Acta Math. Univ. Comenianae
Vol. LXVIII, 2(1999), pp. 257–269

HOMOMORPHISMS OF SEMIHOLONOMIC VERMA MODULES: AN EXCEPTIONAL CASE

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Abstract. It is well known that Verma module homomorphisms correspond to invariant operators on homogeneous spaces, which in certain situations can be regarded as the flat models of specific differential geometries. This can be generalised to curved space by introducing semiholonomic Verma modules, whose homomorphisms give rise to invariant operators on curved space. In this article we investigate from a purely algebraic point of view which Verma module homomorphisms lift to the semiholonomic case for the exceptional Lie algebra E_6.

1. Introduction

Semiholonomic Verma modules were introduced in [**4**]. The motivation was the construction of curved analogues of invariant operators on homogeneous spaces, particularly for conformal geometry. If a homomorphism of Verma modules lifts to a homomorphism of semiholonomic Verma modules then the corresponding invariant operator has a curved analogue. In their article, Eastwood and Slovák give a complete classification of which homomorphisms lift in the case of conformal geometry. Here we investigate the same (algebraic) question for the exceptional Lie algebra E_6.

Our method of constructing semiholonomic lifts will be essentially the same as in the conformal case, namely the translation principle, though here we introduce a variation which we call one-way translations. For our non-existence result we also reduce to the equivalent statement in the conformal case. Unfortunately these methods do not give us a complete classification, but we can almost guess what the complete picture will be.

Rather than duplicate the theory of [**4**], we will assume the reader has a copy of that article to refer to while reading this one, and we will cite results directly from it.

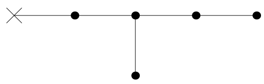
This research was carried out for an MSc thesis at the University of Adelaide, and the author wishes to thank his supervisor Michael Eastwood for his invaluable

Received March 16, 1999.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 17B10, 17B25, 22E47; Secondary 17B35.
On leave from Trinity College, Cambridge. This article describes research undertaken for the author's MSc thesis at the University of Adelaide.

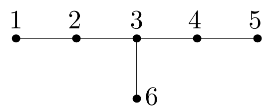
input. Conversations with Toby Bailey and Jan Slovák have been extremely useful. Support in the form of an Australian Postgraduate Award is also gratefully acknowledged.

2. Preliminaries

Let $\mathfrak{g}$ be the exceptional Lie algebra E_6, and let $\mathfrak{p}$ be the parabolic Lie subalgebra with Dynkin diagram



Here (and throughout) we use the same notation as that which appears in [**1**]. Let $\alpha_1, \ldots, \alpha_6$ denote the simple roots of $\mathfrak{g}$, corresponding to the labelling

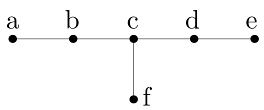


of the nodes of the Dynkin diagram. This root system can be embedded in $\mathbb{C}^8$ if we let

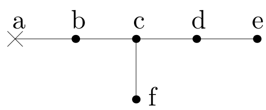
$$\alpha_1 = \frac{1}{2}(\epsilon_1 + \epsilon_8 - \epsilon_2 - \cdots - \epsilon_7), \qquad \alpha_2 = \epsilon_2 - \epsilon_1, \qquad \alpha_3 = \epsilon_3 - \epsilon_2,$$

$$\alpha_4 = \epsilon_4 - \epsilon_3, \qquad \alpha_5 = \epsilon_5 - \epsilon_4, \qquad \alpha_6 = \epsilon_1 + \epsilon_2.$$

Let $\lambda_1, \ldots, \lambda_6$ denote the fundamental weights. An irreducible representation $\mathbb{E}_\lambda$ of $\mathfrak{g}$ will have lowest weight $-\lambda = -(a\lambda_1 + \cdots + f\lambda_6)$[1] where $a, \ldots, f$ are non-negative integers. We will sometimes use



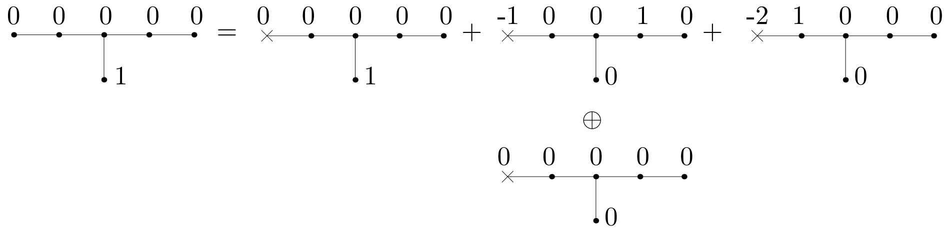
and



to denote both the $\mathfrak{g}$-dominant (respectively $\mathfrak{p}$-dominant) weight and the corresponding $\mathfrak{g}$-module (respectively $\mathfrak{p}$-module). In the latter case a needn't be non-negative, i.e. an irreducible representation of $\mathfrak{p}$ (also denoted $\mathbb{E}_\lambda$) is constructed from the direct sum of the representation of $\mathfrak{so}(10)$ with lowest weight $-(b\lambda_2 + \cdots + f\lambda_6)$ and the representation of the abelian part with lowest weight $-a \in \mathbb{C}$, extended trivially to all of $\mathfrak{p}$.

