ON THE ENDOMORPHISM RING OF A SEMI-INJECTIVE MODULE

S. WONGWAI

Abstract. Let $R$ be a ring. A right $R$-module $M$ is called quasi-principally (or semi-) injective if it is $M$-principally injective. In this paper, we show: (1) The following are equivalent for a projective module $M$: (a) Every $M$-cyclic submodule of $M$ is projective; (b) Every factor module of an $M$-principally injective module is $M$-principally injective; (c) Every factor module of an injective $R$-module is $M$-principally injective. (2) The endomorphism ring $S$ of a semi-injective module is regular if and only if the kernel of every endomorphism is a direct summand. (3) For a semi-injective module $M$, if $S$ is semiregular, then for every $s \in S \setminus J(S)$, there exists a nonzero idempotent $\alpha \in Ss$ such that $\text{Ker}(s) \subset \text{Ker}(\alpha)$ and $\text{Ker}(s(1-\alpha)) \neq 0$. The converse is also considered.

1. Introduction

Let $R$ be a ring. A right $R$-module $M$ is called principally injective if every $R$-homomorphism from a principal right ideal of $R$ to $M$ can be extended to an $R$-homomorphism from $R$ to $M$. This notion was introduced by Camillo [2] for commutative rings. In [7], Nicholson and Yousif studied the structure of principally injective rings and gave some applications. In [9], Sanh and others extended this notion to modules. A right $R$-module $N$ is called $M$-principally injective if every $R$-homomorphism from an $M$-cyclic submodule of $M$ to $N$ can be extended to $M$. In [10], Tansee and Wongwai introduced the dual notion, a right $R$-module $N$ is called $M$-principally projective if every $R$-homomorphism from $N$ to an $M$-cyclic submodule of $M$ can be lifted to an $R$-homomorphism from $N$ to $M$. A module $M$ is called quasi-principally (or semi-) projective if it is $M$-principally projective. Dual to this module and following Wisbauer [12] we consider a semi-injective module.

Throughout this paper, $R$ is an associative ring with identity. Let $M$ be a right $R$-module, the endomorphism ring of $M$ is denoted by $S = \text{End}_R(M)$. A module $N$ is called $M$-generated if there is an epimorphism $M(I) \to N$ for some index set $I$. If $I$ is finite, then $N$ is called finitely $M$-generated. In particular, a submodule $N$ of $M$ is called $M$-cyclic submodule of $M$ if it is isomorphic to $M/X$ for some submodule $X$ of $M$. By the notation $N \subseteq_\oplus M$ ($N \subseteq_\epsilon M$ we mean

Received September 18, 2000; revised March 15, 2001.

2000 Mathematics Subject Classification. Primary 16D50, 16D70, 16D80.

Key words and phrases. Semi-injective modules, Endomorphism rings.
that $N$ is a direct summand (an essential submodule) of $M$. We denote the socle and the singular submodule of $M$ by $\text{Soc}(M)$ and $Z(M)$ respectively, and that $J(M)$ denotes the Jacobson radical of $M$.

2. Principal Injectivity

Definition 2.1. [9] Let $M$ be a right $R$-module. A right $R$-module $N$ is called $M$-principally injective if every $R$-homomorphism from an $M$-cyclic submodule of $M$ to $N$ can be extended to $M$. Equivalently, for any endomorphism $s$ of $M$, every homomorphism from $s(M)$ to $N$ can be extended to a homomorphism from $M$ to $N$. $N$ is called principally injective if it is $R$-principally injective.

Lemma 2.2. Let $M$ and $N$ be $R$-modules. Then $N$ is $M$-principally injective if and only if for each $s \in S = \text{End}_R(M)$, $\text{Hom}_R(M,N)s = \{ f \in \text{Hom}_R(M,N) : f(\text{Ker}(s)) = 0 \}$.

