ANALYSIS OF A SEMIDISCRETE SCHEME FOR SOLVING IMAGE SMOOTHING EQUATION OF MEAN CURVATURE FLOW TYPE

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Abstract. Numerical approximation of a nonlinear diffusion equation of mean curvature flow type is discussed. Convergence and error analysis of a regularized problem is presented.

1. Introduction

In this paper we analyze a semidiscrete numerical method for solving nonlinear diffusion equation

\[ u_t = g(|\nabla u|)|\nabla u|\nabla \cdot \left( \frac{\nabla u}{|\nabla u|} \right) \]

in a domain \( \Omega \subset \mathbb{R}^N \) accompanied with homogeneous Neumann boundary conditions and an initial condition. Equation (1.1) is useful in image processing for selective smoothing of images and shapes. Numerical experiments in processing of 2D and 3D images are presented in [10]. Here, we present analysis of a special semidiscrete scheme for solving (1.1).

Equation (1.1) is a degenerate parabolic equation and is related to the so-called level set equation ((1.1) with \( g(s) \equiv 1 \)) which has been proposed by Osher & Sethian [16],[21] for computation of moving fronts in interfacial dynamics. The level set equation moves each level line (surface) of 2D (3D) image with the velocity proportional to its normal mean curvature field. This causes intrinsic smoothing of level sets. By means of the Perona-Malik function \( g \) (for which a typical choice is, e.g., \( g(s) = 1/(1 + s^2) \)) we control the motion of level sets which are also edges. The smoothing of silhouettes on which the gradient of intensity is large can be slowed down by using \( g \). In analysis and also in computations (see [10]) we use the following Evans-Spruck regularization,
\begin{equation}
\frac{1}{\sqrt{\varepsilon + |\nabla u|^2}} u_t - g(|\nabla u|) \nabla \left( \frac{\nabla u}{\sqrt{\varepsilon + |\nabla u|^2}} \right) = 0 \text{ in } I \times \Omega,
\end{equation}
\begin{equation}
\partial_{\nu} u = 0 \text{ on } I \times \partial \Omega,
\end{equation}
\begin{equation}
u(0,.) = u_0 \text{ in } \Omega,
\end{equation}
where \(1 > \varepsilon > 0\) is a (small) real number, fixed throughout the whole paper and constants in estimates can depend on it. \(I = (0,T)\) is a time-scale interval and \(\Omega \subset \mathbb{R}^N\). Using the ideas of Deckelnick and Dziuk \([5]\) and Frehse’s deformation technique \([8]\) we analyze (for \(N = 2\)) a finite element approximation of the problem (1.2)-(1.4). In \([5]\), the motion of two-dimensional nonparametric surface by its mean curvature, governed by the equation
\begin{equation}
\frac{1}{\sqrt{1 + |\nabla u|^2}} u_t - \nabla \left( \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \text{ in } I \times \Omega,
\end{equation}
is considered, provided \(u = 0\) on \(\partial \Omega\) and starting with smooth initial graph. We adapt their convergence and error estimates results to our situation - equation (1.2) with zero Neumann boundary conditions.

The semidiscrete scheme (Galerkin approximation) for solving (1.2)-(1.4) then reads as follows
\begin{equation}
\int_{\Omega} g(|\nabla u_h|) \sqrt{\varepsilon + |\nabla u_h|^2} + \int_{\Omega} \frac{\nabla u_h \cdot \nabla \phi_h}{\sqrt{\varepsilon + |\nabla u_h|^2}} = 0, \forall \phi_h \in X_h, t \in I,
\end{equation}
\begin{equation}
u_h(0,.) = \nu_{h0},
\end{equation}
where \(u_h(t,.) \in X_h\) is the approximation of \(u\), \(X_h\) is suitable finite element space with grid size parameter \(h\) (see (2.2)) and \(\nu_{h0}\) is a modification to our case of the so-called minimal surface projection of continuous initial data \(u_0\) (see (4.1)).

Our purpose is to prove the convergence of \(u_h\) to \(u\) in some functional spaces. After some notations and assumptions given in Section 2, we present the main results - existence and error estimates - in Section 3. Section 4 is devoted to proofs of theorems.

2. Notations and assumptions

We shall denote the usual norm in Sobolev space \(H^m(\Omega)\) by \(||.||_m\), the norm in \(H^{m,p}(\Omega)\) by \(||.||_{m,p}\) where \(m \geq 0, p \geq 1\); for \(m = 0\) we write \(||.||\) and \(||.||_{L_p}\) respectively. In our theoretical analysis we consider a bounded domain
\begin{equation}
\Omega \subset \mathbb{R}^2 \text{ with } \partial \Omega \in C^6.
\end{equation}
Let \(\tau_h\) be a partition of \(\Omega\) into generalized isoparametric triangles \(T\), i.e. \(T\) is a triangle if \(T\) and \(\partial \Omega\) have at most one point in common, otherwise one of the faces may be curved. The usual regularity condition is fulfilled \([4, \text{ Chapter 2.1}]\). We define the finite dimensional subspace \(X_h\) by
\begin{equation}X_h := \{ v_h \in C(\Omega)| v_h \text{ is linear on each } T \in \tau_h \} \end{equation}
where the isoparametric modification is used in curved elements ([22],[23]). Under these hypotheses, for functions \( v \in H^{k,p}(\Omega), 2 \leq p \leq \infty \), and the corresponding interpolants \( I_h v, I_h : H^{k,p}(\Omega) \to X_h \), the usual approximation and inverse properties hold (see [4, Theorems 3.2.6, 3.3.6]):

\[
|(v - I_h v)|_{j,p} \leq c h^{m-j} ||\nabla^m v||_{L^p}, 0 \leq j \leq 1, m = \min(2,k)
\]

and for \( v_h \in X_h \) we have

\[
||\nabla v_h||_{L^p} \leq c h^{-1} ||v_h||_{L^p}, 1 \leq p \leq \infty \\
||v_h||_{L^\infty} \leq c |\log h|^{1/2} ||v_h||_1.
\]

For the data of (1.2)-(1.4) we assume that

\[
g \in C^4(\mathbb{R}), g(0) = 1, 0 < g(s) \leq 1 \text{ (we admit } g(s) \to 0, \text{ for } s \to \infty),
\]

with bounded derivatives up to 4-th order.

\[
u_0(x) \in C^5(\overline{\Omega}) \text{ satisfying the compatibility conditions}
\]

\[
ed_{\alpha} u_0(x) \big|_{\partial \Omega} = 0, \text{ for } |\alpha| \leq 3.
\]

3. Main results

As we have mentioned above for proving the existence of a solution of the continuous problem in adequate function spaces and obtaining some error estimates for discrete solution we use the ideas and results of Deckelnick and Dziuk [5]. Let us state an existence and uniqueness of a solution result for problem (1.2) - (1.4).

