A DARBOUX PROPERTY OF $I_1$-APPROXIMATE PARTIAL DERIVATIVES

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Abstract. Some Darboux property for functions of two variables is studied. In particular, it is shown that $I_2$-approximately continuous functions and $I_1$-approximate partial derivatives of separately $I_1$-approximately continuous functions are Darboux.

Let $\mathbb{R}(\mathbb{R}^2)$ denote the real line (the plane) and $\mathcal{N}$ the set of all positive integers. All topological notations, except for the case where a topology $T$ is specifically mentioned, are given with respect to the natural topology on $\mathbb{R}$ or $\mathbb{R}^2$.

Let $S_1(S_2)$ denote the $\sigma$-field of sets of $\mathbb{R}(\mathbb{R}^2)$ having the Baire property. $I_1(I_2)$ will denote the $\sigma$-ideal of sets of $\mathbb{R}(\mathbb{R}^2)$ of the first category.

Recall that $0$ is an $I_1$-density point of a set $A \in S_1$ if and only if, for each increasing sequence of positive integers $\{n_m\}_{m \in \mathbb{N}}$, there is a subsequence $\{n_{m_p}\}_{p \in \mathbb{N}}$ such that

$$\{x : \chi_{A \cap [-1,1]}(x) \neq 1\} \in I_1$$

where $n \cdot A = \{nx : x \in A\}$ (see [8] and, for two variables, [2]).

A point $x_0 \in \mathbb{R}$ is said to be an $I_1$-density point of $a \in S_1$ if and only if $0$ is an $I_1$-density point of the set $\{x-x_0 : x \in A\}$.

A point $x_0 \in \mathbb{R}$ is said to be an $I_1$-dispersion point of $A \in S_1$ if and only if $x_0$ is an $I_1$-density point of $\mathbb{R} \setminus A$.

For each $A \in S_1$, we denote

$$\Phi_1(A) = \{x \in \mathbb{R} : x \text{ is an } I_1\text{-density point of } A\},$$

$$\Psi_1(A) = \{x \in \mathbb{R} : x \text{ is an } I_1\text{-dispersion point of } A\}.$$

In [8] it was proved that $T_{I_1} = \{A \in S_1 : A \subset \Phi_1(A)\}$ is a topology on the real line. Every function which is continuous with respect to the $T_{I_1}$-topology is called an $I_1$-approximately continuous function.

We say that $x_0$ is a deep $I_1$-density point of a set $A$ if and only if there exists a closed set $F \subset A \cup \{x_0\}$ such that $x_0 \in \Phi_1(F)$. In [9] it was proved that if $f$ is an $I_1$-approximately continuous function then, for every open set $U$, if $x_0 \in f^{-1}(U)$, then $x_0$ is a deep $I_1$-density point of the set $f^{-1}(U)$.

The following result will be useful (see [5]).

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Lemma 1. Let $G$ be an open subset of the real line; then $0$ is an $I_1$-dispersion point of $G$ if and only if, for each $n \in \mathcal{N}$, there exist $k \in \mathcal{N}$ and a real $\delta > 0$ such that, for any $h \in (0, \delta)$ and $i \in \{1, \ldots, n\}$, there exist two numbers $j, j' \in \{1, \ldots, k\}$ such that
\[
G \cap \left( \left( \frac{i-1}{n} + \frac{j-1}{nk} \right) \cdot h, \left( \frac{i-1}{n} + \frac{j}{nk} \right) \cdot h \right) = 0
\]
and
\[
G \cap \left( \left( \frac{i-1}{n} + \frac{j'}{nk} \right) \cdot h, \left( \frac{i-1}{n} + \frac{j'-1}{nk} \right) \cdot h \right) = \emptyset.
\]

In [2], the definition of an $I_2$-density point of a set $A \in \mathcal{S}_2$ was introduced. The authors obtained analogous results as in [8], on the plane. They defined the topology on the plane in the following way: $I_2 = \{A \in \mathcal{S}_2 : A \subset \Phi_2(A)\}$ where
\[
\Phi_2(A) = \{(x, y) \in \mathbb{R}^2 : (x, y) \text{ is an } I_2\text{-density point of } A\}.
\]
We shall denote by $\Phi_2^+(A)$, for each $A \in \mathcal{S}_2$, the set of $I_2$-density points of the set $A$ with respect to the first quarter on the plane. For the remaining quarters, we use the symbols $\Phi_2^+(A)$, $\Psi_2^+(A)$ and $\Psi_2^-(A)$. By $\Psi_2^+(A)$, $\Psi_2^+(A)$, $\Psi_2^+(A)$ and $\Psi_2^-(A)$ we denote sets of $I_2$-dispersion points of the set $A$ with respect to each quarter on the plane, respectively [2]. Functions which are continuous with respect to the $I_2$-topology will be called $I_2$-approximately continuous.