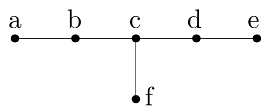
[1]Thus λ is the highest weight of the dual representation $\mathbb{E}_\lambda^*$ and the Verma module $V(\mathbb{E}_\lambda)$.

As a representation of the subalgebra $\mathfrak{p}$, the adjoint representation of $\mathfrak{g}$ on itself decomposes into the composition series



which we write as $\mathfrak{g} = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$. Thus we have $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$ (where $\mathfrak{g}_{\pm 2} = 0$), and a Lie algebra with such a decomposition is called $|1|$-graded. The subalgebra $\mathfrak{p}$ is given by $\mathfrak{g}_0 + \mathfrak{g}_1$, and $\mathfrak{g}_0$ is the Levi part of $\mathfrak{p}$, which further decomposes into a semisimple Lie algebra $[\mathfrak{g}_0, \mathfrak{g}_0]$ isomorphic to $\mathfrak{so}(10)$ and a one-dimensional centre.

Choosing a Serre basis for $\mathfrak{g}$, with Cartan subalgebra generated by $h_1, \ldots, h_6$, we find that the one-dimensional centre of $\mathfrak{g}_0$ is generated by $H = \frac{1}{3}(4h_1 + 5h_2 + 6h_3 + 4h_4 + 2h_5 + 3h_6)$. We obtain a ($\mathfrak{g}_0$-invariant) linear functional on weights from the action of H,

$\ell :$ 

$$\mapsto \frac{1}{3}(4a + 5b + 6c + 4d + 2e + 3f),$$

and this functional is constant on the weights of a given irreducible $\mathfrak{p}$-module. For example, $[H, \mathfrak{g}_i] = i\mathfrak{g}_i$. A general irreducible $\mathfrak{g}-$module $\mathbb{E}$ decomposes into a composition series of some finite length n,

$$\mathbb{E} = \mathbb{E}_\alpha + \mathbb{E}_{\alpha+1} + \cdots + \mathbb{E}_{\alpha+n},$$

where each $\mathbb{E}_{\alpha+j}$ is a (not necessarily irreducible) representation of $\mathfrak{g}_0$ which has H-eigenvalue $\alpha + j$ (where α is some number). The actions of $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ moves us backwards and forwards (respectively) along the series. Regarded as $\mathfrak{p}$-modules, we get a $\mathfrak{p}$-invariant inclusion

$$\mathbb{E}_{\alpha+n} \to \mathbb{E} = \mathbb{E}_\alpha + \mathbb{E}_{\alpha+1} + \cdots + \mathbb{E}_{\alpha+n},$$

and a $\mathfrak{p}$-invariant projection

$$\mathbb{E} = \mathbb{E}_\alpha + \mathbb{E}_{\alpha+1} + \cdots + \mathbb{E}_{\alpha+n} \to \mathbb{E}_\alpha.$$

3. Verma Modules and Semiholonomic Verma Modules[2]

Let $\mathbb{E}$ be a $\mathfrak{p}$-module. We shall use the notation $V(\mathbb{E})$ (and $\bar{V}(\mathbb{E})$) to denote the Verma module (respectively, semiholonomic Verma module) associated to $\mathbb{E}$.

[2]In the following sections, the reader should consult [**4**] for the definitions of Verma modules and semiholonomic Verma modules, and for proofs of the relevant theory.

Then our main purpose is to find when a $\mathfrak{g}$-module homomorphism

$$V(\mathbb{F}) \to V(\mathbb{E})$$

lifts to a homomorphism of the semiholonomic Verma modules,

$$\bar{V}(\mathbb{F}) \to \bar{V}(\mathbb{E}).$$

The classification of all $\mathfrak{g}$-module homomorphisms $V(\mathbb{F}) \to V(\mathbb{E})$ has been carried out by Boe and Collingwood [**2**].