**Theorem 3.1.** Let (2.1), (2.5) and (2.6) be satisfied. Then there exists a time \( T > 0 \) such that (1.2)-(1.4) has a unique solution \( u \in L^\infty(I; H^5(\Omega)) \cap L^2(I; H^6(\Omega)) \) with bounded derivatives up to 4-th order.

\[
u_t \in L^\infty(I; H^4(\Omega)) \cap L^2(I; H^5(\Omega)) \text{ and } u_{tt} \in L^\infty(I; H^1(\Omega)) \cap L^2(I; H^2(\Omega)).
\]

For the Galerkin approximation \( u_h \) given by (1.5)-(1.6) and its relation to the continuous solution \( u \) from Theorem 3.1 we have

**Theorem 3.2.** Let (2.1), (2.5) and (2.6) be satisfied. There exists \( h_0 > 0 \) such that problem (1.5)-(1.6) has a unique solution \( u_h \in L^\infty(I, L^2(\Omega)) \cap L^2(I, H^4(\Omega)) \) for all \( 0 < h \leq h_0 \). Furthermore, we have the following error estimates:

\[
\sup_{(0,T)} \|u - u_h\| \leq ch^2 |\log h|^2, \quad \left( \int_0^T ||\nabla(u - u_h)||^2 \right)^{1/2} \leq ch,
\]

\[
\sup_{(0,T)} \|u_t - u_{h,t}\| \leq ch |\log h|, \quad \left( \int_0^T ||\nabla(u_t - u_{h,t})||^2 \right)^{1/2} \leq ch |\log h|.
\]

These statements will be consequences of results obtained by deformation technique introduced by Frehse [8] which has been used also in [5]. We consider
the following family of initial-boundary value problems depending on a parameter \( \sigma \in [0, 1] \):

\[
\frac{u_t^\sigma}{(1-\sigma)^{1/2}} + \frac{u^\sigma}{(1-\sigma)^{1/2}} \cdot \nabla u^\sigma = 0 \quad \text{in} \quad I \times \Omega, \quad (P^\sigma)
\]

\[
\partial_t u^\sigma = 0 \quad \text{on} \quad I \times \partial \Omega,
\]

\[
u^\sigma(0,.) = u_0 \quad \text{in} \quad \Omega.
\]

The corresponding Galerkin approximation then reads as

\[
\int_\Omega (1-\sigma)^{1/2} \cdot u_h^\sigma \cdot \nabla \varphi_h + \int_\Omega \frac{\nabla u_h^\sigma \cdot \nabla \varphi_h}{\sqrt{\varepsilon + |\nabla u_h^\sigma|^2}} = 0, \quad \forall \varphi_h \in X_h, t \in I, \quad (P_h^\sigma)
\]

where \( \bar{u}_{h,0} \) is defined as in (4.1).

We can prove the existence result for the continuous problem (P^\sigma)

**Theorem 3.3.** Let (2.1), (2.5) and (2.6) be satisfied. Then there exists a unique solution \( u^\sigma \in L_\infty(I; H^5(\Omega)) \cap L_2(I; H^6(\Omega)) \) to problem (P^\sigma), provided that \( T > 0 \) is small enough.

In case \( \sigma = 1 \), (P^\sigma) is our original problem (1.2)-(1.4), so if we prove the Theorem 3.3, Theorem 3.1 is also proved. In case \( \sigma = 0 \), (P^\sigma) is deformed into

\[
u_t - \nabla \cdot \left( \frac{\nabla u}{\sqrt{\varepsilon + |\nabla u|^2}} \right) = 0 \quad \text{in} \quad I \times \Omega
\]

\[
\partial_t u = 0 \quad \text{on} \quad I \times \partial \Omega
\]

\[
u(0,.) = u_0 \quad \text{in} \quad \Omega.
\]

This equation is still nonlinear but its elliptic part is in the divergence form. Therefore we first investigate problem (3.1) and its Galerkin approximation \( u_h \) given by

\[
\int_\Omega u_{h,t} \varphi_h + \int_\Omega \frac{\nabla u_h \cdot \nabla \varphi_h}{\sqrt{\varepsilon + |\nabla u_h|^2}} = 0, \quad \forall \varphi_h \in X_h, t \in I,
\]

\[
u_h(0,.) = \bar{u}_{h,0}.
\]

We obtain the following result which itself gives the error estimates for the finite element approximation of widely used regularization of pure anisotropic diffusion introduced by Osher & Rudin [15].

**Theorem 3.4.** Let (2.1), (2.6) be satisfied. Let \( u \) be a solution to (3.1) and let \( u_h \) be a discrete solution given by (3.2). Then

\[
\sup_{(0,T)} ||\nabla u_h||_{L_\infty} \leq c,
\]

\[
\sup_{(0,T)} ||u - u_h|| \leq c h^2 \log h, \quad \left( \int_0^T ||\nabla (u_t - u_{h,t})||^2 \right) \leq c h^2 \log h^2.
\]
Finally, for $h \leq 1$ and $\gamma > 0, k_1 > 0$ we define a set $\Theta_h \subseteq [0, 1]$ by
\[
\Theta_h := \{ \sigma \in [0, 1] \mid (P^\sigma_h) \text{ has a solution } u^\sigma_h \text{ on } I \text{ and }
||\nabla u^\sigma_h||_{L^\infty} < 2\gamma, \int_0^T ||\nabla (u^\sigma_t - u^\sigma_{h,t})||^2 < k_1^2 h^2 |\log h|^2 \}
\]
where $\gamma$ is a uniform upper bound on $||\nabla u^\sigma||_{L^\infty}$ for $\sigma \in [0, 1]$. We prove the following result.

**Theorem 3.5.** For each $h \leq h_0$ (it may depend on the data of the problem and $k_1$) the set $\Theta_h$ is nonempty, open and closed with respect to $[0, 1]$ and therefore must coincide with $[0, 1]$.

Since $u^1 = u$, Theorem 3.1 is a direct consequence of Theorem 3.3. Theorem 3.5 together with the fact that $u^1_h = u_h$ will be used in the proof of Theorem 3.2.

4. Proofs of theorems

**Proposition 4.1.** For every $u \in L^\infty(I; H^2(\Omega)) \cap L^2(I; H^3(\Omega))$ with $u_t \in L^\infty(I; H^2(\Omega)) \cap L^2(I; H^3(\Omega))$, $u_{tt} \in L^\infty(I; H^1(\Omega)) \cap L^2(I; H^2(\Omega))$ and for all $0 \leq h \leq h_0$, $h_0$ sufficiently small, there exists a unique function $\bar{u}_h, \bar{u}_h(t, \cdot) \in X_h$ (for a.e. $t \in I$), such that for every $\varphi_h \in X_h$

\[
\int_\Omega \bar{u}_h \varphi_h + \int_\Omega \frac{\nabla \bar{u}_h \cdot \nabla \varphi_h}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} = \int_\Omega u \varphi_h + \int_\Omega \frac{\nabla u \cdot \nabla \varphi_h}{\sqrt{\varepsilon + |\nabla u|^2}}
\]

and the error between $u$ and $\bar{u}_h$ can be estimated as follows

\[
\sup_{(0, T)} ||u - \bar{u}_h|| + \sup_{(0, T)} ||\nabla (u - \bar{u}_h)|| \leq C h^2,
\]

\[
\sup_{(0, T)} ||u_t - \bar{u}_{h,t}||_{L^\infty} + \sup_{(0, T)} ||\nabla (u_t - \bar{u}_{h,t})||_{L^\infty} \leq C h^2 |\log h|,
\]

\[
\sup_{(0, T)} ||u_t - \bar{u}_{h,t}|| \leq C h^2 |\log h|^2, \quad \sup_{(0, T)} ||u_{tt} - \bar{u}_{h,tt}|| \leq C h,
\]

\[
(\int_0^T ||\nabla (u_{tt} - \bar{u}_{h,tt})||^2)^{1/2} \leq C h |\log h|,
\]

\[
(\int_0^T ||u_{tt} - \bar{u}_{h,tt}||^2)^{1/2} \leq C h |\log h|.
\]

**Remark:** The definition of so-called surface projection $\bar{u}_h$ is different as in [5] due to Neumann boundary condition (see also [20]).

