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UNIFORMLY WIGGLY DOMAINS AND RADIAL VARIATION

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1. Introduction

In a recent article [**5**] P. W. Jones and the author verified the so called Anderson conjecture for conformal maps on the unit disk; i.e., we proved that for each conformal map φ on the unit disk there exists an angle β such that

$$\int_0^1 |\varphi''(re^{i\beta})|\, dr < \infty. \tag{1.1}$$

The conjecture was formulated by Anderson in 1971 [**1**] and popularized by Ch. Pommerenke's book [**8**] and N. Makarov's survey [**7**]. For an early discussion of Anderson's conjecture see also the collection of problems edited by Ch. Pommerenke in 1972 [**9**].

In this note we present a class of conformal maps for which (1.1) can be given an elementary proof. We say that a domain Ω is uniformly wiggly if the corresponding Riemann map $\varphi\colon \mathbf{D} \to \Omega$ satisfies the following condition.

(1.2) For each $w \in \mathbf{D}$ there exists $z \in \mathbf{D}$ with $|z - w| \le (1 - |w|)/2$ and $(1 - |z|)|\varphi''(z)|/|\varphi'(z)| > \delta$, where $\delta > 0$ is independent of w.

In [**4**] one finds a geometric description of these domains that motivates the terminology. We consider these domains here for the following reasons.

(1) Important and well known examples like the von Koch curve, hyperbolic Julia sets and limit sets of degenerate Kleinian groups bound uniformly wiggly domains. (See [**7**], [**10**], [**3**].)
(2) A lower estimate in the law of the interated logarithm for $b = \log|\varphi'|$ holds precisely when Ω is uniformly wiggly. (See [**7**].)
(3) The special case of uniformly wiggly domains illustrates the proof of the general case given in [5]; in particular the role played by the stopping time Lipschitz domains and the estimates coming from J. Bourgain's theorem [**2**].

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2. Uniformly Wiggly Domains and Radial Variation

We start the proof by first collecting simple consequences of condition (1.2).

Let $n = n(R, \delta)$ be a large integer, its value will be specified later. For $z = re^{i\theta}$ let

$$G(z) = \left\{ \rho e^{i\psi} : |\psi - \theta| < 1 - r,\ (1-r)2^{-n} \leq (1-\rho) < 2(1-r) \right\}.$$

Let $R(z)$ denote the boundary of $G(z)$ and let

$$O(z) = R(z) \cap \{w : 1 - |w| > (1 - |z|)/4\}$$

and

$$U(z) = R(z) \setminus O(z).$$

Let h_z be the harmonic measure of $G(z)$ evaluated in z. Using square functions, Koebe's distortion estimates and (1.2) we verify that that $b = \log |\varphi'|$ satisfies the following estimates. (See [**4**].)

$$\left(\int_{R(z)} |b(w) - b(z)|^2 \, dh_z(w) \right)^{1/2} \geq C_0 n, \tag{2.1}$$

$$\left(\int_{R(z)} |b(w) - b(z)|^4 \, dh_z(w) \right)^{1/4} \leq C_0 n, \tag{2.2}$$

$$\left(\int_{O(z)} |b(w) - b(z)|^2 \, dh_z(w) \right)^{1/2} \leq C_0, \tag{2.3}$$

$$\int_{R(z)} (b(w) - b(z)) dh_z(w) = 0. \tag{2.4}$$

Below we use the following simple observation which in the special case of uniformly wiggly domains replaces the following components used in [**5**] for the proof of the general case: Bourgain's estimates from [**2**], Beurling's estimates on harmonic measure and the analysis of stopping time Lipschitz-domains.

Lemma 1. *Let (R, μ) be a probability space and let $b \colon R \to \mathbb{R}$ be square integrable, R^- denotes $\{w : b(w) \leq 0\}$. Let $O \subseteq R$ and $U = R \setminus O$. Suppose that the following conditions hold:*

$$\left(\int_R b^2 \, d\mu \right)^{1/2} \leq C_2 \int_R |b| \, d\mu, \tag{2.5}$$

$$\int_O b^2 \, d\mu \leq \tau \int_R b^2 \, d\mu, \tag{2.6}$$

$$\int_R b \, d\mu = 0. \tag{2.7}$$

Then, the following lower estimate holds

$$\int_{U\cap R^-} b^2\,d\mu \geq \left(\frac{C_2^{-2}}{4} - \tau\right)\int_R b^2\,d\mu. \tag{2.8}$$

Proof. By (2.5) and (2.7) we have

$$\left(\int_{R^-} b^2\,d\mu\right)^{1/2} \geq \int_{R^-} |b|\,d\mu = \frac{1}{2}\int_R |b|\,d\mu \geq \frac{C_2^{-1}}{2}\left(\int_R b^2\,d\mu\right)^{1/2}. \tag{2.9}$$

