frac{|Q|}{w(Q)} \right)^{1/p} \leq C,$$

which together with the estimate for $J$ yields the desired result. This finishes the proof of Theorem 2.

Proof of Theorem 3. By the equality

$$R_{m+1}(A; x, y) = Q_{m+1}(A; x, y) + \sum_{|\alpha| = m} \frac{1}{\alpha!} (x - y)^\alpha (D^\alpha A(x) - D^\alpha A(y))$$

and similar to the proof of Lemma 2, we get

$$g^A_\mu(f)(x) \leq \tilde{g}^A_\mu(f)(x) + C \sum_{|\alpha| = m} \int \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x - y|^n}|f(y)|dy,$$

by Theorem 1 and 2 with [3], we obtain

$$w(\{x \in R^n : \tilde{g}^A_\mu(f)(x) > \lambda\})$$

$$\leq w(\{x \in R^n : \tilde{g}^A_\mu(f)(x) > \lambda/2\})$$

$$+ w(\{x \in R^n : \sum_{|\alpha| = m} \int \frac{|D^\alpha A(x) - D^\alpha A(y)|}{|x - y|^n}|f(y)|dy > C\lambda\})$$

$$\leq C||f||_{H^1(w)}/\lambda.$$

This completes the proof of Theorem 3.

Proof of Theorem 4. (i) It suffices to show that there exists a constant $C > 0$ such that for every $H^1(w)$-atom $a$ with supp $a \subset Q = Q(x_0, d)$, there is

$$||g^A_\mu(a)||_{L^1(w)} \leq C.$$
Let $\tilde{A}(x) = A(x) - \sum_{|a| = m} \frac{1}{a!} (D^\alpha A)_{Q} x^\alpha$. We write, by the vanishing moment of $a$ and for $u \in 3Q \setminus 2Q$,

$$F_1^A(a)(x, y) = \chi_{4Q}(x) F_1^A(a)(x, y)$$

$$+ \chi_{4Q}(x) \int_{R^n} \left[ \frac{R_m(\tilde{A}; x, z) \psi_t(y - z) - R_m(\tilde{A}; x, u) \psi_t(y - u)}{|x - z|^m} \right] a(z) dz$$

$$- \sum_{|a| = m} \frac{1}{a!} \int_{R^n} \left[ \frac{(x - z)^\alpha}{|x - z|^m} - \frac{(x - u)^\alpha}{|x - u|^m} \right] \psi_t(y - z) D^\alpha A(z) a(z) dz$$

then

$$g_\mu^A(a)(x)$$

$$\leq \chi_{4Q}(x) \left( \int_{R^n} \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} F_1^A(a)(x, y) \right)$$

$$+ \chi_{4Q}(x) \left( \int_{R^n} \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} F_1^A(a)(x, y) \right)$$

$$+ \sum_{|a| = m} \frac{1}{a!} \int_{R^n} \left[ \frac{(x - z)^\alpha}{|x - z|^m} - \frac{(x - u)^\alpha}{|x - u|^m} \right] \psi_t(y - z) D^\alpha A(z) a(z) dz$$

$$= L_1(x) + L_2(x) + L_3(x, u) + L_4(x, u).$$

By the $L^\infty(w)$-boundedness of $g_\mu^A$, we get

$$\int_{R^n} L_1(x) w(x) dx = \int_{4Q} g_\mu^A(a)(x) w(x) dx \leq ||g_\mu^A(a)||_{L^\infty(w)(4Q)}$$

$$\leq C||a||_{L^\infty(w)(Q)} \leq C;$$
Similar to the proof of Theorem 1, we obtain

\[ \int_{\mathbb{R}^n} L_2(x)w(x)dx \leq C \]

and

\[ \int_{\mathbb{R}^n} L_3(x, u)w(x)dx \leq C. \]

Thus, using the condition of \( L_4(x, u) \), we obtain

\[ \int_{\mathbb{R}^n} g_A^\mu(a)(x)w(x)dx \leq C. \]