**Proof.** From equation (4.1) we immediately have

\[
\int_\Omega (u - \bar{u}_h) \varphi_h + \int_\Omega \frac{\nabla (u - \bar{u}_h) \cdot \nabla \varphi_h}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} = \int_\Omega \left( \frac{1}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} - \frac{1}{\sqrt{\varepsilon + |\nabla u|^2}} \right) \nabla u \cdot \nabla \varphi_h.
\]
We take \( \varphi_h = I_h u - \bar{u}_h \in X_h \) and after some rearrangement we obtain
\[
\int_{\Omega} |u - \bar{u}_h|^2 + \int_{\Omega} \frac{1}{\sqrt{\varepsilon + |\nabla u_h|^2}} |\nabla (u - \bar{u}_h)|^2 \\
= \int_{\Omega} \left( \frac{1}{\sqrt{\varepsilon + |\nabla u_h|^2}} - \frac{1}{\sqrt{\varepsilon + |\nabla u|^2}} \right) \nabla u \cdot \nabla (I_h u - \bar{u}_h) \\
+ \int_{\Omega} (u - \bar{u}_h)(u - I_h u) + \int_{\Omega} \frac{\nabla (u - \bar{u}_h) \cdot \nabla (u - I_h u)}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} = I_1 + I_2 + I_3.
\]

We estimate
\[
|I_1| \leq \int_{\Omega} \frac{\nabla u \cdot \nabla (I_h u - \bar{u}_h) \cdot \nabla (u - \bar{u}_h)}{\sqrt{\varepsilon + |\nabla u|^2 \sqrt{\varepsilon + |\nabla u_h|^2} + \sqrt{\varepsilon + |\nabla \bar{u}_h|^2}}} \\
\leq \gamma \int_{\Omega} \frac{\nabla (I_h u - u) \cdot \nabla (u - \bar{u}_h)}{\sqrt{\varepsilon + |\nabla u_h|^2}} + \gamma \int_{\Omega} \frac{\nabla (u - \bar{u}_h)^2}{\sqrt{\varepsilon + |\nabla u_h|^2}} \\
\leq \gamma (1 + \delta_1) \int_{\Omega} \frac{\nabla (u - \bar{u}_h)^2}{\sqrt{\varepsilon + |\nabla u_h|^2}} + C C_{\delta_1} \|\nabla (u - I_h u)\|^2,
\]
\[
|I_2| \leq \delta_2 \|u - \bar{u}_h\|^2 + C_{\delta_2} \|u - I_h u\|^2,
\]
\[
|I_3| \leq \delta_3 \int_{\Omega} \frac{\nabla (u - \bar{u}_h)^2}{\sqrt{\varepsilon + |\nabla u_h|^2}} + C_{\delta_3} \int_{\Omega} \frac{\nabla (u - I_h u)^2}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} \\
\leq \delta_3 \int_{\Omega} \frac{\nabla (u - \bar{u}_h)^2}{\sqrt{\varepsilon + |\nabla u_h|^2}} + C C_{\delta_3} \|\nabla (u - I_h u)\|^2
\]
where \( \gamma = \max_{\Omega} \frac{\nabla u}{\sqrt{\varepsilon + |\nabla u|^2}} < 1 \). Then, for \( \delta_i, i = 1, 2, 3 \) sufficiently small, we obtain
\[
\|u - \bar{u}_h\|^2 + \int_{\Omega} \frac{\nabla (u - \bar{u}_h)^2}{\sqrt{\varepsilon + |\nabla u_h|^2}} \leq C \|u - I_h u\|^2.
\]

Using (2.3) and the regularity of \( u \) we have
\[
\|u - \bar{u}_h\|^2 + \int_{\Omega} \frac{\nabla (u - \bar{u}_h)^2}{\sqrt{\varepsilon + |\nabla u_h|^2}} \leq C h^2 \|u\|^2 \leq Ch^2.
\]

Now, one can obtain (see also [11]) that
\[
\|\nabla \bar{u}_h\|_{L_\infty} \leq C.
\]
and so we derive the estimate for \( \|\nabla (u - \bar{u}_h)\| \) in (4.2).

The estimate for \( \|u - \bar{u}_h\| \) in (4.2) and estimates in (4.3) can be proved in similar way as in [18, Theorem 1] and it’s mentioned modification, (see also [9]) with respect to the definition of \( \bar{u}_h \) see also [20, Theorem 1], for linear case with Neumann boundary condition. The proof is rather technical so we omit them here.

Next we will use the abbreviation
\[
F(p) = \frac{p}{\sqrt{\varepsilon + |p|^2}} \quad (p \in \mathbb{R}^2).
\]
Let us differentiate (4.1) with respect to $t$ and get
\begin{equation}
(4.9) \quad \int_{\Omega} (u - \bar{u}_h) v_h + \int_{\Omega} F'(\nabla u) \nabla u \cdot \nabla v - \int_{\Omega} F'(\nabla \bar{u}_h) \nabla \bar{u}_h \cdot \nabla v_h = 0.
\end{equation}
We take $\varphi_h = I_h u_t - \bar{u}_{h,t}$ and using the properties of $F$ and $u$ we successively obtain
\begin{align*}
||u_t - \bar{u}_{h,t}||^2 + & \int_{\Omega} F'(\nabla \bar{u}_h) |\nabla (u_t - \bar{u}_{h,t})|^2 \\
= & \int_{\Omega} (u_t - \bar{u}_{h,t})(u_t - I_h u_t) + \int_{\Omega} F'(\nabla \bar{u}_h) \nabla (u_t - \bar{u}_{h,t}) \cdot \nabla (u_t - I_h u_t) \\
+ & \int_{\Omega} (F'(\nabla \bar{u}_h) - F'(\nabla u)) \nabla u_t \cdot \nabla (I_h u_t - \bar{u}_{h,t}) \\
\leq & \delta_1 ||u_t - \bar{u}_{h,t}||^2 + C_{\delta_1} ||u_t - I_h u_t||^2 + C_1 \int_{\Omega} |\nabla (u_t - \bar{u}_{h,t})| |\nabla (u_t - I_h u_t)| \\
+ & C_2 ||\nabla u_t||_{L^\infty} \int_{\Omega} |\nabla (u - \bar{u}_h)|| \nabla (I_h u_t - \bar{u}_{h,t})|
\end{align*}
Finally, using the properties of $u$ and strict positivity of $F'$, then for $\delta_i$, $i = 1, 2, 3$, sufficiently small, we obtain
\begin{align*}
||u_t - \bar{u}_{h,t}||^2 + ||\nabla (u_t - \bar{u}_{h,t})||^2 & \leq C(||u_t - I_h u_t||^2 + ||\nabla (u - \bar{u}_h)||^2),
\end{align*}
and using (2.3), (4.2) and the properties of $u$ we derive
\begin{align*}
||u_t - \bar{u}_{h,t}||^2 + ||\nabla (u_t - \bar{u}_{h,t})||^2 & \leq C_1 h^2 ||u_t||^2 + C h^2
\end{align*}
uniformly for $t$ and the estimate for $||\nabla (u_t - \bar{u}_{h,t})||$ in (4.4) is completed. The rest of (4.4) can be proved in the similar way as in [5]. Let $v$ be the solution of the linear equation
\begin{equation*}
v - \nabla (F'(\nabla u) \nabla v) = u_t - \bar{u}_{h,t} \text{ in } \Omega
\end{equation*}
with zero Neumann boundary condition. We have
\begin{align*}
||u_t - \bar{u}_{h,t}||^2 & = (v, u_t - \bar{u}_{h,t}) + (F'(\nabla u) \nabla v, \nabla (u_t - \bar{u}_{h,t}).
\end{align*}
Using (4.9), (2.3) and the well know estimates of $v$ (see [12]) we derive
\begin{align*}
||u_t - \bar{u}_{h,t}||^2 & \leq c h ||u_t - \bar{u}_{h,t}||^2 + \int_{\Omega} F'(\nabla u) \nabla (u_t - \bar{u}_{h,t}) \cdot \nabla (v - I_h v) \\
& + \int_{\Omega} (F'(\nabla \bar{u}_h) - F'(\nabla u)) \nabla \bar{u}_{h,t} \cdot \nabla I_h v
\end{align*}
and after some rearrangement, for $h \leq h_0$, $h_0$ sufficiently small, we obtain practically in the same way as in [5] with respect to zero Neumann boundary condition and the estimates for $v$ [12, Chapter 3]:
\begin{align*}
||u_t - \bar{u}_{h,t}||^2 & \leq C h^2 ||\nabla v|| + c h^2 ||\nabla u_t|| ||\nabla v||
\end{align*}
Integrating this inequality and using the boundedness of $|||\cdot|||$ obtain estimates (4.5), (4.6).