Proof. Clearly, $\text{Hom}_R(M,N)s \subset \{ f \in \text{Hom}_R(M,N) : f(\text{Ker}(s)) = 0 \}$. Let $f \in \text{Hom}_R(M,N)$ such that $f(\text{Ker}(s)) = 0$. This leads to $\text{Ker}(f) = 0$. Then there is an $R$-homomorphism $\varphi : s(M) \rightarrow N$ such that $\varphi s = f$. Since $N$ is $M$-principally injective, there exists an $R$-homomorphism $t : M \rightarrow N$ such that $ts = \varphi$ where $s : s(M) \rightarrow M$ is the inclusion map. Hence $f = ts$ and therefore $f \in \text{Hom}_R(M,N)s$.

Conversely, let $\varphi : s(M) \rightarrow N$ be an $R$-homomorphism. Then $\varphi s \in \text{Hom}_R(M,N)$ and $\varphi s(\text{Ker}(s)) = 0$. By assumption, we have $\varphi s = us$ for some $u \in \text{Hom}_R(M,N)$. This shows that $N$ is $M$-principally injective. \qed

Example 2.3. Let $R = \left( \begin{array}{cc} F & 0 \\ 0 & F \end{array} \right)$ where $F$ is a field, $M_R = \left( \begin{array}{cc} F & 0 \\ 0 & F \end{array} \right)$ and $N_R = \left( \begin{array}{cc} 0 & 0 \\ 0 & F \end{array} \right)$. Then

(1) $N$ is not $M$-injective.

(2) $N$ is $M$-principally injective.

Proof. (1) Define $\varphi : \left( \begin{array}{cc} 0 & 0 \\ 0 & F \end{array} \right) \rightarrow \left( \begin{array}{cc} 0 & 0 \\ 0 & F \end{array} \right)$ with $\varphi(\left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right)) = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$. It is clear that $\varphi$ is an $R$-isomorphism. For any homomorphism $\alpha : \left( \begin{array}{cc} F & 0 \\ 0 & F \end{array} \right) \rightarrow \left( \begin{array}{cc} 0 & 0 \\ 0 & F \end{array} \right)$ with $\alpha(\left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)) = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$ for some $x \in F$, then $\alpha(\left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)) = \alpha(\left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)) = \left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$ for every $\left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right) \in \left( \begin{array}{cc} F & 0 \\ 0 & F \end{array} \right)$, so that $\alpha = 0$. Therefore $N$ is not $M$-injective. 

(2) It follows from (1) that $\left( \begin{array}{cc} 0 & 0 \\ 0 & 0 \end{array} \right)$ is not $M$-cyclic submodule of $M$. Hence only 0 and $M$ are $M$-cyclic submodules of $M$, thus $N$ is $M$-principally injective. \qed

Clearly, every $X$-cyclic submodule of $X$ is an $M$-cyclic submodule of $M$ for every $M$-cyclic submodule $X$ of $M$. Thus we have the following

Proposition 2.4. $N$ is $M$-principally injective if and only if $N$ is $X$-principally injective for every $M$-cyclic submodule $X$ of $M$. In particular, if $X$ is a direct summand of $M$ and $N$ is $M$-principally injective, then $N$ is both $X$-principally injective and $M/X$-principally injective.
**Proposition 2.5.** Let $M$ and $N$ be $R$-modules. Then $M$ is $N$-principally projective and every $N$-cyclic submodule of $N$ is $M$-principally injective if and only if $N$ is $M$-principally injective and every $M$-cyclic submodule of $M$ is $N$-principally projective.

**Proof.** ($\Rightarrow$) Let $f$ be an endomorphism of $N$, $X$ an $M$-cyclic submodule of $M$ and let $g : X \to f(N)$ be an $R$-homomorphism. Then $g$ is extended to an $R$-homomorphism $h : M \to f(N)$ so $h$ is lifted to an $R$-homomorphism $t : M \to N$. Thus $t$ is lifted $g$ where $\iota : X \to M$ is the inclusion map.