In a similar way as Lemma 1 we may prove the following

Lemma 2. Let $G$ be an open set on the plane; then $(0,0) \in \Psi_2^+(G)$ if and only if, for each $n \in \mathcal{N}$, there exist $k \in \mathcal{N}$ and a real number $\delta > 0$ such that, for any $h \in (0, \delta)$ and $i, i' \in \{1, \ldots, n\}$, there exist two numbers $j, j' \in \{1, \ldots, k\}$ such that
\[
G \cap \left( \left( \frac{i-1}{n} + \frac{j-1}{nk} \right) \cdot h, \left( \frac{i-1}{n} + \frac{j}{nk} \right) \cdot h \right)
\]
\[
\times \left( \left( \frac{i'-1}{n} + \frac{j'-1}{nk} \right) \cdot h, \left( \frac{i'-1}{n} + \frac{j'}{nk} \right) \cdot h \right) = \emptyset.
\]

The definition of a separately $I_1$-approximately continuous function was introduced in the obvious manner in [10] and was considered in [10] and [1].

In [6], the definition of the $I_1$-approximative derivative of a function $f$ of one variable was introduced. Many properties of $I_1$-approximate derivatives and $I_1$-differentiable functions were considered there.

Definition 3 ([6]). Let $f : \mathbb{R} \to \mathbb{R}$ have the Baire property in a neighbourhood of $x_0$. The upper $I_1$-approximate limit of $f$ at $x_0$ ($I_1\limsup_{x \to x_0} f(x)$) is the greatest lower bound of the set $\{y : \{x : f(x) > y\} \text{ has } x_0 \text{ as an } I_1\text{-dispersion point}\}$. The lower $I_1$-approximate limit, the right-hand and left-hand upper and lower $I_1$-approximate limits are defined similarly. If $I_1\limsup_{x \to x_0} f(x) = I_1\liminf_{x \to x_0} f(x)$, their common value will be called the $I_1$-approximate limit of $f$ at $x_0$ and denoted by $I_1\lim f(x_0) f(x)$.

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $(x_0, y_0) \in \mathbb{R}^2$. Put
\[
U_{(x_0,y_0)}(x) = \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} \quad \text{for } x \in \mathbb{R}, x \neq x_0.
\]
Definition 4 ([6]). Let \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) be any function defined in some neighborhood of \((x_0, y_0) \in \mathbb{R}^2\) and having there the Baire property in the direction of the \(ox\) axis. We define the upper right \(I_1\)-approximate partial derivative of \(f\) at \((x_0, y_0)\) in the direction of \(ox\) as the corresponding extreme limit of \(U(x_0, y_0, x)\) as \(x\) tends to \(x_0\) from the right. The other extreme \(I_1\)-approximate partial derivatives in the direction of \(ox\) are defined similarly. If all these derivatives are equal and finite, we call their common value the \(I_1\)-approximate partial derivative of \(f\) at \((x_0, y_0)\) and denote it by \(f_{I_1,x}(x_0, y_0)\).

In a similar way we can define the partial \(I_1\)-approximate derivative in the direction of the \(oy\) axis.

The partial \(I_1\)-approximate derivatives are considered in [3] and [4].

Definition 5. Let \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \). We shall say that \( f \) has the Darboux property if and only if, for each open interval \( J \subset \mathbb{R}^2 \), \( f(J) \) is a connected set.

Definition 6 ([7]). A set \( D \subset \mathbb{R}^2 \) is Darboux if and only if

- for each \( x \in D \), there exists a closed interval \( I \) such that \( x \in I \) and \( \text{int}(I) \subset D \),
- for two points \( x, y \in D \), there are \( k \in \mathbb{N} \) and \( Q_1, Q_2, \ldots, Q_k \) such that, for each \( i \in \{1, \ldots, k\} \), \( \text{int}(\text{cl}(Q_i)) \subset Q_i \subset D \), \( \text{cl}(Q_i) \) is a closed interval, \( x \in Q_1 \), \( y \in Q_k \) and \( Q_i \cap Q_{i+1} \neq \emptyset \) for \( i = 1, \ldots, k - 1 \).