Using (4.3), (2.3), the properties of $u$ due to the properties of $\phi$ and $\bar{u}_h$, we get

Now we shall treat the second derivative with respect to $t$. After differentiation of (4.9) we obtain

$$
\int_{\Omega} (u - \bar{u}_h)\nabla \varphi_h + \int_{\Omega} (F'(\nabla u)\nabla u_{tt} - F'(\nabla \bar{u}_h)\nabla \bar{u}_{h,tt}) \nabla \varphi_h
$$

(4.10)

Inserting $\varphi_h = \bar{u}_{h,tt}$ into (4.10), in a similar way like above we get

$$
\int_{0}^{T} ||\bar{u}_{h,tt}||^2 + \int_{0}^{T} ||\nabla \bar{u}_{h,tt}||^2
$$

\leq C \int_{0}^{T} (||u_{tt}||^2 + ||\nabla u_{tt}||^2 + ||\nabla \bar{u}_h||^2 L_\infty ||\nabla \bar{u}_{h,t}||^2 L_\infty + ||\nabla u||^2 L_\infty ||\nabla u_{tt}||^2 L_\infty ) \leq C
$$
due to the properties of $u$ and $\bar{u}_h$ and using (4.7) and (4.4).

Now we put $\varphi_h = I_h u_{tt} - \bar{u}_{h,tt}$ in (4.10). We get

$$
||u_{tt} - \bar{u}_{h,tt}||^2 + ||\nabla (u_{tt} - \bar{u}_{h,tt})||^2 \leq C \int_{\Omega} (u_{tt} - \bar{u}_{h,tt})(I_h u_{tt} - u_{tt})
$$

Using (4.3), (2.3), the properties of $u$ and $\bar{u}_h$ we get

$$
||u_{tt} - \bar{u}_{h,tt}||^2 + ||\nabla u_{tt} - \bar{u}_{h,tt}||^2 \leq ch^2 ||u_{tt}||^2 + ch^2 \log h \||\nabla \bar{u}_{h,tt}||||u_{tt}||^2
$$

Integrating this inequality and using the boundedness of $||u_{tt}||^2$ and $||\nabla \bar{u}_{h,tt}||$ we obtain estimates (4.5), (4.6).}

\textbf{Proof. (Proof of Theorem 3.4.)} From (3.1) and the definition of $\bar{u}_h$ we have

$$
\int_{\Omega} u_{t} \varphi_h + \int_{\Omega} \frac{\nabla \bar{u}_h \nabla \varphi_h}{\sqrt{\varepsilon + ||\nabla \bar{u}_h||^2}} = \int_{\Omega} (u - \bar{u}_h) \varphi_h, \quad \varphi_h \in X_h, \ t \in I.
$$

Taking the difference of (4.11) and (3.2) we obtain

$$
\int_{\Omega} (\bar{u}_{h,t} - u_{h,t}) \varphi_h + \int_{\Omega} \frac{\nabla (\bar{u}_h - u_h) \nabla \varphi_h}{\sqrt{\varepsilon + ||\nabla \bar{u}_h||^2}}
$$
\[
\int_\Omega (\bar{u}_h,t - u_t) \varphi + \int_\Omega \left( \frac{1}{\sqrt{\varepsilon + |\nabla u_h|^2}} - \frac{1}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} \right) \nabla \bar{u}_h \nabla \varphi + \int_\Omega (u - \bar{u}_h) \varphi
\]

Now, we choose \( \varphi_h = \bar{u}_h - u_h \) to obtain

\[
\frac{1}{2} \frac{d}{dt} ||\bar{u}_h - u_h||^2 + \int_\Omega \left( \frac{1}{\sqrt{\varepsilon + |\nabla u_h|^2}} - \frac{1}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} \right) ||\nabla (\bar{u}_h - u_h)|| \nabla \bar{u}_h ||\nabla \varphi_h\
\leq ||\bar{u}_h,t - u_t|| ||\bar{u}_h - u_h|| + ||u - \bar{u}_h|| ||\bar{u}_h - u_h||
\]

\[
+ \int_\Omega \frac{1}{\sqrt{\varepsilon + |\nabla u_h|^2}} - \frac{1}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} ||\nabla (\bar{u}_h - u_h)|| \nabla \bar{u}_h ||\nabla \varphi_h\
\leq ||\bar{u}_h - u_h||^2 + ||u_t - \bar{u}_h,t||^2 + ||u - \bar{u}_h||^2 + \alpha \int_\Omega \frac{|\nabla (\bar{u}_h - u_h)|^2}{\sqrt{\varepsilon + |\nabla u_h|^2}}\
\]

with \( \alpha < 1 \), where (4.7) has been used. Using Gronwall’s lemma and results of Proposition 4.1 we obtain

\[
\sup_{(0,T)} ||(\bar{u}_h - u_h)||^2 + \int_0^T \int_\Omega \frac{|\nabla (\bar{u}_h - u_h)|^2}{\sqrt{\varepsilon + |\nabla u_h|^2}} \leq C h^4 |\log h|^4,
\]

which together with (4.2) implies the second inequality of Theorem. We can also conclude

\[
||\nabla (\bar{u}_h - u_h)||_{L^\infty} \leq C_1 h^{-1} ||\nabla (\bar{u}_h - u_h)|| \leq C_2 h^{-2} ||(\bar{u}_h - u_h)|| \leq C |\log h|^2
\]

uniformly for a.e. \( t \in [0, T] \) and therefore

(4.12) \[
\sup_{(0,T)} ||\nabla u_h||_{L^\infty} \leq C |\log h|^2.
\]

Now differentiating (4.11) and (3.2) with respect to \( t \) and taking the difference of the resulting equations we obtain

\[
\int_\Omega (\bar{u}_{h,t} - u_{t,t}) \varphi + \int_\Omega \frac{\nabla (\bar{u}_{h,t} - u_{t,t}) \nabla \varphi_h}{\sqrt{\varepsilon + |\nabla u_h|^2}}
\]

\[
= \int_\Omega (\bar{u}_{h,t} - u_{t,t}) \varphi + \left( \frac{1}{\sqrt{\varepsilon + |\nabla u_h|^2}} - \frac{1}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} \right) \nabla \bar{u}_h \nabla \varphi_h
\]

\[
+ \int_\Omega \frac{\nabla u_h \nabla \varphi_h}{\sqrt{\varepsilon + |\nabla u_h|^2}} \nabla \bar{u}_h \nabla (\bar{u}_{h,t} - u_{t,t})
\]

\[
+ \int_\Omega \left( \frac{\nabla \bar{u}_h \nabla \varphi_h}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} \right)^2 \bar{u}_h - \frac{\nabla u_h \nabla \varphi_h}{\sqrt{\varepsilon + |\nabla u_h|^2}} \nabla u_h \right) \nabla \bar{u}_{h,t}
\]