Theorem 1. *The Verma module homomorphisms for the exceptional Lie algebra E_6 (with parabolic subalgebra $\mathfrak{p}$) are classified by the patterns in Figure* 1.[3] *Each Verma module occurs in precisely one such pattern from which the $\mathfrak{g}$-module homomorphisms can be read off.*

The patterns are parametrised by the highest weight

$$\lambda = \begin{tikzcd} \overset{\mathrm{a}}{\bullet} \arrow[r, dash] & \overset{\mathrm{b}}{\bullet} \arrow[r, dash] & \overset{\mathrm{c}}{\bullet} \arrow[r, dash] \arrow[d, dash] & \overset{\mathrm{d}}{\bullet} \arrow[r, dash] & \overset{\mathrm{e}}{\bullet} \\ & & \bullet\,\mathrm{f} & & \end{tikzcd}$$

of the first Verma module, and we shall call this the λ-pattern. If we denote the weight $\lambda_1 + \cdots + \lambda_6$ by δ, then $\lambda + \delta$ must be $\mathfrak{g}$-dominant, and the highest weights of the other Verma modules in the pattern are obtained by the affine action of the Weyl group on λ. If $\lambda+\delta$ lies strictly inside the dominant Weyl chamber we call the pattern regular. If it lies on precisely one wall we call the pattern singular, and we need to omit some of the Verma modules corresponding to non-$\mathfrak{p}$-dominant highest weights. If it lies on more than one wall we get no non-trivial homomorphisms. These three situations correspond to the integers $a, \ldots, f$ being all non-negative, precisely one being equal to -1, and at least two being equal to -1 respectively.

The short arrows which act between adjacent levels are known as the standard homomorphisms, and the longer arrows are known as the non-standard homomorphisms. In the 0-pattern the standard homomorphisms are all first order and the non-standard homomorphisms are of higher order.

In order to prove the existence of certain lifts, we shall use the semiholonomic version of the translation principle, Theorem 4 of [**4**]. Our initial data will be:

- the standard homomorphisms in the 0-pattern,
- the standard homomorphisms in the singular $-\lambda_1$-pattern, $-\lambda_2$-pattern, etc.

Note that all of these Verma module homomorphisms are first order, which means that the existence of lifts for them is immediate. To use homomorphisms can be obtained from this initial data in the holonomic case. We consider several cases.

[3] Note that we have omitted the V's so as not to overcrowd the diagram.