Definition 7. Let \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \). We shall say that \( f \) is Darboux if and only if, for every Darboux set \( Q \), \( f(Q) \) is a connected set.

Definition 8. Let \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \). We shall say that \( f \) is a connected function if and only if, for every connected set \( A \), \( f(A) \) is connected.

By [2], we have the following theorem.

Theorem 9. Let \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \) be an \(I_2\)-approximately continuous function. Then \( f \) has the Darboux property.

Corollary 10. Every open interval is a connected set with respect to the \(T_2\)-topology.

Proposition 11. Every Darboux set is connected with respect to the \(T_2\)-topology.

Proof. It is enough to prove that each set \( Q \subset \mathbb{R}^2 \), such that \( \text{cl}(Q) \) is a closed interval and \( \text{int}(\text{cl}(Q)) \subset Q \), is connected with respect to \(T_2\). We put \( A = \text{int}(\text{cl}(Q)) \) and assume that \( Q \setminus A \neq \emptyset \). We observe that if \( (x, y) \in Q \setminus A \), then \( (x, y) \in \Phi_{1}^{+}(A) \) or \( (x, y) \in \Phi_{2}^{-}(A) \) or \( (x, y) \in \Phi_{2}^{-}(A) \). Therefore, for each \( U \in T_2 \), such that \( (x, y) \in U \), \( U \cap A = \emptyset \).

We suppose that there exist two sets \( U_1, U_2 \in T_2 \) such that \( Q \cap U_1 \neq \emptyset \), \( Q \cap U_2 \neq \emptyset \), \( Q \cap U_1 \cap U_2 = \emptyset \) and \( Q \cap (U_1 \cup U_2) = \emptyset \). Since \( A \) is \(T_2\)-connected, therefore \( A \subset U_1 \) or \( A \subset U_2 \). We assume that \( A \subset U_1 \). Thus \( \emptyset \neq U_2 \cap A \subset U_2 \cap U_1 \cap Q \), a contradiction. Hence every Darboux set is \(T_2\)-connected. \(\square\)
Theorem 12. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be an $\mathcal{I}_2$-approximately continuous function. Then $f$ is a Darboux function.

Proposition 13. There exists a set $A \subset \mathbb{R}^2$ such that $A$ is connected with respect to the natural topology and $A$ is not connected with respect to the $\mathcal{T}_2$-topology.

Proof. It is enough to show that there exist two disjoint nonempty sets $A_1$ and $A_2$ such that $A_1 \in \mathcal{T}_2$, $A_2 \in \mathcal{T}_2$, and $A_1 \cup A_2$ is a connected set with respect to the natural topology.

Let

$$A_1 = \left\{(x, y) \in \mathbb{R}^2 : -\frac{1}{2} < y < \frac{1}{2} x^2\right\}$$

and

$$A_2 = (\mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 : -x^2 \leq y \leq x^2\}) \cup \{(0, 0)\}.$$  

Then $A_1 \in \mathcal{T}_2$ and $A_1 \cup A_2$ is a connected set with respect to the natural topology. We shall show that $A_2 \in \mathcal{T}_2$. Since $A_2 \setminus \{(0, 0)\}$ is an open set we only prove that $(0, 0) \in \Phi_2(A_2)$. It is obvious that $(0, 0) \in \Phi_2^+(A_2)$ and $(0, 0) \notin \Phi_2^-(A_2)$.

Let $n \in \mathcal{N}$. We put $k = 2$ and $\delta = \frac{1}{2n}$. Let $0 < h < \delta$, $(i_1, i_2) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$ and

$$(x_0, y_0) \in \left(\frac{i_1 - 1}{n}h, \frac{2i_1 - 1}{2n}h\right) \times \left(\frac{2i_2 - 1}{2n}h, \frac{i_2}{n}h\right).$$

Then $y_0 > \frac{2i_2 - 1}{2n}h > (2i_2 - 1)h^2 \geq \delta^2$ and $0 < x_0 < h$. Thus $y_0 > x_0^2$ and $(x_0, y_0) \in A_2$. Therefore there exists $(j_1, j_2) = (1, 2) \in \{1, 2\} \times \{1, 2\}$ such that

$$\left(\frac{i_1 - 1}{n} + \frac{j_1 - 1}{nk}\right) \cdot h, \left(\frac{i_1 - 1}{n} + \frac{j_1}{nk}\right) \cdot h$$

$$\times \left(\left(\frac{i_2 - 1}{n} + \frac{j_2 - 1}{nk}\right) \cdot h, \left(\frac{i_2 - 1}{n} + \frac{j_2}{nk}\right) \cdot h\right) \subset A_2.$$ 

Hence, by Lemma 2, $(0, 0) \in \Phi_2^+(A_2)$. In a similar way we can prove that $(0, 0) \notin \Phi_2^-(A_2)$ and the proof of the proposition is completed.

