EXISTENCE OF CONSERVATION LAWS IN NILPOTENT CASE

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ABSTRACT. Using the Spencer-Goldschmidt version of the Cartan-Kähler theorem, we prove the local existence of conservation laws for analytical quasi-linear systems of two independent variables in the nilpotent and 2-cyclic case.

INTRODUCTION

A conservation law for a (1-1) tensor field $h$ on a manifold $M$, $\dim M = n$, is a 1-form $\theta$ which satisfies $d\theta = 0$ and $dh^*\theta = 0$, where $h^*$ is the transpose of $h$: $h^*\theta := \theta \circ h$. Conservation laws arise, for example, in the following classical problem. Consider a system of $n$ quasi-linear equations in two independent variables:

\begin{equation}
\frac{\partial x^i}{\partial u} + h^i_j(x) \frac{\partial x^j}{\partial v} = 0 \quad (i, j = 1, \ldots, n).
\end{equation}

If $\theta := \lambda_i(x)dx^i$ is a conservation law with respect to the (1-1) tensor field $h$ defined by the matrix $h^i_j$, there exist locally two functions $f$ and $g$ so that $\theta = df$ and $h^*\theta = dg$, (i.e. $\lambda_i = \frac{\partial f}{\partial x^i}$ and $h^i_j\lambda_i = \frac{\partial g}{\partial x^j}$), and we have

\[ 0 = \lambda_i \frac{\partial x^i}{\partial u} + \lambda_i h^i_j(x) \frac{\partial x^j}{\partial v} = \frac{\partial f}{\partial x^j} \frac{\partial x^j}{\partial u} + \frac{\partial g}{\partial x^j} \frac{\partial x^j}{\partial v} = 0. \]

Then for any solution $x^i(u, v)$ of the system (*), we have

\[ \frac{\partial f(x(u, v))}{\partial u} + \frac{\partial g(x(u, v))}{\partial v} = 0, \]

and it contains a conservation law in the sense of Lax [10].

Locally, giving a conservation law is equivalent to giving a function $f$ such that $(dh^*d)(f) = 0$. Thus the study of the local existence of conservation laws is equivalent (in an analytic context) to the study of the formal integrability of the differential operator $dh^*d$.

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This problem has already been studied by Osborn, who, using Cartan’s theory of exterior differential systems, showed the existence of conservation laws when \( h \) has constant coefficients in a suitable coordinate system \((7)\).

In a paper published in 1964, Osborn \((8)\) proved the formal Integrability of the operator \( dh^* d \) in the case when \( h \) is cyclic and if there exists a generator \( v^1 \) such that \( v^1, \ldots, h^{n-1} v^1 \) commutes in the sense of the square bracket.

Using the theory presented by Spencer and Goldschmidt \((4, 9)\), we improve in \((2)\) the case when \( h \) is cyclic, by getting rid of the supplementary condition given by Osborn. Recently, we show in \((5)\) the following theorem:

**Theorem.** Suppose that \( h \) is nilpotent of order \( p \), \((p \geq 2)\), analytic and such that \([h, h] = 0\). Fix \( x_0 \in M \). Then there exists a neighborhood \( U \) of \( x_0 \) such that any \( x \in U \) admits a “complete system” of conservation laws (i.e. every \( \omega_0 \in T^*_x (M) \) can be prolonged in a germ of conservation laws) if and only if \( \ker h, \ker h^2, \ldots, \ker h^{p-1} \) are involutive.

In this case the operator \( dh^* d \) is completely integrable \((5)\).

The main result of the present paper, whose essential ideas were given in \((5)\), can be expressed as following theorem:

**Theorem.** Suppose that \( h \) is nilpotent of order \( p \), \((p \geq 2)\), analytic, \([h, h] = 0\) and such that \( \dim (\text{Im} h^{p-1}) \geq \dim (\ker h) - 1 \). Fix \( x_0 \in M \). Then there exists a neighborhood \( U \) of \( x_0 \) such that any \( x \in U \) admits a “complete system” of conservation laws.

**Corollary.** Suppose that \( h \) is nilpotent of order \( p \), \((p \geq 2)\), analytic, \([h, h] = 0\) and such that \( h \) is 2 cyclic, \((1,1)\) form. Fix \( x_0 \in M \). Then there exists a neighborhood \( U \) of \( x_0 \) such that any \( x \in U \) admits a “complete system” of conservation laws.