\[
+ \int_\Omega (u_t - \bar{u}_{h,t}) \nabla \varphi_h.
\]
We take \( \varphi_h = \bar{u}_{h,t} - u_{h,t} \) and similarly as above and as in [5] we get
\[
\frac{1}{2} \frac{d}{dt} ||\bar{u}_{h,t} - u_{h,t}||^2 + \frac{\varepsilon}{2(\varepsilon + \sup_{(0,T)} ||\nabla u_h||_{L^\infty})} \int_{\Omega} |\nabla (\bar{u}_{h,t} - u_{h,t})|^2 \sqrt{\varepsilon + ||\nabla u_h||^2} \\
\leq \frac{1}{2} ||u_{tt} - \bar{u}_{h,tt}||^2 + ||\bar{u}_{h,t} - u_{h,t}||^2 + \frac{1}{2} ||u_t - \bar{u}_{h,t}||^2 \\
+ C(\varepsilon + \sup_{(0,T)} ||\nabla u_h||_{L^\infty})^2 ||\nabla \bar{u}_{h,t}||_{L^\infty}.
\]
Integrating it with respect to \( t \), estimating \( ||(\bar{u}_{h,t} - u_{h,t})(0)|| \) as in [5] and using (4.4)-(4.5) we obtain
\[
||\bar{u}_{h,t} - u_{h,t}||^2 + \frac{\varepsilon}{2(\varepsilon + \sup_{(0,T)} ||\nabla u_h||_{L^\infty})} \int_0^t \int_{\Omega} |\nabla (\bar{u}_{h,t} - u_{h,t})|^2 \sqrt{\varepsilon + ||\nabla u_h||^2} \\
\leq Ch^2 |\log h|^8 + \int_0^t ||\bar{u}_{h,t} - u_{h,t}||^2.
\]
If we apply Gronwall’s lemma, we have
\[
\sup_{(0,T)} ||\bar{u}_{h,t} - u_{h,t}||^2 \leq ch^2 |\log h|^8,
\]
and using (4.12) we get
\[
\int_0^T \int_{\Omega} |\nabla (\bar{u}_{h,t} - u_{h,t})|^2 \sqrt{\varepsilon + ||\nabla u_h||^2} \leq Ch^2 |\log h|^{12},
\]
from which we have
\[
||\nabla (\bar{u}_h - u_h(t))||^2 \leq Ch^3 |\log h|^{10}
\]
and
\[
||\nabla u_h(t)||_{L^\infty} \leq C + Ch^{-1} ||\nabla (\bar{u}_h - u_h(t))|| \leq C.
\]
Now, using this result in similar way as in [5] we can obtain
\[
\int_0^T ||\nabla (u_t - u_{h,t})||^2 \leq Ch^2 |\log h|^2.
\]
Proposition 4.1 gives the estimates for \( u^\sigma - \bar{u}_h^\sigma \), the next assertion will gives us some useful relations between \( u^\sigma - u_h^\sigma \) and \( \bar{u}_h^\sigma - u_h^\sigma \) which we will use in the proof of Theorem 3.5. \( \square \)
Proposition 4.2. Let \( u_h^\sigma \) be a solution of \((P^\sigma_h)\) satisfying \( \|\nabla u_h^\sigma\|_{L_\infty} \leq 2\gamma \). Denote \( e^\sigma = u^\sigma - u_h^\sigma \) and \( e_h^\sigma = \bar{u}^\sigma_h - u_h^\sigma \). Then

\[
\begin{align*}
(4.13) \quad & \sup_{(0,T)} \|e_h^\sigma\|^2 \leq c_1 h^4 |\log h|^4 \exp(c_1 \int_0^T \|\nabla e_h^\sigma\|^2), \\
(4.14) \quad & \int_0^T \|\nabla e_h^\sigma\|^2 \leq c_1 h^4 |\log h|^4 \left( 1 + \exp(c_1 \int_0^T \|\nabla e_h^\sigma\|^2) \right), \\
(4.15) \quad & \sup_{(0,T)} \|e_h^\sigma\|^2 \leq c_2 (h^2 |\log h|^2 + \sup_{(0,T)} \|\nabla e^\sigma\|^2 + (h^2 |\log h|^2) \right)
+ \sup_{(0,T)} \|\nabla e^\sigma\|^2 \int_0^T \|\nabla e_h^\sigma\|^2 \exp(c_2 \int_0^T \|\nabla e_h^\sigma\|^2), \\
(4.16) \quad & \int_0^T \|\nabla e_h^\sigma\|^2 \leq c_2 \left( h^2 |\log h|^2 + \sup_{(0,T)} \|\nabla e^\sigma\|^2 \right) \int_0^T \|\nabla e_h^\sigma\|^2 \\
+ h^2 |\log h|^2 + \sup_{(0,T)} \|\nabla e^\sigma\|^2 \left( 1 + \exp(c_2 \int_0^T \|\nabla e_h^\sigma\|^2) \int_0^T \|\nabla e_h^\sigma\|^2 \right).
\end{align*}
\]

Proof. The proof is similar to the one in [5]. In order to simplify the presentation we only look at the case \( \sigma = 1 \) and we omit this upper index. We can write

\[ e = u - u_h = (u - \bar{u}_h) + (\bar{u}_h - u_h) =: \bar{e} + e_h. \]

Now, from definition of \((P^\sigma)\), for \( \sigma = 1 \), we have

\[ \int_{\Omega} \frac{u_t \varphi_h}{\sqrt{\varepsilon + \|\nabla u\|^2 g(\|\nabla u\|)}} + \int_{\Omega} \frac{\nabla u \nabla \varphi_h}{\sqrt{\varepsilon + |\nabla u|^2}} = 0, \quad \forall \varphi_h \in X_h, t \in I. \]

By definition of \( \bar{u}_h \) we get

\[
(4.17) \quad \int_{\Omega} \frac{u_t \varphi_h}{\sqrt{\varepsilon + |\nabla u|^2 g(\|\nabla u\|)}} + \int_{\Omega} \frac{\nabla \bar{u}_h \nabla \varphi_h}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} = \int_{\Omega} (u - \bar{u}_h) \varphi_h, \quad \forall \varphi_h \in X_h, t \in I.
\]

Taking the difference of (4.17) and \((P^1_h)\) we obtain

\[
(4.18) \quad \int_{\Omega} \frac{\bar{e}, t \varphi_h}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2 g(\|\nabla \bar{u}_h\|)}} + \int_{\Omega} \left( \frac{\nabla \bar{u}_h}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} - \frac{\nabla u_h}{\sqrt{\varepsilon + |\nabla u_h|^2}} \right) \nabla \varphi_h
= \int_{\Omega} (u - \bar{u}_h) \varphi_h - \int_{\Omega} u_t \varphi_h \left( \frac{1}{\sqrt{\varepsilon + |\nabla u|^2 g(\|\nabla u\|)}} - \frac{1}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2 g(\|\nabla \bar{u}_h\|)}} \right)
- \int_{\Omega} \frac{\bar{e}, t \varphi_h}{\sqrt{\varepsilon + |\nabla u_h|^2 g(\|\nabla u_h\|)}}.
\]
We use the function $F$ defined in the proof of Proposition 4.1. We also define $G: \mathbb{R}^2 \to \mathbb{R}$ by

$$G(p) = \frac{1}{\sqrt{\varepsilon + |p|^2} g(|p|)}.$$  

In the same way as in [5], using the mean value theorem, we have

$$\frac{\nabla \bar{u}_h}{\sqrt{\varepsilon + |\nabla \bar{u}_h|^2}} - \frac{\nabla u_h}{\sqrt{\varepsilon + |\nabla u_h|^2}} = \int_0^1 F'(s \nabla \bar{u}_h + (1 - s) \nabla u_h) ds \nabla e_h$$

and we can define the bilinear form

$$a^h(v, w) = \int_\Omega \left( \int_0^1 F'(s \nabla \bar{u}_h + (1 - s) \nabla u_h) ds \nabla v \right) \cdot \nabla w.$$  

Due to the properties of $F$, $a^h$ is symmetric and from the fact that $||\nabla \bar{u}_h||_{L_\infty} \leq 2\gamma$, $||\nabla u_h||_{L_\infty} \leq 2\gamma$ we can prove

$$a^h(v, v) \geq c_0(\gamma) ||\nabla e||^2.$$

Similarly as above, if we denote

$$b^h = \int_0^1 G'(s \nabla u + (1 - s) \nabla u_h) ds,$$

we can write

$$\frac{1}{\sqrt{\varepsilon + |\nabla u|^2} g(|\nabla u|)} - \frac{1}{\sqrt{\varepsilon + |\nabla u_h|^2} g(|\nabla u_h|)} = b^h \cdot \nabla e.$$