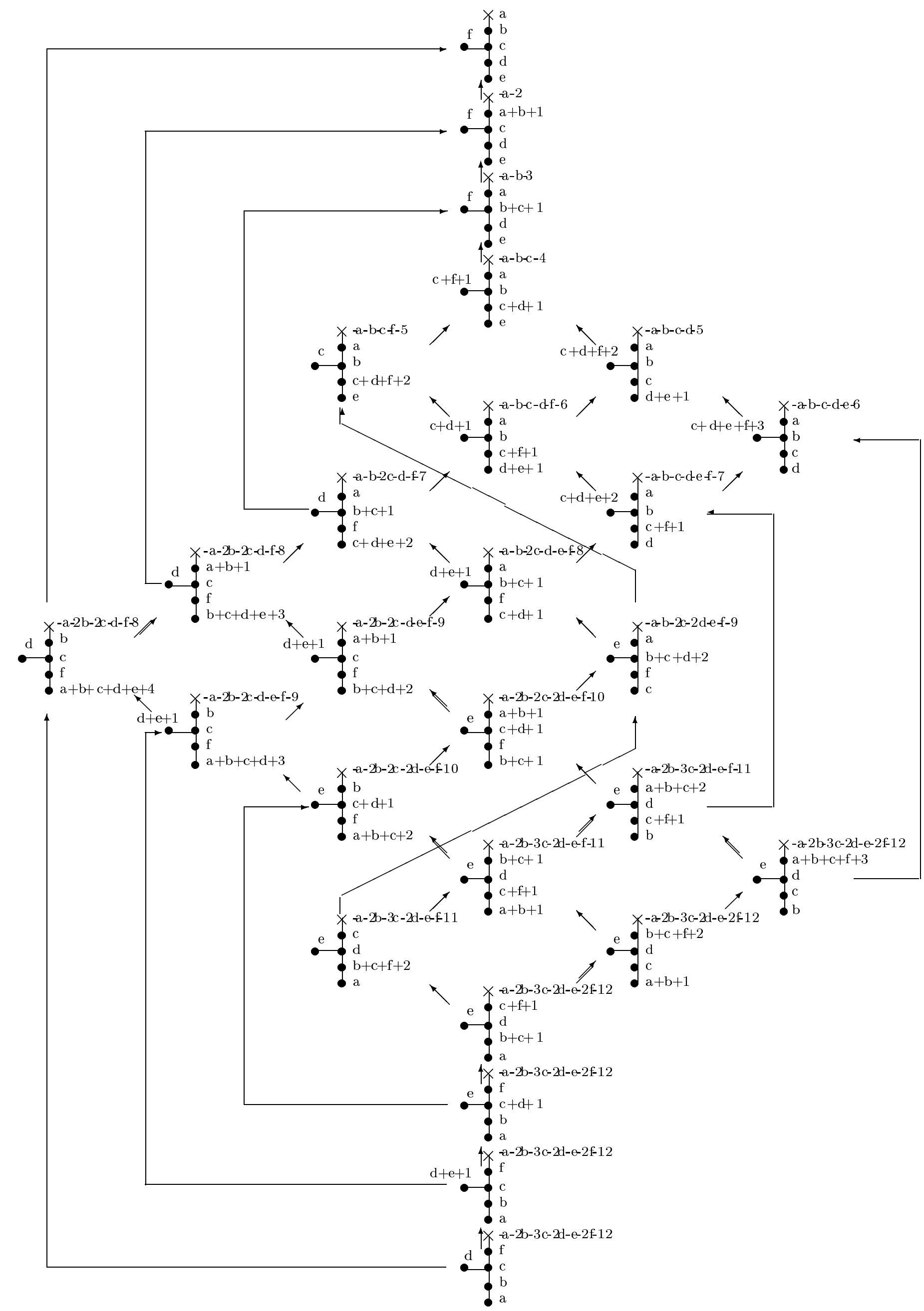


Figure 1. Homomorphisms of Verma modules.

Regular Standard Homomorphisms

Our starting point is the standard homomorphisms in the 0-pattern. By translating with the $\mathfrak{g}$-module $\mathbb{W} = \mathbb{E}_{\lambda_1}$ and its dual $\mathbb{W}^* = \mathbb{E}^*_{\lambda_1} = \mathbb{E}_{\lambda_5}$, we will show that we can obtain the standard homomorphisms in a pattern with any of $a, \ldots, f$ increased by one. Proposition 9 of [**4**] will ensure all of these translations result in non-zero homomorphisms, so inductively we obtain all regular standard homomorphisms.

A homomorphism in a λ-pattern must look like

$$V(\mathbb{E}_{w'.\lambda}) \to V(\mathbb{E}_{w.\lambda}),$$

for some w and w' in the Weyl group $\mathcal{W}$, acting affinely on the weight λ. Since by assumption $\lambda + \delta$ lies strictly inside the dominant Weyl chamber, $w.\lambda + \delta$ and $w'.\lambda + \delta$ must lie in the Weyl chambers corresponding to w and w' respectively.

Consider the $\mathfrak{p}$-modules occurring in the decomposition of the tensor product $\mathbb{E}_{w.\lambda} \otimes \mathbb{W}$. All of the weights in $\mathbb{W}$ are obtained from the action of the Weyl group on the lowest weight $-\lambda_1$, so they have the same Euclidean length $\sqrt{4/3}$ (as calculated from our embedding of the weight system in $\mathbb{C}^8$ given in Section 2). On the other hand, the walls of the Weyl chambers are the planes perpendicular to the simple roots $\alpha_1, \ldots, \alpha_6$, and thus the weight $\lambda + \delta$ (and hence also $w.\lambda + \delta$) will be distance

$$(\lambda + \delta).\alpha_i / |\alpha_i| \geq 1/\sqrt{2}$$

from the walls. In particular, $w.\lambda + \delta$ will be distance at least $\sqrt{2}$ from another regular weight inside an adjacent Weyl chamber, so the weights we are translating by (of length $\sqrt{4/3}$) are not long enough to move us into another chamber.

So if $-\eta$ is the lowest weight of a $\mathfrak{p}$-module in $\mathbb{E}_{w.\lambda} \otimes \mathbb{W}$, then $\eta + \delta$ will be inside (or at worst on the boundary) of the Weyl chamber corresponding to w. In particular, all the $\mathfrak{p}$-modules in the decomposition of this tensor product must have distinct central characters (two distinct weights cannot be in the same Weyl group orbit if they lie in, or on the boundary of, the same Weyl chamber). Of course, the same is true for $\mathbb{E}_{w'.\lambda} \otimes \mathbb{W}$, and if we replace $\mathbb{W}$ by $\mathbb{W}^*$. Indeed whenever we translate one of these regular homomorphisms with $\mathbb{W}$ or $\mathbb{W}^*$, everything will split off with unique central character. This is precisely the hypothesis of Proposition 9 of [**4**], which thus tells us that all of these translations result in non-zero homomorphisms.