Proposition 14. There exists a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f$ is $\mathcal{I}_2$-approximately continuous and is not a connected function.

Proof. Let $A_1, A_2$ be defined in the same way as in Proposition 13. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function at each $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ such that $f(A_1) = \{1\}$ and $f(A_2) = \{0\}$. Since $(0, 0) \in \Phi_2(A_2)$ we have that $f$ is $\mathcal{I}_2$-approximately continuous on $\mathbb{R}^2$. By $f(A_1 \cup A_2) = \{0, 1\}$, we know that $f$ is not connected.

Lemma 15. Let $[a, b] \subset \mathbb{R}$ and let $A_1, A_2$ be two nonempty sets having the Baire property such that $[a, b] = A_1 \cup A_2$. Then $A_1 \cap (\{a, b\} \setminus \Psi_1(A_1)) \neq \emptyset$ or $A_2 \cap (\{a, b\} \setminus \Psi_1(A_2)) \neq \emptyset$.

Proof. First we assume that $A_1 \cap A_2 \notin \mathcal{I}_1$. Then, by [8], $(a, b) \cap A_1 \cap A_2 \cap \Phi_1(A_1 \cap A_2) \neq \emptyset$ and we choose $x_0 \in (a, b) \cap A_1 \cap A_2 \cap \Phi_1(A_1 \cap A_2)$. Then $x_0 \in A_1 \cap ((a, b) \setminus \Psi_1(A_2))$. 
Now, let $A_1 \cap A_2 \in \mathcal{I}_1$. We put $B_1 = (A_1 \setminus (A_1 \cap A_2)) \cap (a, b)$ and $B_2 = A_2 \cap (a, b)$. Then, by [8], $\Psi_1(B_1) = \Psi_2(A_1)$ and $\Psi_1(B_2) = \Psi_1(A_2)$. We suppose that $B_1 \subset \Psi_1(B_2)$ and $B_2 \subset \Psi_1(B_1)$. Then $B_1 \subset \Phi(B_1)$ and $B_2 \subset \Phi(B_2)$. Hence $B_1, B_2$ are open sets with respect to the $\mathcal{T}_{\mathcal{I}_1}$-topology. $B_1 \cup B_2 = (a, b)$ and $B_1 \cap B_2 = \emptyset$. This is impossible since $(a, b)$ is a connected set the with respect to the $\mathcal{T}_{\mathcal{I}_1}$-topology [8]. Thus $B_1 \cap ((a, b) \setminus \Psi_1(B_2)) \neq \emptyset$ or $B_2 \cap ((a, b) \setminus \Psi_1(B_1)) \neq \emptyset$, and $A_1 \cap ((a, b) \setminus \Psi(A_2) \neq \emptyset$ or $A_2 \cap ((a, b) \setminus \Psi(A_1)) \neq \emptyset$.

**Lemma 16.** Let $f, g : \mathbb{R} \to \mathbb{R}$ be $\mathcal{I}_1$-approximately continuous functions. If $0$ is not an $\mathcal{I}_1$-dispersion point of a set $A \in \mathcal{S}_1$, then there exists a sequence $\{y_n\}_{n \in \mathbb{N}} \subset A$ such that $\lim_{n \to \infty} y_n = 0$, $\lim_{n \to \infty} f(y_n) = f(0)$ and $\lim_{n \to \infty} g(y_n) = g(0)$.

**Proof.** We may assume that $0$ is not a right-side $\mathcal{I}_1$-dispersion point of the set $A \in \mathcal{S}_1$. By Lemma 1, there exists $n \in \mathbb{N}$ such that, for any $k \in \mathbb{N}$ and a real $\delta > 0$, there exist $h = h(k, \delta) \in (0, \delta)$ and $i = i(h) \in \{1, \ldots, n\}$ such that, for each $j \in \{1, \ldots, k\}$,

$$\left(\frac{(i-1)k + j - 1}{nk} h, \frac{(i-1)k + j}{nk} h\right) \cap A \neq \emptyset.$$  