Introducing the smooth function $b := G'(\nabla u)$, it is easy to see that

$$|b - b^h| \leq c_1(\gamma)||\nabla e||, \quad |b| \leq c_2(\gamma).$$

With these abbreviations (4.18) can be written as

$$\int_\Omega \frac{e_h \cdot \varphi_h}{\sqrt{\varepsilon + |\nabla u_h|^2} g(|\nabla u_h|)} + a^h (e_h, \varphi_h)$$

$$= \int_\Omega (u - \bar{u}_h) \varphi_h - \int_\Omega u_h b^h \nabla e \varphi_h - \int_\Omega \frac{\varepsilon_t \varphi_h}{\sqrt{\varepsilon + |\nabla u_h|^2} g(|\nabla u_h|)}.$$

Now setting $\varphi_h = e_h$ in (4.21) and using (4.19) we get

$$\frac{1}{2} \frac{d}{dt} \int_\Omega \frac{e_h^2}{\sqrt{\varepsilon + |\nabla u_h|^2} g(|\nabla u_h|)} + c_0 ||\nabla e_h||^2$$

$$\leq \frac{1}{2} \int_\Omega (\sqrt{\varepsilon + |\nabla u_h|^2})^3 g(|\nabla u_h|) \nabla u_h \cdot \nabla u_h, t$$

$$- \frac{1}{2} \int_\Omega \frac{e_h^2 g(|\nabla u_h|)_{t_t}}{\sqrt{\varepsilon + |\nabla u_h|^2} g^2(|\nabla u_h|)} - \int_\Omega (\bar{u}_h - u) e_h$$

$$= I_1 + I_2 + I_3 + I_4 + I_5.$$
The term $I_1$ we estimate in similar way as in [5], but the inequality
\begin{equation}
||\varphi||_{L^4} \leq c(||\varphi||_{H^1})^{1/2}(||\varphi||_{L^2})^{1/2}
\end{equation}
is used for $\varphi \in H^1(\Omega)$. We get

$$ |I_1| \leq C \int_{\Omega} |e_h|^2 |\nabla u_{h,t}| \leq C ||e_h||_{L^4}^2 ||\nabla u_{h,t}|$$

$$ \leq C ||e_h|| ||\nabla u_h|| (||\nabla u_t|| + ||\nabla e_t||) \leq \delta ||\nabla e_h||^2 + C_\delta (||\nabla u_t||^2 + ||\nabla e_t||^2)||e_h||^2. $$

Using the properties of $u$ and $g$, the term $I_2$ we also obtain

$$ |I_2| \leq C \int_{\Omega} |e_h|^2 |\nabla u_{h,t}| $$

and then we continue as above. Employing Proposition 4.1, we have

$$ |I_3| \leq Ch^2 + \frac{1}{2}||e_h||^2. $$

We rewrite $I_4$ into the form

$$ I_4 = \int_{\Omega} u_t e_h (b - b^h) \cdot \nabla e - \int_{\Omega} u_t e_h b \cdot \nabla e = I_{41} + I_{42}. $$

To estimate the term $I_{41}$ we can proceed similarly as in [5], and using continuous embedding, (4.20) and Proposition 4.1 we have

$$ |I_{41}| \leq Ch^4 |\log h|^2 + \delta ||\nabla e_h||^2 + C_\delta ||e_h||^2. $$

$I_{42}$ we estimate using the properties of $b$, $u$ and Proposition 1

$$ |I_{42}| \leq c_2(\gamma)||u_t||_{L^\infty} \int_{\Omega} |e_h||\nabla e| $$

$$ \leq C(||e_h|| ||\nabla \bar{\varepsilon}|| + ||e_h|| ||\nabla e_h||) \leq Ch^2 + C_\delta ||e_h||^2 + \delta ||\nabla e_h||^2. $$

Finally, $I_5$ yields

$$ |I_5| \leq C ||\bar{\varepsilon}||^2 + C ||e_h||^2 \leq ch^4 |\log h|^4 + C ||e_h||^2. $$

Now, integrating (4.22) from 0 to $t$, taking into account the estimates of terms $I_1, \ldots, I_5$, and the fact that $e_h(0) = 0$ we get

$$ ||e_h||^2 + \int_0^t ||\nabla e_h||^2 \leq Ch^4 |\log h|^4 + C \int_0^t (1 + ||\nabla e_t||^2) ||e_h||^2. $$

Then Gronwall’s lemma gives

$$ \sup_{(0,T)} ||e_h(t)||^2 \leq Ch^4 |\log h|^4 \exp(c \int_0^T ||\nabla e_t||^2) $$

and the proofs of (4.13) and (4.14) are complete.

In order to prove (4.15) and (4.16) we differentiate (4.21) with respect to $t$. Then we have

$$ \int_{\Omega} \frac{e_{h,t} \varphi_h}{\sqrt{\varepsilon + ||\nabla u_h||^2 g(||\nabla u_h||)}} + a(e_{h,t},\varphi_h) $$
Using (4.20) and (4.24) we get

\[
\int_{\Omega} \phi \left( \frac{\nabla u_h \nabla u_{h,t}}{\sqrt{\varepsilon + |\nabla u_h|^2}} + \frac{g(|\nabla u_h|_t)}{\sqrt{\varepsilon + |\nabla u_h|^2 g^2(|\nabla u_h|)}} \right) - a^\theta_t(e_h, \phi_h) - \int_{\Omega} (\bar{u}_h - u_t) \phi_h - \int_{\Omega} u_{tt} b^h \nabla e \phi_h - \int_{\Omega} u_t b^h \nabla e \phi_h
\]

\[
\int_{\Omega} \xi_t \phi_h \left( \frac{\nabla u_h \nabla u_{h,t}}{\sqrt{\varepsilon + |\nabla u_h|^2}} + \frac{g(|\nabla u_h|_t)}{\sqrt{\varepsilon + |\nabla u_h|^2 g^2(|\nabla u_h|)}} \right)
\]

Now we take \(\phi_h = e_{h,t}\) and similarly as above we get

\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} \frac{e^2_{h,t}}{\sqrt{\varepsilon + |\nabla u_h|^2 g(|\nabla u_h|)}} + c_0 ||\nabla e_{h,t}||^2
\]

\[
\leq - \frac{1}{2} \int_{\Omega} \left( \frac{\varepsilon}{\sqrt{\varepsilon + |\nabla u_h|^2}} \right) \nabla u_h \nabla u_{h,t} - \frac{1}{2} \int_{\Omega} \frac{e^2_{h,t} g(|\nabla u_h|_t)}{\sqrt{\varepsilon + |\nabla u_h|^2 g^2(|\nabla u_h|)}}
\]

(4.25)

\[
-a^\theta_t(e_h, e_{h,t}) - \int_{\Omega} (\bar{u}_h - u_t) e_{h,t} - \int_{\Omega} u_{tt} e_{h,t} b^h \nabla e
\]

\[
- \int_{\Omega} u_t b^h \nabla e e_{h,t} - \int_{\Omega} u_{tt} b^h \nabla e e_{h,t} - \int_{\Omega} \frac{\xi_t e_{h,t}}{\sqrt{\varepsilon + |\nabla u_h|^2}} \nabla u_h \nabla u_{h,t}
\]

\[
\int_{\Omega} \frac{\nabla u_h \nabla u_{h,t}}{\sqrt{\varepsilon + |\nabla u_h|^2}} + \frac{g(|\nabla u_h|_t)}{\sqrt{\varepsilon + |\nabla u_h|^2 g^2(|\nabla u_h|)}}
\]