We now outline the inductive step. The lowest weight of $\mathbb{W}$ is $-\lambda_1$, and hence $-w\lambda_1$ will also occur as a weight of this $\mathfrak{g}$-module. Thus there will be a $\mathfrak{p}$-module in the tensor product $\mathbb{E}_{w.\lambda} \otimes \mathbb{W}$ with lowest weight $-w.\lambda - w\lambda_1 = -w.(\lambda + \lambda_1)$. The same is true if we replace w by w', and hence translating with $\mathbb{W}$ gives us a non-zero homomorphism

$$V(\mathbb{E}_{w'.(\lambda+\lambda_1)}) \to V(\mathbb{E}_{w.(\lambda+\lambda_1)}).$$

Of course, adding λ_1 to λ simply increases a by one. Similarly, using $\mathbb{W}^*$ instead of $\mathbb{W}$ increases e by one.

We can increase each of b, c, d, and f by one in several steps. For example, first increase a by one then translate with $\mathbb{W}$ again. Since the weights $\lambda_1 - \lambda_2$ and $w(\lambda_1 - \lambda_2)$ occur amongst the weights of $\mathbb{W}$, the tensor product $\mathbb{E}_{w.(\lambda+\lambda_1)} \otimes \mathbb{W}$, will contain a component with lowest weight $-w.(\lambda+\lambda_1) + w(\lambda_1 - \lambda_2) = -w.(\lambda + \lambda_2)$. Thus we have increased b by one.

We have shown that all the standard homomorphisms in the regular patterns can be obtained by translating from the regular initial data. In fact, the above argument applies equally well to the non-standard homomorphisms in the regular patterns, but we first need some initial data, namely the non-standard homomorphisms in the 0-pattern. We will see that some of these can be obtained by translating from singular patterns.

Singular Standard Homomorphisms

Essentially the same argument as above shows that all the standard homomorphisms in the singular patterns can be obtained inductively by translating from the singular initial data. The main difference between the regular and singular cases is that with the former we are dealing with weights strictly inside Weyl chambers whereas with the latter the weights lie on walls. Nevertheless, it is still possible to show that when we translate weights along walls, the relevant modules will split off with unique central characters. Hence Proposition 9 of [**4**] is applicable as before, and translation results in non-zero homomorphisms.

Singular Non-Standard Homomorphisms

As with the regular case, all the non-standard homomorphisms in the singular patterns can be obtained by translating, but only when we first have some initial data to begin with. In fact, all of this initial data (namely the non-standard operators in the singular $-\lambda_1$-pattern, $-\lambda_2$-pattern, etc.) can be obtained by translating from the standard singular initial data. This can be exhibited in a case by case way. For example, when we translate the standard homomorphism

$$V(\mathbb{E}_{-5\lambda_1+\lambda_4}) \to V(\mathbb{E}_{-4\lambda_1+\lambda_6})$$

from the $-\lambda_4$-pattern by $\mathbb{W}$, one of the (non-zero) homomorphisms which we get is the non-standard homomorphism

$$V(\mathbb{E}_{-5\lambda_1+\lambda_5}) \to V(\mathbb{E}_{-3\lambda_1})$$

from the $-\lambda_3$-pattern. Further translations result in all of the singular non-standard initial data. Note that while it appears that we will eventually arrive at regular non-standard initial data in this way, in fact this is precisely the point at which translation breaks down (in the final translation the relevant modules do not split off with unique central characters, and hence we cannot apply Proposition 9 of [**4**] to ensure non-vanishing of the resulting homomorphisms).

4. One-way Translation

So far we have relied solely on Proposition 9 of [**4**] to ensure that our translations give us non-zero homomorphisms, but even when the relevant modules do not always split off with unique central characters we may still get non-zero homomorphisms. We will describe such a situation, which results in one-way translations, i.e. translating in one direction gives a non-zero homomorphism but translating back gives zero. This situation occurs when we try to translate between singular and regular homomorphisms.