From the identity $R^- = \{R^- \cap U\} \cup \{R^- \cap O\}$ it follows that $R^- \cap U \subseteq R^- \setminus O$. Hence, by (2.7) and (2.9)

$$\int_{R^-\cap U} b^2\,d\mu \geq \int_{R^-} b^2\,d\mu - \int_O b^2\,d\mu \geq \frac{c_2}{4}\int_R b^2\,d\mu - \tau\int_R b^2\,d\mu.$$

□

Remarks. 1. Condition (2.5) follows from

$$\left(\int_R b^4\,d\mu\right)^{1/4} \leq C_4\left(\int_R b^2\,d\mu\right)^{1/2}. \tag{2.10}$$

We can take $C_2 = C_4^6$. Indeed, Hölder's inequality applied to $u^2 = |u|^{2/3}|u|^{4/3}$ gives

$$\int_R b^2\,d\mu \leq \left(\int_R |b|\,d\mu\right)^{2/3}\left(\int_R b^4\,d\mu\right)^{1/3}.$$

By (2.10),

$$\int_R b^2\,d\mu \leq C_4^4\left(\int_R |b|\,d\mu\right)^{2/3}\left(\int_R b^2\,d\mu\right)^{1/3}.$$

Or

$$\left(\int_R b^2 d\mu\right)^{1/2} \leq C_4^4\left(\int_R |b| d\mu\right)^{2/3}.$$

This gives $C_2 = C_4^6$.

2. Let $\varphi\colon \mathbf{D} \to \Omega$ satisfy (1.2), and let $b = \log|\varphi'|$. For $z \in \mathbf{D}$ let w_z be the harmonic measure on the boundary of $G(z)$, evaluated at z. Then the function $w \to b(w) - b(z)$ on the probability space $(R(z), h_z)$ satisfies the hypothesis of Lemma 1. This follows from (2.1)–(2.4), and the previous remark.

Theorem 1. *When Ω is uniformly wiggly then the Riemann map $\varphi : \mathbf{D} \to \Omega$ satisfies* (1.1).

Remark. As we pointed out above, this theorem holds true for any conformal map. (See [**5**].) The point made in this paper is that for uniformly wiggly domains we obtain an elementary proof which is given below. Its ingredients are the measure theoretic Lemma 2 and Remark 2.

Proof of Theorem 1. Fix $\eta > 0$, then for n large enough it follows from the meanvalue property of harmonic functions that there exists $z_1 \in \mathbb{D}$ with $2^{-n} \geq |z_1| > 1 - 1/8$ and

$$b(z_1) - b(0) < -\eta.$$

Suppose we have already constructed $z_1, z_2, \ldots, z_l$ so that for $k \leq l$,

$$\begin{aligned} &b(z_k) - b(z_{k-1}) < -\eta, \\ &z_k \in U(z_{k-1}) \text{ and } 1 - |z_k| < \frac{1}{2}(1 - |z_{k-1}|). \end{aligned} \tag{2.11}$$

Then consider $G(z_l)$. It follows from Remark 2 and Lemma 1 that there exists $z_{l+1} \in U(z_l)$ such that $b(z_{l+1}) - b(z_l) < -\eta$. Summing the corresponding telescoping series gives

$$b(z_k) < -k\eta \tag{2.12}$$

for $k \in \mathbf{N}$. Next, by (2.11) the sequence z_k converges to a point in $\mathbb{T}$, call it $e^{i\beta}$. We fix k, and let $L_k = \{z \in (0, e^{i\beta}) : |z_k| \geq |z| \geq |z_{k+1}|\}$. Again by (2.11), for $z \in L_k$ we have $|z - z_k| \leq (1 - |z|)2^n$. Now we recall that $|\nabla b(z)|(1 - |z|) \leq 6$, for $z \in \mathbf{D}$, (see [**8**]). By (2.12) this gives $b(z) < -k\eta + n$, for $z \in L_k$, and $\int_{L_k} |\nabla b(\zeta)|\,|d\zeta| \leq 24n$. Summing over k we have, for $z \in (0, e^{i\beta})$,

$$b(z) < -\frac{1}{4n} \int_{(0,z)} |\nabla b(\zeta)|\,|d\zeta| + 24n.$$

A simple change of variables transforms this estimate into (1.1). Indeed, we observe that $2|\varphi''| = |\nabla b| e^b$ and write,

$$\begin{aligned} 2\int_0^1 |\varphi''(re^{i\beta})|\,dr &= \int_0^1 |\nabla b(re^{i\beta})| e^{b(re^{i\beta})}\,dr \\ &\leq c_1 \int_0^1 |\nabla b(re^{i\beta})| e^{-c_2 \int_0^r |\nabla b(\rho e^{i\beta})|\,d\rho}\,dr \\ &\leq c_3. \end{aligned}$$

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