($\Leftarrow$) Let $\varphi : s(M) \to t(N)$ be an $R$-homomorphism where $s, t$ are endomorphisms of $M$ and $N$, respectively. Then $\varphi$ lifts to an $R$-homomorphism $\hat{\varphi} : s(M) \to N$ and so $\hat{\varphi}$ is extended to an $R$-homomorphism $\alpha : M \to N$, it is clear that $\alpha$ is an extension of $\varphi$. \qed

A ring $R$ is called a left (resp. right) **PP-ring** if each of its principal left (resp. right) ideal is projective. This is equivalent to the fact that, for each $a \in R$ there is an idempotent $e$ such that $t_R(a) = Re$ (resp. $r_R(a) = eR$).

**Theorem 2.6.** The following are equivalent for a projective module $M$:

1. Every $M$-cyclic submodule of $M$ is projective;
2. Every factor module of an $M$-principally injective module is $M$-principally injective;
3. Every factor module of an injective $R$-module is $M$-principally injective.

**Proof.** (1) $\Rightarrow$ (2) Let $N$ be an $M$-principally injective module, $X$ a submodule of $N$ and let $\varphi : s(M) \to N/X$ be an $R$-homomorphism. By (1), there exists an $R$-homomorphism $\hat{\varphi} : s(M) \to N$ such that $\varphi = \eta \hat{\varphi}$ where $\eta : N \to N/X$ is the natural epimorphism. Since $N$ is $M$-principally injective, there exists an $R$-homomorphism $t : M \to N$ which is an extension of $\hat{\varphi}$ to $M$. Then $\eta t$ is an extension of $\varphi$ to $M$.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Let $t(M)$ be an $M$-cyclic submodule of $M$, $h : A \to B$ an epimorphism and let $\alpha : t(M) \to B$ be $R$-homomorphism. Embed $A$ in an injective module $E$. Then $B \simeq A/\text{Ker}(h)$ is a submodule of $E/\text{Ker}(h)$; we may view $\alpha : t(M) \to E/\text{Ker}(h)$, which by hypothesis we can extend to $\hat{\alpha} : M \to E/\text{Ker}(h)$. Since $M$ is projective, $\alpha$ can be lifted to $g : M \to E$. It is clear that $g(t(M)) \subset A$. Therefore we have lifted $\alpha$. \qed

**Corollary 2.7.**[12, Exercises 39.17(4)] The following are equivalent for a ring $R$:

1. $R$ is a right PP-ring;
2. Every factor module of a principally injective module is principally injective;
3. Every factor module of an injective $R$-module is principally injective.

**Definition 2.8.** A right $R$-module $M$ is called **semi-injective** if it is $M$-principally injective.
In general, we have:

injective \implies quasi-injective \implies semi-injective \implies direct-injective.

Recall that an \( R \)-module \( M \) is said to be \textbf{direct-injective} if for any direct summand \( D \) of \( M \), every monomorphism \( f : D \to M \) splits. \textbf{Direct-projective} modules are defined dually. A submodule \( N \) of \( M \) is called a \textbf{fully invariant} submodule of \( M \) if \( s(N) \subseteq N \) for every \( s \in S \).

3. The Endomorphism Ring and its Jacobson Radical

Write

\[ \triangle = \{ s \in S : \ker(s) \subseteq M \}, \quad \hat{\triangle} = \{ s \in S : \ker(1 + ts) = 0 \text{ for all } t \in S \}. \]

It is known that \( \triangle \) is an ideal of \( S \) [5, Lemma 3.2]. Since \( \ker(s) \cap \ker(1 + ts) = 0 \), \( \triangle \subseteq \hat{\triangle} \). It is well-known that, for a quasi-continuous module \( M \), \( M \) is continuous if and only if \( S/\triangle \) is regular and \( J(S) = \triangle \) [5, Proposition 3.15]. We now investigate when \( J(S) = \triangle \).

Following [12], an \( R \)-module \( M \) is called \textbf{\( \pi \)-injective} if, for all submodules \( U \) and \( V \) of \( M \) with \( U \cap V = 0 \), there exists \( f \in S \) with \( U \subseteq \ker(f) \) and \( V \subseteq \ker(1 - f) \). A module \( M \) is called a \textbf{self-generator} if it generates all its submodules.