We estimate terms \(I_1\) and \(I_2\) as above and obtain

\[
|I_1| + |I_2| \leq \delta ||\nabla e_{h,t}||^2 + C_\delta (||\nabla e_t||^2 + 1)||e_{h,t}||^2.
\]

For the term \(I_3\) we realize

\[
\int_0^1 F'(s \nabla \bar{u}_h + (1 - s) \nabla u_h) ds \leq c(|\nabla u_t| + |\nabla \xi_t| + |\nabla e_t|)
\]

and again as in [5] we get

\[
|I_3| \leq \delta ||\nabla e_{h,t}||^2 + C_\delta ||u_t||_t^2 ||\nabla e_h||^2 + C_\delta ||\nabla \xi_t||^2 + C_\delta ||\nabla e_{h,t}||_\infty^2 ||\nabla e_t||^2.
\]

The term \(I_4\) is easy to estimate because

\[
|I_4| \leq C(||\bar{u}_h - u_t||^2 + ||u_{h,t}||^2),
\]

and

\[
I_5 = - \int_{\Omega} u_{tt} (b^h - b) \nabla e e_{h,t} + \int_{\Omega} u_{tt} b \nabla e e_{h,t} = I_{51} + I_{52}.
\]

Using (4.20) and (4.24) we get

\[
|I_{51}| \leq \delta ||\nabla e_{h,t}||^2 + C_\delta ||u_{tt}||_t^2 ||e_{h,t}||^2 + C_\delta ||\nabla e||^2 ||u_{tt}||^2,
\]

\[
|I_{52}| \leq c_2(\gamma)||u_{tt}||_L^2 \int_{\Omega} |e_{h,t}| ||\nabla e|| \leq C||e_{h,t}||^2 + C||\nabla e_t||^2 + C||\nabla e_h||^2.
\]
From the inequality
\[ |b^h_t| \leq C(||\nabla u_t|| + ||\nabla e_t||) \]
we obtain as in [5]
\[ |J_0| \leq C||\nabla e||^2 + C||u_t||^2 ||e_h||^2 + C||\nabla e||^2 ||\nabla e_t||^2 + C||e_h||^2. \]

From the properties of \( b \) we get
\[ |I_7| + |I_8| \leq C||u_t||_\infty ||\nabla e||_\infty ||\nabla e_h|| + ||e_t||_\infty ||e_h||_\infty + ||e_h||_2 \]
\[ \leq C||\nabla e_h||^2 + C||e_h||^2 + C(||\nabla e_t||^2 + ||e_t||^2). \]

Finally, in the last term we use the properties of \( g \) and as we get
\[ |J_9| \leq C \int_\Omega |\varepsilon_t||e_h| ||\nabla u_h| + C \int_\Omega |\varepsilon_t||e_h|. \]

Using the results of Proposition 4.1 we obtain
\[ |I_9| \leq C h^4 \log h^4 + C_5(1 + ||u_t||_2^2 ||e_h||)^2 + C_5h^2 \log h^4 ||\nabla e_t||^2. \]

Now integrating (4.25) from 0 to \( t \) and using the estimates for \( I_1, \ldots, I_9 \) with \( \delta \)
sufficiently small we obtain
\[ ||e_{h,t}||^2 + \int_0^t ||\nabla e_{h,t}||^2 \leq c||e_{h,t}(0)||^2 + c \int_0^t (||\varepsilon_{t,\ell}||^2 + ||\nabla e_{t,\ell}||^2) \]
\[ + C \int_0^t (1 + ||u_t||_3^2 + ||u_{tt}||_3^2 + ||\nabla e_t||^2)||e_h||_2^2 \]
\[ + C \int_0^t (||u_t||_3^2 + ||u_{tt}||_1^2 + 1)(h^2 + \sup_{(0,T)} ||\nabla e||)^2 \]
\[ + C (\sup_{(0,T)} ||\nabla e||_\infty^2 + h^2 \log h)^2 \int_0^T ||\nabla e_t||^2 + C h^4 \log h^4. \]

Because (see also [5])
\[ ||e_{h,t}(0)|| \leq Ch, \]
we get
\[ ||e_{h,t}||^2 + \int_0^t ||\nabla e_{h,t}||^2 \leq Ch + C \sup_{(0,T)} ||\nabla e||^2 + C h^2 \log h^2 \]
\[ + \sup_{(0,T)} ||\nabla e||_\infty^2 \int_0^T ||\nabla e_t||^2 + C \int_0^t (1 + ||u_t||_3^2 + ||u_{tt}||_1^2 + ||\nabla e_t||^2)||e_h||_2^2. \]

Applying Gronwall’s lemma we prove (4.15) and similarly (4.16). \( \square \)

\textbf{Proof. (Proof of Theorem 3.5.)} First, \( \Theta_h \) is not empty, because 0 \( \in \Theta_h \) by
Theorem 3.4. We prove that \( \Theta_h \) is open. As in [5], let \( \sigma \in \Theta_h \), i.e \( (P_h^\sigma) \) is solvable.
Using the implicit function theorem it can be shown that \( (P_h^\sigma) \) has a solution for
all \( \mu \) in a neighborhood of \( \sigma \). Because the same is true for \( u^\sigma \) we obtain the strict inequalities

\[
||\nabla u^\mu_n||_{L^\infty} < 2\gamma, \quad \int_0^T ||\nabla(u^\mu_n - u^\mu_{n,t})||^2 < k^2_1 h^2|\log h|,
\]

provided \( \mu \) lies in a neighborhood of \( \sigma \). Finally we prove that \( \Theta_h \) is closed. Let \( \{\sigma_n\}_{n \in N} \subset \Theta_h, \sigma_n \to \sigma, n \to \infty \). Because of continuous dependence of \( u^\sigma_n, u^\sigma \) on \( \sigma \) we immediately get

\[
(4.26) \quad ||\nabla u^\sigma_n||_{L^\infty} \leq 2\gamma, \quad \int_0^T ||\nabla(u^\sigma_n - u^\sigma_{n,t})||^2 \leq k^2_1 h^2|\log h|.
\]

Furthermore, \( u^\sigma_n \) is the unique solution of \( (P^\sigma_n) \). It remains to show the strict inequalities in (4.26). For this purpose we use results of Proposition 4.2. We infer from (4.14) and (4.26) that

\[
(4.27) \quad \int_0^T ||\nabla e^\sigma_n||^2 \leq c_1 h^4|\log h|^4 (1 + \exp(c_1 k^2_1 h^2|\log h|^2)) k^2_1 h^2|\log h|^2 \\
\leq c_1 h^4|\log h|^4,
\]

provided \( h \leq h_0 \) and \( h_0^2|\log h_0| \leq c_1^{-1} k_1^{-2} \). With the help of (4.26), (4.27) and Proposition 4.1, since \( e^\sigma_n(0) = 0 \) we have

\[
||\nabla e^\sigma_n||_{L^\infty} \leq C(1 + k_1)|h| |\log h|^3.
\]

Then using (2.4) and Proposition 4.1 we also have

\[
||\nabla e^\sigma_n||_{L^\infty} \leq C(1 + k_1)|h| |\log h|^3.
\]

Combining these results with Proposition 4.1 we get

\[
(4.28) \quad ||\nabla e^\sigma||^2 \leq Ch^2 + c k_1 |\log h|^3 \quad (4.29) \quad ||\nabla e^\sigma||_{L^\infty} \leq c \sqrt{1 + k_1} |h_1| |\log h|^3 \leq C(1 + k_1)|h| |\log h|^3
\]

for \( h \leq h_0 \). So we immediately obtain

\[
||\nabla u^\sigma_n||_{L^\infty} \leq ||\nabla u^\sigma||_{L^\infty} + ||\nabla e^\sigma||_{L^\infty} \leq \gamma + c \sqrt{1 + k_1} h^{1/2}|\log h|^{3/2} < 2\gamma,
\]

for \( h \leq h_1 \leq h_0 \) and \( c \sqrt{1 + k_1} h^{1/2}|\log h_1|^{3/2} < \gamma \). Combining (4.16), (4.26), (4.28) and (4.29) we have

\[
\int_0^T ||\nabla e^\sigma_{n,t}||^2 \\
\leq c(h^2 |\log h|^2 + k_1 |h|^3 |\log h|^3 + (h^2 |\log h|^2 + (1 + k_1)|h| |\log h|^3)k^2_1 h^2 |\log h|^2) \\
\leq ch^2 |\log h|^2 (1 + (1 + k_1)|h| |\log h|^3).
\]