Let $p \in \mathbb{N}$. We put $C_p = \{y : |f(y) - f(0)| < \frac{1}{p}\}$ and $B_p = \{y : |g(y) - g(0)| < \frac{1}{p}\}$. Since $f$ and $g$ are $\mathcal{I}_1$-approximately continuous, $0$ is a deep $\mathcal{I}_1$-density point of $C_p \cap B_p$. Therefore, by Lemma 1, there exist $k_1 \in \mathbb{N}$ and $\delta_1 > 0$ such that, for any $i \in \{1, \ldots, n\}$ and $h \in (0, \delta_1)$, there exists $j = j(i, h) \in \{1, \ldots, k_1\}$ such that

$$\left(\frac{(i-1)k_1 + j - 1}{nk_1} h, \frac{(i-1)k_1 + j}{nk_1} h\right) \subset C_p \cap B_p.$$

Let $\delta_0 = \min\left(\frac{1}{p}, \delta_1\right)$. We put $h = h(k_1, \delta_0)$, $i = i(h)$ and $j = j(i, h)$. Then we may choose

$$y_p \in \left(\frac{(i-1)k_1 + j - 1}{nk_1} h, \frac{(i-1)k_1 + j}{nk_1} h\right) \cap A \subset C_p \cap B_p.$$

Thus $0 < y_p < \frac{1}{p}$, $|f(y_p) - f(0)| < \frac{1}{p}$ and $|g(y_p) - g(0)| < \frac{1}{p}$.

Hence $\lim_{p \to \infty} y_p = 0$, $\lim_{p \to \infty} f(y_p) = f(0)$ and $\lim_{p \to \infty} g(y_p) = g(0)$. 

**Theorem 17.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a separately $\mathcal{I}_1$-approximately continuous function. If $f$ is $\mathcal{I}_1$-approximately differentiable with respect to $x$ at every point, then $f_{x,x}$ is a Darboux function.

**Proof.** By the assumption and by the result of [10], we have that $f$ has the Baire property. Therefore, by [3], $f_{x,x}$ has the Baire property, too.

First, we show that if $I = [a, b] \times [c, d]$, then $f_{x,x}(I)$ is a connected set. If it is not true, there exists $x_0 \in \mathbb{R}$ and two nonempty sets $A$ and $B$ having the Baire property, such that $I = A \cup B$ and $f_{x,x}(A) \subset (-\infty, x_0)$ and $f_{x,x}(B) \subset (x_0, +\infty)$. For $y \in [c, d]$, let $H_y = \{(x, y) : x \in [a, b]\}$. Since $f_{x,x}(x, y)$, as a function of $x$, has Darboux property, [6], we have that $f_{x,x}(H_y)$ is a connected set. Then $H_y \subset A$ or $H_y \subset B$. Hence there exist $A_1, A_2$ such that $A = [a, b] \times A_1$ and $B = [a, b] \times A_2$. By Lemma 15, we may assume that there exists a point $y_0 \in A_1$ which is not an $\mathcal{I}_1$-dispersion point of $A_2$. Thus, by the above and the $\mathcal{I}_1$-approximate
continuity of the functions \( f(a, y) \) and \( f(b, y) \) as functions of \( y \), we may choose a sequence \( \{y_n\}_{n \in \mathbb{N}} \subset A_2 \) such that \( \lim_{n \to \infty} y_n = y_0 \), \( \lim_{n \to \infty} f(b, y_n) = f(b, y_0) \) and \( \lim_{n \to \infty} f(a, y_n) = f(a, y_0) \) (see Lemma 16). Since, for each \( n \in \mathbb{N} \), \( f(x, y_n) \) is \( \mathcal{I}_1 \)-approximately differentiable as a function of \( x \), by the mean-value property [6], we have that there exists \( z_n \in (a, b) \) such that
\[
\frac{f(b, y_n) - f(a, y_n)}{b - a} = f_{\mathcal{I}_1, x}(z_n, y_n).
\]
Hence
\[
\lim_{n \to \infty} f_{\mathcal{I}_1, x}(z_n, y_n) = \frac{f(b, y_0) - f(a, y_0)}{b - a}.
\]
Applying the mean-value property to the function \( f(x, y) \), we can find \( z_0 \in (a, b) \) such that
\[
\frac{f(b, y_0) - f(a, y_0)}{b - a} = f_{\mathcal{I}_1, x}(z_0, y_0).
\]
Hence
\[
\lim_{n \to \infty} f_{\mathcal{I}_1, x}(z_n, y_n) = f_{\mathcal{I}_1, x}(z_0, y_0).
\]
Since \( \{z_n, y_n\}_{n \in \mathbb{N}} \subset B \), we have that \( \{f_{\mathcal{I}_1, x}(z_n, y_n)\}_{n \in \mathbb{N}} \subset f_{\mathcal{I}_1, x}(B) \subset (a, \infty) \) and \( f_{\mathcal{I}_1, x}(z_0, y_0) \geq x_0 \). This contradicts the fact that \( f_{\mathcal{I}_1, x}(z_0, y_0) \in f(A) \subset (-\infty, x_0) \).