**Proposition 3.1.** Let \( M \) be semi-injective.

1. \( J(S) = \hat{\triangle} \).
2. If \( S \) is local, then \( J(S) = \{ s \in S : \ker(s) \neq 0 \} \).
3. If \( S/\triangle \) is regular, then \( J(S) = \triangle \).
4. If \( S/J(S) \) is regular, then \( S/\triangle \) is regular if and only if \( J(S) = \triangle \).
5. If \( \text{Im} s \subseteq M \) where \( s \in S \), then any monomorphism \( t : s(M) \to M \) can be extended to a monomorphism in \( S \).
6. If \( M \) is uniform, then \( S \) is a local ring and \( J(S) = \triangle \).
7. For \( s \in S \), if \( M \) is uniform and \( s \) is left invertible, then \( s \) is invertible.
8. \( M \) is uniform if and only if \( S \) is local and \( M \) is \( \pi \)-injective.
9. If \( M \) is uniform, then \( Z(S_S) \subseteq J(S) \).

**Proof.** (1) For any \( s \in J(S) \) and \( t \in S \), \( g(1 + ts) = 1_M \) for some \( g \in S \). Thus \( \ker(1 + ts) = 0 \), and hence \( J(S) \subseteq \hat{\triangle} \). On the other hand, if \( \ker(1 + s) = 0 \), then \( \ell_S(\ker(1 + ts)) = S \). By Lemma 2.2, we have \( S = S(1 + ts) \) which implies \( 1_M = g(1 + ts) \) for some \( g \in S \). It follows that \( s \in J(S) \).

(2) Since \( S \) is local, \( Ss \neq S \) for any \( s \in J(S) \). If \( \ker(s) = 0 \), then \( \alpha : s(M) \to M \) given by \( \alpha(s(m)) = m \) for any \( m \in M \) is an \( R \)-homomorphism. Since \( M \) is semi-injective, let \( \beta \in S \) be an extension of \( \alpha \) to \( M \). It follows that \( \beta s = 1_M \) so \( Ss = S \), which is a contradiction. This shows that \( J(S) \subseteq \{ s \in S : \ker(s) \neq 0 \} \). The other inclusion is clear.

(3) Clearly, \( \triangle \subseteq J(S) \). If \( s \in J(S) \), then \( (1 - sa)s = s - sos \in \triangle \) for some \( \alpha \in S \). Since \( (1 - sa) \) has a left inverse, \( s \in \triangle \). This shows that \( J(S) \subseteq \triangle \).

(4) This follows from (3).
(5) Since $M$ is semi-injective, there exists $g \in S$ such that $gs = ts$. Thus $\text{Im}(s) \cap \text{Ker}(g) = 0$. Since $\text{Im}(s) \subseteq M$, $\text{Ker}(g) = 0$.

(6) Since $M$ is direct-injective, $S$ is local provided that $M$ is uniform [12, 41.22]. It follows that $J(S) = \triangle$ by (2).

(7) Since $s$ has a left inverse, $\text{Ker}(s) = 0$. Follows from (6) and (2), we have $s \notin J(S)$ hence $s$ is invertible.

(8) The necessity is trivial. For the sufficiency, let $U$ and $V$ be submodules of $M$ such that $U \cap V = 0$. As $M$ is $\pi$-injective, we can choose $f \in S$ so that $U \subseteq \text{Ker}(f)$ and $V \subseteq \text{Ker}(1 - f)$. Note that either $f$ or $1 - f$ belong to $J(S)$. If $f \in J(S)$, then $g(1 - f) = 1$ for some $g \in S$. Thus $\text{Ker}(1 - f) = 0$, and it follows that $V = 0$. Otherwise, $U = 0$.

(9) Let $s \in Z(S)$. Then $\text{Ker}(s) \neq 0$. For any $t \in S$ we have $\text{Ker}(s) \cap \text{Ker}(1 + ts) = 0$, then $\text{Ker}(1 + ts) = 0$. Hence $s \in J(S)$ by (1).

\begin{proof}
\end{proof}

Proposition 3.2. Suppose $M$ is a semi-injective and $\pi$-injective module. If $S$ is semiperfect, then $M = \bigoplus_{i=1}^{n} U_i$, where $U_i$ is uniform and semi-injective for each $i$.