Now, we use (4.4) to obtain

\[
\int_0^T ||\nabla e^\sigma_t||^2 \leq 2 \left( \int_0^T ||\nabla e^\sigma_{n,t}||^2 + \int_0^T ||\nabla (u^\sigma_n - u^\sigma_{n,t})||^2 \right)
\]
\[ \leq c h^2 |\log h|^2 (1 + (k_1 + 1)^3 h |\log h|^3). \]

Let us fix \( k_1 > 2c \) and choose \( h_2 \leq h_1 \) so small that \( (1 + k_1)^3 h_2 |\log h_2|^3 \leq 1 \). Then
\[ \int_0^T ||\nabla e_k^\tau||^2 \leq k_1 h^2 |\log h|^2, \]
which is the second inequality we have had to prove. So \( \sigma \in \Theta_h \) and the set is closed.

\[ \square \]

**Proof.** (Proof of Theorem 3.2.) The existence of a solution \( u_h \) is a consequence of Theorem 3.5, existence of this discrete solution and its properties we can obtain also due the properties of Galerkin approximation of elliptic operator (see also [18]). The fourth error estimate is fulfilled due to Theorem 3.5, since \( \Theta_h = [0, 1] \). To obtain the others we can use the results of Propositions 4.1 and 4.2. So
\[ \sup_{(0,T)} ||u - u_h|| \leq \sup_{(0,T)} ||\varepsilon|| + \sup_{(0,T)} ||e_h|| \leq Ch^2 + Ch^2(eCf_0^T ||\nabla e||^2)^{1/2} \leq Ch^2, \]
due to Theorem 3.5, and in a similar way we obtain the rest. \( \square \)

**Proof.** (Proof of Theorem 3.3.) Here, we briefly describe only the main ideas of the proof. First we denote
\[ a'_{ij}(p) := \frac{g(\sigma|p|)}{(\varepsilon + |p|^2)^{3/2}} \left( \frac{\varepsilon + |p|^2}{(1 - \sigma)^3 (1 + |p|^2)^2} \right) \delta_{ij} (\varepsilon + |p|^2) - pjp_1 \]
where \( \delta_{ij} \) denotes Kronecker’s symbol. We can write the differential equation of problem \((P^\sigma)\) in the form
\[ u_t - a'_{ij}(\nabla u)u_{x,i} = 0. \]

First, we linearize \((P^\sigma)\) expanding \( a'_{ij} \) around \( \nabla u_0 \) and after that we change variable \( v = u - u_0 \) to obtain
\[ v_t - a''_{ij}(\nabla u_0)v_{x,i} = a''_{ij}(\nabla u_0)u_{0,x,i}v_{x,j} + a''_{ij}(\nabla u_0)v_{x,i}v_{x,j} + r_{ij}^o(\nabla u_0) v + u_0, x, x, j \equiv F^o(v), \]
where
\[ r_{ij}^o(\nabla u_0, \nabla v) = \int_0^1 (1 - s)a''_{ij, p, p}(\nabla u_0 + s \nabla v) ds v_{x,i} v_{x,j}. \]

Setting \( a''_{ij}(x) := a''_{ij}(\nabla u_0) \) and \( b''_j := -a''_{ij, p, p}(\nabla u_0)u_{0,x,i} \), we have
\[ v_t - a''_{ij} v_{x,i} + b''_j v_{x,j} = F^o(v) \text{ in } I \times \Omega, \quad (L^o) \]
\[ \partial_n v = 0 \text{ on } I \times \partial \Omega, \]
\[ v(0) = v_0 \text{ in } \Omega. \]
It is clear that \( u \) is a solution of \((P^\sigma)\) if and only if \( v = u - u_0 \) solves \((L^\sigma)\). Now we analyze the following linear problem

\[
\begin{align*}
v_t - a_{ij} v_{x_i,x_j} + b_i v_{x_i} &= f \text{ in } I \times \Omega, \\
\partial_s v &= 0 \text{ on } I \times \partial \Omega, \\
v(0) &= v_0 \text{ in } \Omega. 
\end{align*}
\]

(4.30)

For this problem we use the results of [12, Chapter 4]. We obtain that under the assumptions on the data, the linear problem (4.30) has a unique solution \( v \in L_\infty(I; H^3(\Omega)) \cap L_2(I; H^6(\Omega)) \) with \( v_t \in L_\infty(I; H^5(\Omega)) \cap L_2(I; H^8(\Omega)) \), \( v_{tt} \in L_\infty(I; H^4(\Omega)) \cap L_2(I; H^7(\Omega)) \) and moreover

\[
||v||_2 \leq c(||v_0||_2^{(5)} + ||f||_2^{(4)})
\]

where norms are denoted as in [12]: \( v \in W^{2l,1}(Q_T) \) is a function \( v \in L_2(Q_T) \) such that \( v \) has generalized derivative of \( D_r^l D_s^r v \) for all \( r, s; 2r + s \leq 2l \) with the norm

\[
||v||_2^{(2l)} = \sum_{j=0}^{2l} \left( \sum_{2r+s=j} ||D_r^l D_s^r v||_{L_2(Q_T)} \right).
\]

Now, similarly as in [5], we use the Banach fixed point theorem for existence the solution of \((L_\sigma)\). We will consider the Banach space \( X = C^0(I; H^1(\Omega)) \cap L_2(I; H^2(\Omega)) \) with the norm

\[
||v||_X^2 = \sup_{[0,T]} ||v(t)||_2^2 + \int_0^T ||v(s)||_2^2 ds.
\]

For \( 0 < T \leq 1 \), \( M > 0 \) we define

\[
R_{T,M} := \{ v \in X | v(0) = 0, \partial_s v(t, .)|_{\partial \Omega} = 0, 0 \leq t \leq T, \\
v \in L_\infty(I; H^3(\Omega)) \cap L_2(I; H^6(\Omega)), v_t \in L_\infty(I; H^5(\Omega)) \cap L_2(I; H^8(\Omega)), \\
v_{tt} \in L_\infty(I; H^4(\Omega)) \cap L_2(I; H^7(\Omega)), ||v||_2^{(6)} \leq M^2 \}.
\]

Let us introduce the map \( S : R_{T,M} \to X \) which assigns to a function \( u \in R_{T,M} \) the unique solution \( v \) of the linear problem

\[
\begin{align*}
v_t - a_{ij} v_{x_i,x_j} + b_i v_{x_i} &= F^\sigma(u) \text{ in } I \times \Omega, \\
\partial_s v &= 0 \text{ on } I \times \partial \Omega, \\
v(0) &= v_0 \text{ in } \Omega. 
\end{align*}
\]

Now the aim is to prove that \( S \) has a fixed point, provided \( T \) is sufficiently small. This proof is rather technical and long and is practically the same as in [5] so we omit it here. First, it was shown, that for arbitrary \( u \in R_{T,M} \) its image \( S(u) \) is in \( R_{T,M} \) too, so \( S(R_{T,M}) \subset R_{T,M} \). Then the proof that \( S \) is a contraction is presented. This fixed point is a solution of \((L^\sigma)\) so we have the solution of \((P^\sigma)\) as well. \( \Box \)
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