First we will show that translating from the regular homomorphism $V(\mathbb{F}) \to V(\mathbb{E})$ to the singular homomorphism $V(\mathbb{F}_1) \to V(\mathbb{E}_1)$ will result in the zero homomorphism. In the tensor product

$$V(\mathbb{E} \otimes \mathbb{W}) = V(\mathbb{E}) \otimes \mathbb{W}^*,$$

$V(\mathbb{E}_1)$ will split off with unique central character, as we have translated from a regular highest weight inside a Weyl chamber[4] to a singular highest weight on a wall of that Weyl chamber. The same is true if we replace $\mathbb{E}$ with $\mathbb{F}$, and hence we get a new homomorphism

$$\begin{tikzcd} V(\mathbb{F}_1) \arrow[r] & V(\mathbb{F} \otimes \mathbb{W}) \arrow[d] \arrow[dl] \\ V(\mathbb{E}_1) & V(\mathbb{E} \otimes \mathbb{W}). \arrow[l] \end{tikzcd}$$

It follows from the tautological isomorphism (11) in [**4**] that the diagonal homomorphism is equivalent to

$$\begin{tikzcd} & V(\mathbb{F}) \arrow[d] \arrow[dl] \\ V(\mathbb{E}_1 \otimes \mathbb{W}^*) & V(\mathbb{E}). \arrow[l] \end{tikzcd}$$

Now $V(\mathbb{E})$ must occur in the decomposition of $V(\mathbb{E}_1 \otimes \mathbb{W}^*)$, but it cannot have unique central character. Indeed, we are translating from a singular weight on a wall to a regular weight inside a chamber, so there must be a second module $V(\mathbb{E}')$ whose highest weight is related to the highest weight of $V(\mathbb{E})$ by a reflection in that wall. In particular, these modules have the same central character. So assume we get a composition series that looks like either

$$V(\mathbb{E}_1 \otimes \mathbb{W}^*) = (\cdots) + (V(\mathbb{E}') \oplus \cdots) + (V(\mathbb{E}) \oplus \cdots)$$

[4]Here, and in what follows, we will talk about highest weights when we really mean the weight translated by δ.

or

$$V(\mathbb{E}_1 \otimes \mathbb{W}^*) = (V(\mathbb{E}') \oplus \cdots) + (V(\mathbb{E}) \oplus \cdots) + (\cdots),$$

where we have not shown modules with other central characters. Thus we have an invariant inclusion

$$V(\mathbb{E}') \to V(\mathbb{E}_1 \otimes \mathbb{W}^*)$$

and an invariant projection

$$V(\mathbb{E}_1 \otimes \mathbb{W}^*) \to V(\mathbb{E}),$$

but note that these Verma modules do not split off completely. In particular, the composition of the bottom two homomorphisms in

$$\begin{tikzcd} & & V(\mathbb{F}) \arrow[dl] \arrow[d] \\ V(\mathbb{E}) & V(\mathbb{E}_1 \otimes \mathbb{W}^*) \arrow[l] & V(\mathbb{E}) \arrow[l] \end{tikzcd}$$

must be zero, and hence the diagonal homomorphism must factor through $V(\mathbb{E}')$:

$$\begin{tikzcd} & V(\mathbb{F}) \arrow[d] \\ V(\mathbb{E}_1 \otimes \mathbb{W}^*) & V(\mathbb{E}') \arrow[l] \end{tikzcd}$$

If no non-zero homomorphism $V(\mathbb{F}) \to V(\mathbb{E}')$ exists (this can be checked from the classification of homomorphisms of Verma modules, Theorem 2), it follows that the diagonal homomorphism must necessarily be zero, and hence translating from the regular homomorphism to the singular homomorphism gives us zero.

We now show that translating from the singular homomorphism to the regular homomorphism gives a non-zero homomorphism, and so we have an example of a one-way translation. We have already made an assumption about the form of the composition series for $V(\mathbb{E}_1 \otimes \mathbb{W}^*)$. We further assume that the composition series for $V(\mathbb{F}_1 \otimes \mathbb{W}^*)$ looks like either

$$V(\mathbb{F}_1 \otimes \mathbb{W}^*) = (\cdots) + (V(\mathbb{F}) \oplus \cdots) + (V(\mathbb{F}') \oplus \cdots)$$

or

$$V(\mathbb{F}_1 \otimes \mathbb{W}^*) = (V(\mathbb{F}) \oplus \cdots) + (V(\mathbb{F}') \oplus \cdots) + (\cdots).$$

In other words, the necessary inclusions and projections of Verma modules will exist for us to translate the singular homomorphism to obtain

$$\begin{tikzcd} V(\mathbb{F}) \arrow[r] & V(\mathbb{F}_1 \otimes \mathbb{W}^*) \arrow[dl] \arrow[d] \\ V(\mathbb{E}) & V(\mathbb{E}_1 \otimes \mathbb{W}^*). \arrow[l] \end{tikzcd}$$

Again using the isomorphism (11) of [**4**], the diagonal homomorphism is equivalent to

$$\begin{tikzcd}
& V(\mathbb{F}_1) \arrow[dl] \arrow[d] \\
V(\mathbb{E}\otimes\mathbb{W}) & V(\mathbb{E}_1). \arrow[l]
\end{tikzcd}$$