Proof. Since $S$ is semiperfect and $M$ is semi-injective, $M = U_1 \oplus \ldots \oplus U_n$, where each $\text{End}_R(U_i)$ is local. Note that $U_i$ is semi-injective. Each $U_i$ is $\pi$-injective [12, 41.20], thus by Proposition 3.1(8) we see that $U_i$ is uniform.

The following proposition is modified from [1, Lemma 18.21]

Proposition 3.3. If $\text{Soc}(M) \subseteq M$, then

1. $\triangle = \ell_S(\text{Soc}(M))$, and
2. $S/\triangle$ is embedded in $\text{End}_R(\text{Soc}(M))$ as a subring.

Proof. (1) Let $s \in \triangle$. Then $\text{soc}(M) \subseteq \text{Ker}(s)$, it follows that $s(\text{Soc}(M)) = 0$. If, on the other hand, $s(\text{Soc}(M)) = 0$, then $\text{Ker}(s) \subseteq M$ and $s \in \triangle$.

(2) For each $s \in S$, let $\theta(s)$ be a map from $\text{Soc}(M)$ into itself defined by $(\theta(s))(x) = s(x)$. Since $\text{Soc}(M)$ is fully invariant in $M$, it follows that $\theta(s) \in \text{End}_R(\text{Soc}(M))$ and $\theta : S \rightarrow \text{End}_R(\text{Soc}(M))$ is a ring homomorphism. Clearly, $\text{Ker}(\theta) = \triangle$ and the proof is complete.

\begin{proof}
\end{proof}

Corollary 3.4. If $M$ is semi-injective and a self-generator and if $\text{Soc}(M) \subseteq M$, then

1. $J(S) = \ell_S(\text{Soc}(M))$, and
2. $S/J(S) \simeq \text{End}_R(\text{Soc}(M))$.

Proof. (1) As $M$ is semi-injective and a self-generator, we have $J(S) = \triangle$ by [9, Theorem 2.13].

(2) Since $M$ is semi-injective, every $R$-homomorphism in $\text{End}_R(\text{Soc}(M))$ can be extended to an $R$-homomorphism in $S$. By (1) and Proposition 3.3(2), it follows that $S/J(S)$ is isomorphic to $\text{End}_R(\text{Soc}(M))$ as rings.

\begin{proof}
\end{proof}

Proposition 3.5. Let $M$ be a semi-injective module.

1. If $\text{Im}(s)$ is a simple right $R$-module, $s \in S$, then $Ss$ is a simple left $S$-module.
(2) If \( s_1(M) \oplus \cdots \oplus s_n(M) \) is direct, \( s_1, \ldots, s_n \in S \), then \( S(s_1 + \cdots + s_n) = Ss_1 + \cdots + Ss_n \).

Proof. (1) Let \( A \) be a nonzero submodule of \( Ss \) and \( 0 \neq \alpha s \in A \). Then \( SsA \subset A \). Since \( \text{Im}(s) \) is simple, \( \ker(\alpha) \cap \text{Im}(s) = 0 \). Define \( g : \alpha s(M) \to M \) by \( g(\alpha s(m)) = s(m) \) for every \( m \in M \). It is obvious that \( g \) is an \( R \)-homomorphism. Since \( M \) is semi-injective, there exists a homomorphism \( h \in S \) such that \( h(\alpha s) = g(\alpha s) \). Therefore \( h(\alpha s) = s \) so \( s \in SsA \). It follows that \( SsA = Sa \) and hence \( A = Sa \).