(ii) For any cube \( Q = Q(x_0, d) \), let \( \tilde{A}(x) = A(x) - \sum_{|\alpha| = m} \frac{1}{\alpha!}(D^\alpha A)_{Q}x^\alpha. \) We write, for \( f = f\chi_{4Q} + f\chi_{(4Q)^c} = f_1 + f_2 \) and \( u \in 3Q \setminus 2Q \),

\[ \tilde{F}_t^A(f)(x, y) = \tilde{F}_t^A(f_1)(x, y) + \int_{\mathbb{R}^n} \frac{R_{m}(\tilde{A}; x, z)}{|x - z|^m} \psi_t(y - z)f_2(z)dz \]

\[ - \sum_{|\alpha| = m} \frac{1}{\alpha!}(D^\alpha A(x) - (D^\alpha A)_Q) \int_{\mathbb{R}^n} \frac{(x - z)^\alpha}{|x - z|^m} - \frac{(u - z)^\alpha}{|u - z|^m} \psi_t(y - z)f_2(z)dz \]

\[ - \sum_{|\alpha| = m} \frac{1}{\alpha!}(D^\alpha A(x) - (D^\alpha A)_Q) \int_{\mathbb{R}^n} \frac{(u - z)^\alpha}{|u - z|^m} \psi_t(y - z)f_2(z)dz, \]

then

\[ \left| \tilde{g}_n^\mu(f)(x) - g_\mu \left( \frac{R_{m}(\tilde{A}; x_0, \cdot)}{|x_0 - \cdot|^m} f_2 \right)(x_0) \right| \]

\[ = \left\| \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} \tilde{F}_t^A(f)(x, y) \right\| \]

\[ - \left\| \left( \frac{t}{t + |x_0 - y|} \right)^{n\mu/2} F_t \left( \frac{R_{m}(\tilde{A}; x_0, \cdot)}{|x_0 - \cdot|^m} f_2 \right)(x_0) \right\| \]

\[ \leq \left\| \left( \frac{t}{t + |x - y|} \right)^{n\mu/2} \tilde{F}_t^A(f)(x, y) \right\| \]

\[ - \left( \frac{t}{t + |x_0 - y|} \right)^{n\mu/2} F_t \left( \frac{R_{m}(\tilde{A}; x_0, \cdot)}{|x_0 - \cdot|^m} f_2 \right)(x_0) \]
\[
\leq \left\| \left( \frac{t}{t+|x-y|} \right)^{\mu/2} \tilde{F}_t^A(f_1)(x,y) \right\| + \left\| \int_{\mathbb{R}^n} \left[ \left( \frac{t}{t+|x-y|} \right)^{\mu/2} \frac{R_m(\tilde{A}; x, z)}{|x-z|^m} \psi_t(y-z) \right. \right.
\]
\[
- \left. \left( \frac{t}{t+|x_0-y|} \right)^{\mu/2} \frac{R_m(\tilde{A}; x_0, z)}{|x_0-z|^m} \psi_t(x_0-z) \right] f_2(z)dz \right\| + \left\| \left( \frac{t}{t+|x-y|} \right)^{\mu/2}
\]
\[
\cdot \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)_Q) \int_{\mathbb{R}^n} \left[ \frac{(y-z)^\alpha}{|y-z|^m} - \frac{(u-z)^\alpha}{|u-z|^m} \right] \psi_t(y-z)f_2(z)dz \right\| + \left\| \left( \frac{t}{t+|x-y|} \right)^{\mu/2}
\]
\[
\cdot \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A(x) - (D^\alpha A)_Q) \int_{\mathbb{R}^n} \frac{(u-z)^\alpha}{|u-z|^m} \psi_t(y-z)f_2(z)dz \right\|
\]
\[
= M_1(x) + M_2(x) + M_3(x, u) + M_4(x, u).
\]

By the $L^\infty(w)$-boundedness of $\tilde{g}_\mu^A$, we get
\[
\frac{1}{w(Q)} \int_Q M_1(x)w(x)dx \leq ||\tilde{g}_\mu^A(f_1)||_{L^\infty(w)} \leq C||f||_{L^\infty(w)};
\]

Similar to the proof of Theorem 1, we obtain
\[
\frac{1}{w(Q)} \int_Q M_2(x)w(x)dx \leq C||f||_{L^\infty(w)}
\]
and
\[
\frac{1}{w(Q)} \int_Q M_3(x, u)w(x)dx \leq C||f||_{L^\infty(w)}.
\]
Thus, using the condition of $M_4(x, u)$, we obtain
\[
\frac{1}{w(Q)} \int_Q \tilde{g}_\mu^A(f)(x) - g_{\mu} \left( \frac{R_m(\tilde{A}; x_0, \cdot)}{|x_0 - \cdot|^m} f_2 \right)(x_0) |w(x)|dx \leq C||f||_{L^\infty(w)}.
\]
This completes the proof of Theorem 4.

Acknowledgement. The author would like to express his deep gratitude to the referee for his valuable comments and suggestions.
ENDPOINT ESTIMATES FOR LITTLEWOOD-PALEY OPERATORS

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Liu Lanzhe, College of Mathematics and Computer, Changsha University of Science and Technology, Changsha 410077, P. R. of China, e-mail: lanzheliu@263.net