To complete the proof, it suffices to show that, for each set \( Q \) such that int (cl (\( Q \))) \( \subset Q \) and cl (\( Q \)) is a closed interval, \( f_{\mathcal{I}_1, x}(Q) \) is a connected set. If \( Q \) is an open interval then \( Q = \bigcup_{n \in \mathbb{N}} [a_n, b_n] \times [c_n, d_n] \) where, for each \( n \in \mathbb{N} \), \([a_n, b_n] \times [c_n, d_n] \subset [a_{n+1}, b_{n+1}] \times [c_{n+1}, d_{n+1}] \). Since \( f_{\mathcal{I}_1, x}([a_n, b_n] \times [c_n, d_n]) \) is a connected set for each \( n \in \mathbb{N} \), therefore \( f(I_1, x)(Q) \) is a connected set, too. If \( Q \) is not an open interval, we may assume that there exists \( p_0 \in Q \setminus \text{int} (Q) \). Let \( I = [a, b] \times [c, d] \) be an interval included in cl (\( Q \)), having \( p_0 \) as a vertex. Say, \( p_0 = (a, d) \). We want to show that \( f_{\mathcal{I}_1, x}(\text{int} (I) \cup \{p_0\}) \) is connected. Since \( \text{int} (I) \) is an open interval, \( f_{\mathcal{I}_1, x}(\text{int} (I)) \) is connected. Thus the proof will be completed if we show that \( f_{\mathcal{I}_1, x}(p_0) \) is a limit of a sequence of points of \( f_{\mathcal{I}_1, x}(\text{int} (I)) \). Since \( f_{\mathcal{I}_1, x}(x, d) \) has the Darboux property, there exists a sequence \( \{x_n\}_{n \in \mathbb{N}} \subset (a, b) \) such that \( \lim_{n \to \infty} x_n = a \) and \( \lim_{n \to \infty} f_{\mathcal{I}_1, x}(x_n, d) = f_{\mathcal{I}_1, x}(a, d) \).

Let \( n \in \mathbb{N} \). Then, by our assumption, there exists \( z_n \in (a, b) \) \( \setminus \{x_n\} \) such that
\[
\frac{|f(z_n, d) - f(x_n, d)|}{z_n - x_n} - f_{\mathcal{I}_1, x}(x_n, d) < \frac{1}{3n}.
\]
We assume that \( z_n > x_n \). On the other hand, by the \( \mathcal{I}_1 \)-approximate continuity of \( f(z_n, y) \) and \( f(x_n, y) \) as functions of \( y \), there exists \( y_n \in (c, d) \) such that
\[
|f(x_n, d) - f(x_n, y_n)| < \frac{1}{3n}|x_n - z_n|
\]
and
\[
|f(z_n, d) - f(z_n, y_n)| < \frac{1}{3n}|x_n - z_n|.
\]
Then we have
\[
\frac{|f(x_n, y_n) - f(z_n, y_n)|}{x_n - z_n} - f_{\mathcal{I}_1, x}(x_n, d) < \frac{1}{n}.
\]
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By the mean-value theorem for $I_1$-approximate derivatives (see [6]), we can choose a point $t_n \in (x_n, z_n)$ such that $f(x_n, y_n) - f(z_n, y_n) = f_{I_1,x}(t_n, y_n)(x_n - z_n)$. Then we have

$$|f_{I_1,x}(t_n, y_n) - f_{I_1,x}(x_n, d)| < \frac{1}{n}.$$ 

Hence we have the sequence $\{(t_n, y_n)\}_{n \in \mathbb{N}} \subset \text{int}(I)$ satisfying for each $n \in \mathbb{N}$,

$$|f_{I_1,x}(t_n, y_n) - f_{I_1,x}(x_n, d)| < \frac{1}{n}.$$ 

Therefore $\lim_{n \to \infty} f_{I_1,x}(t_n, y_n) = f_{I_1,x}(a, d)$. □

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