(2) Let \( s_1 s_1 + \cdots + s_n \in Ss_1 + \cdots + Ss_n \). For each \( i \), define \( \varphi_i : (s_1 + \cdots + s_n)(M) \to M \) by \( \varphi_i((s_1 + \cdots + s_n)(m)) = s_i(m) \) for every \( m \in M \). Since \( s_1(M) \oplus \cdots \oplus s_n(M) \) is direct, \( \varphi_i \) is well-defined, so it is clear that \( \varphi_i \) is an \( R \)-homomorphism. Then there exists an \( R \)-homomorphism \( \hat{\varphi}_i \in S \) which is an extension of \( \varphi_i \). Then \( s_i = \varphi_i(s_1 + \cdots + s_n) = \hat{\varphi}_i(s_1 + \cdots + s_n) \in S(s_1 + \cdots + s_n) \) for every \( i = 1, \ldots, n \). Consequently, \( s_1 s_1 + \cdots + s_n s_n \in S(s_1 + \cdots + s_n) \). Hence \( Ss_1 + \cdots + Ss_n \subset S(s_1 + \cdots + s_n) \). The other inclusion always holds.

We call a module \( M \) a duo module if every submodule of \( M \) is fully invariant. \( M \) is said \([11]\) to have the summand intersection property (SIP) if the intersection of two direct summands is again a direct summand. The module \( M \) is said \([4]\) to have the summand sum property (SSP) if the sum of any two summands of \( M \) is again a summand.

We prove a similar result here for a semi-injective module \( M \), with the (SIP) and (SSP). Note that every direct summand of \( M \) is of the form \( s(M) \) for some \( s \in S \).

**Proposition 3.6.** Every duo and semi-injective module has the (SIP) and (SSP).

Proof. Write \( M = s(M) \oplus K \) and \( M = t(M) \oplus L \). Since \( M \) is duo, \( s(M) = s(t(M) \oplus L) = s(t(M) + s(L) \subset (s(M) \cap t(M)) \cup (s(M) \cap L) = s(M) \cap t(M) \oplus (s(M) \cap L) \subset s(M) \). Then \( s(M) \cap t(M) \subset M \). Now we write \( M = s(M) \oplus t(M) \oplus N \). Then \( t(M) = t(M) \cap (s(M) \cap t(M) \oplus N) = s(M) \cap t(M) \oplus t(M) \cap N \) by the Modular law. So \( s(M) + t(M) = s(M) + (s(M) \cap t(M) \oplus t(M) \cap N) = (s(M) + t(M) \cap N) = (s(M) + t(M)) \cap N \). Since \( s(M) \) and \( t(M) \cap N \) are direct summands, \( s(M) + t(M) \) is a direct summand of \( M \) by \((C_3)\). \( \square \)

Following \([6]\) a ring \( R \) is called semiregular if \( R/J(R) \) is regular and idempotents can be lifted modulo \( J(R) \). Equivalently, \( R \) is semiregular if and only if for each element \( a \in R \), there exists \( e^2 = e \in Ra \) such that \( a(1 - e) \in J(R) \).

**Theorem 3.7.** For a semi-injective module \( M \), if \( S \) is semiregular, then \((\ast)\) holds, where \((\ast)\) is the condition

\((\ast)\): For every \( s \in S \setminus J(S) \), there exists a nonzero idempotent \( \alpha \in Ss \) such that \( \ker(s) \subset \ker(\alpha) \) and \( \ker(s(1 - \alpha)) \neq 0 \).

If, in addition, \( S \) is local, then the converse is true.
Proof. Let \( s \in S \setminus J(S) \). Then there exists \( \alpha^2 = \alpha \in Ss \) such that \( s(1 - \alpha) \in J(S) \). Then \( \alpha \neq 0 \) and \( \text{Ker}(s) \subset \text{Ker}(\alpha) \). If \( \text{Ker}(s(1 - \alpha)) = 0 \), then \( gs(1 - \alpha) = 1_M \) for some \( g \in S \) by the semi-injectivity of \( M \). It follows that \( \alpha = 0 \), a contradiction.

The converse follows from Proposition 3.1(2). \( \square \)

Acknowledgment. The author is grateful to Professor S. Dhompongsa for many helpful comments and suggestions. The author also wishes to thank an anonymous referee for his or her suggestions which led to substantial improvements of this paper.

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S. Wongwai, Department of Mathematics, Faculty of Science, Rajamangala Institute of Technology, Pathumthani 12110, Thailand, e-mail: wsrum@access.rit.ac.th