If this diagonal homomorphism was zero, then the composition

$$\begin{tikzcd}
& & V(\mathbb{F}_1) \arrow[dl] \arrow[d] \\
V(\mathbb{E}_1) & V(\mathbb{E}\otimes\mathbb{W}) \arrow[l] & V(\mathbb{E}_1) \arrow[l]
\end{tikzcd}$$

would be too. However, $V(\mathbb{E}_1)$ splits off from $V(\mathbb{E}\otimes\mathbb{W})$ with unique central character, so the composition of the bottom two homomorphisms is the identity on $V(\mathbb{E}_1)$. It would then follow that the original singular homomorphism $V(\mathbb{F}_1)\to V(\mathbb{E}_1)$ that we are translating from is zero, which is absurd. Hence these diagonal homomorphisms must be non-zero, i.e.

$$\begin{tikzcd}
& V(\mathbb{F}_1\otimes\mathbb{W}^*) \arrow[dl] \arrow[d] \\
V(\mathbb{E}) & V(\mathbb{E}_1\otimes\mathbb{W}^*) \arrow[l]
\end{tikzcd}$$

is non-zero. Finally, if we assume that no homomorphism $V(\mathbb{F}')\to V(\mathbb{E})$ exists, then when we compose the diagonal homomorphism with the inclusion

$$V(\mathbb{F})\to V(\mathbb{F}_1\otimes\mathbb{W}^*)$$

we necessarily get a non-zero composition $V(\mathbb{F})\to V(\mathbb{E})$.

Thus under the right conditions, it will be possible to translate the singular homomorphism $V(\mathbb{F}_1)\to V(\mathbb{E}_1)$ to obtain the regular one $V(\mathbb{F})\to V(\mathbb{E})$, even though translating in the reverse direction gives zero.

For example, translation of the non-standard homomorphism

$$V(\mathbb{E}_{-12\lambda_1+3\lambda_2+\lambda_5})\to V(\mathbb{E}_{-7\lambda_1+\lambda_3+2\lambda_6})$$

from the singular $(\lambda_2-\lambda_6)$-pattern by $\mathbb{W}$ to get the regular non-standard initial homomorphism

$$V(\mathbb{E}_{-11\lambda_1+2\lambda_2+\lambda_4})\to V(\mathbb{E}_{-7\lambda_1+\lambda_4+2\lambda_6})$$

fits precisely into this scenario.

Using this one-way translation we can obtain all but five of the regular initial non-standard homomorphisms. These are the five homomorphisms

$$V(\mathbb{E}_{-8\lambda_1+4\lambda_5}) \to V(\mathbb{E}_0), \qquad V(\mathbb{E}_{-12\lambda_1}) \to V(\mathbb{E}_{-8\lambda_1+4\lambda_5}),$$
$$V(\mathbb{E}_{-9\lambda_1+2\lambda_3}) \to V(\mathbb{E}_{-5\lambda_1+2\lambda_4}), \qquad V(\mathbb{E}_{-11\lambda_1+2\lambda_4}) \to V(\mathbb{E}_{-9\lambda_1+2\lambda_3}),$$

and

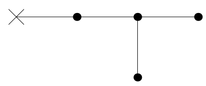
$$V(\mathbb{E}_{-12\lambda_1+3\lambda_2}) \to V(\mathbb{E}_{-6\lambda_1+3\lambda_6})$$

from the 0-pattern. Apart from the families of regular non-standard homomorphisms corresponding to these exceptional homomorphisms (i.e. from arbitrary regular λ-patterns), all other regular homomorphisms and all singular homomorphisms can be obtained by translating from the standard initial data.

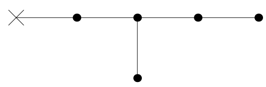
Since the homomorphisms in the standard initial data are all first order, they trivially lift to the semiholonomic case. Then the semiholonomic translation principle, Theorem 4 of [**4**], tells us that every homomorphism which can be obtained from the initial data by translation also lifts to the semiholonomic case.[5] Therefore at this stage the only homomorphisms for which we don't know whether a lift to the semiholonomic case exists are the five exceptional families mentioned above.

5. The Non-Existence of Lifts of the First Exceptional Family

It is enough to show that a lift to the semiholonomic case does not exist for the first exceptional homomorphism $V(\mathbb{E}_{-8\lambda_1+4\lambda_5}) \to V(\mathbb{E}_0)$. Notice that this homomorphism looks remarkably like the long homomorphism in the eight-dimensional conformal case. Ignoring the fifth nodes of the terms in the classifying pattern, what we get is precisely the classifying pattern for the parabolic Lie subalgebra

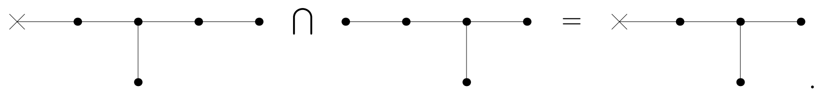


sitting inside $\mathfrak{so}(10,\mathbb{C})$, which is the eight-dimensional conformal case studied in [**4**]. Indeed, $\mathfrak{so}(10,\mathbb{C})$ itself sits inside $\mathfrak{g}$ as a Lie subalgebra: if $X \in \mathfrak{g}$ belongs to the root space $X_{a_1\alpha_1+\cdots+a_6\alpha_6}$, then the subalgebra $\{X \in \mathfrak{g} | a_5 = 0\}$ with Cartan subalgebra $\langle h_1, h_2, h_3, h_4, h_6\rangle$ is isomorphic to $\mathfrak{so}(10,\mathbb{C})$. Furthermore, the parabolic



[5] Theorem 4 of [**4**] applies to usual applications of the translation principle, but the argument applies equally well to one-way translations.

intersects this $\mathfrak{so}(10,\mathbb{C})$ subalgebra in the parabolic

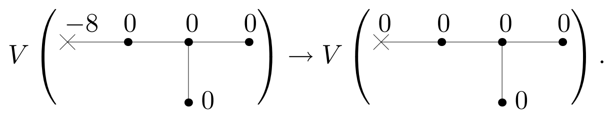


Now let us rewrite the highest weights of the relevant Verma modules in terms of the simple roots $\{\alpha_1,\ldots,\alpha_6\}$ instead of the fundamental weights $\{\lambda_1,\ldots,\lambda_6\}$:

$$-8\lambda_1+4\lambda_5=(-8,-8,-8,-4,0,-4) \quad \text{and} \quad 0=(0,0,0,0,0,0).$$

Recall that to find a Verma module homomorphism $V(\mathbb{E}_{-8\lambda_1+4\lambda_5})\to V(\mathbb{E}_0)$, we need to find a maximal (that is, killed by all raising operators) element $v=q\otimes w$ in $V(\mathbb{E}_0)$ with the appropriate weight, where w is a highest weight vector for $\mathbb{E}_0^*$ and q is an element of $\bigodot^*\mathfrak{g}_{-1}$ (or in the semiholonomic case, an element of $\bigotimes^*\mathfrak{g}_{-1}$). We see from above that achieving the appropriate weight will mean that the element q can involve only elements in $\mathfrak{g}_{-1}$ which belong to root spaces $X_{-a_1\alpha_1-\cdots-a_6\alpha_6}$ with $a_5=0$. In other words, q only involves lowering operators belonging to the subalgebra isomorphic to $\mathfrak{so}(10,\mathbb{C})$. Furthermore, we know a priori that v is killed by raising operators which do not lie in the $\mathfrak{so}(10,\mathbb{C})$ subalgebra (such raising operators must necessarily commute past the lowering operators occurring in q, and then kill the maximal weight vector w). Thus if v is maximal for $\mathfrak{so}(10,\mathbb{C})$ then it is maximal for all of $\mathfrak{g}$.

Suppose it is possible to choose such a maximal element v. Then precisely the same element q could be used to construct a maximal weight vector in the eight-dimensional conformal case. Thus there would exist a lift to the semiholonomic case of the so-called long homomorphism:



However, we know from Proposition 5 of [**4**] that the long homomorphism does not admit a lift to the semiholonomic case in any of the even-dimensional conformal cases. This contradiction implies that there cannot exist a lift to the semiholonomic case of the exceptional homomorphism $V(\mathbb{E}_{-8\lambda_1+4\lambda_5})\to V(\mathbb{E}_0)$.

6. Final Comments

As far as the other exceptional families are concerned, the situation is not so clear. While there appears to be similarities with the long homomorphisms in conformal geometry, the approach considered above only works for the first exceptional homomorphism. However, if we consider the geometric interpretation of Verma module homomorphisms as invariant operators then we can make some further observations. For example, there is a geometric notion of adjoint operators, and a recent result of Eastwood [**3**] shows that, at least for the holonomic case, this can be formulated purely in algebraic terms. A semiholonomic analogue of this result would imply the equivalence of the existence of lifts of the first and second exceptional homomorphisms, and of the third and fourth exceptional homomorphisms. In particular, we could conclude that the second exceptional family does not admit lifts to the semiholonomic case.

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