1. **Algebraic preliminaries**

Using Frölicher-Nijenhuis formalism \((3)\), we know that for any point \( x \in M \) and for any \((1,1)\) tensor field \( h \) there exists a neighborhood \( U \) of \( x \) such that \( h \) decomposes \( TU \) as a direct sum of the cyclic subspaces \( V_i, i = 1, \ldots, s \) stable for \( h \), (i.e. the restriction of \( h \) to \( V_i \) is cyclic) \((1, 7, 8)\). Let \( q_i \) designate the dimension of \( V_i \) at \( x \) and at any point in \( U \). We suppose that \( V_i, i = 1, \ldots, s \) are arranged in such a way that \( q_1 \geq q_2 \geq \cdots \geq q_s \). In this and following section we design by \( v_i^1 \) a generator of \( V_i \) (for \( i = 1, \ldots, s \)) and denote \( v_i^{\alpha} := h^{\alpha-1} v_i^1, \alpha_i = 1, \ldots, q_i \). The vectors \( \{(v_1^{\alpha})_{\alpha_1 = 1, \ldots, q_1}, \ldots, (v_s^{\alpha})_{\alpha_s = 1, \ldots, q_s}\} \equiv \{v_i^{\alpha}\}_{i=1, \ldots, s, \alpha = 1, \ldots, q_i} \) form a basis of \( TU \) which called “adapted” to the decomposition into cyclic subspaces. By convention, we write \( v_i^\beta = 0 \) for \( \beta > q_i \).
Proposition 1.1. If $h$ is nilpotent of order $p$ and $r \in \{1, \ldots, p\}$, we have:

1. $\ker h^r$ is generated by $\{v_i^{\alpha+q_i-r}\}_{\alpha=1,\ldots,q_i}^{i=1,\ldots,s}$
2. $\text{Im } h^r$ is generated by $\{v_i^{\alpha+q_i}\}_{\alpha=1,\ldots,q_i}^{i=1,\ldots,s}$
3. $\dim \ker h = s$.

Proof. Conformally to the introduction of this section we can write the following table (5), which explain the relation between the elements of the set $\{v_i^{\alpha}\}_{\alpha=1,\ldots,q_i}^{i=1,\ldots,s}$. In fact:

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Cyclic Subspaces</th>
<th>Sequence Defined by $h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_1$</td>
<td>$V_1$</td>
<td>$v_1^1 \overset{h}{\rightarrow} v_1^2 \overset{h}{\rightarrow} \ldots \overset{h}{\rightarrow} v_1^q_i \overset{h}{\rightarrow} 0$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$q_i$</td>
<td>$V_i$</td>
<td>$v_i^1 \overset{h}{\rightarrow} v_i^2 \overset{h}{\rightarrow} \ldots \overset{h}{\rightarrow} v_i^q_i \overset{h}{\rightarrow} 0$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$q_s$</td>
<td>$V_s$</td>
<td>$v_s^1 \overset{h}{\rightarrow} \ldots \overset{h}{\rightarrow} v_s^q_s \overset{h}{\rightarrow} 0$</td>
</tr>
</tbody>
</table>

We prove this proposition by simple application of this table (5).

Definition 1. We call a Nijenhuis-manifold $(M, h)$ every $C^\infty$ manifold $M$ equipped with a $(1-1)$ tensor field $h$ such that $[h, h] = 0$. $[h, h]$ being the Nijenhuis square bracket of $h$ defined by:

$$\frac{1}{2}[h, h](X,Y) := [hX, hY] + h^2[X, Y] - h[hX, Y] - h[X, hY] \quad \forall X, Y \in TM.$$

Proposition 1.2. On the Nijenhuis-manifold $(M, h)$ we have:

$$h^\alpha[X, Y] = -\sum_{j=1}^{\alpha-1} [h^{\alpha-j}X, h^jY] + \sum_{j=0}^{\alpha-1} h[h^{\alpha-j-1}X, h^jY]$$

$\forall \alpha = 1, \ldots$ and $\forall X, Y \in TM$.

Proof. It is easy to prove by induction the proposition, which holds when $[h, h] = 0$. In fact, it is true for $\alpha = 2$. Suppose it is true up the order $\alpha - 1$. Then
∀X, Y ∈ TM; ∀α = 1, 2, . . . we have:
\[ h^α[X, Y] = hh^{α-1}[X, Y] = -\sum_{j=1}^{α-2} h[h^{α-j-1}X, h^jY] + \sum_{j=0}^{α-2} h^2[h^{α-j-2}X, h^jY] \]
\[ = -\sum_{j=1}^{α-2} h[h^{α-j-1}X, h^jY] - \sum_{j=1}^{α-1} [h^{α-j}X, h^jY] + \sum_{j=0}^{α-1} h[h^{α-j-1}X, h^jY] \]
\[ + \sum_{j=1}^{α-1} h[h^{α-j-1}X, h^jY] - h[X, h^{α-1}Y] \]
\[ = -\sum_{j=1}^{α-1} [h^{α-j}X, h^jY] + \sum_{j=0}^{α-1} h[h^{α-j-1}X, h^jY]. \]

\[ \square \]

2. Complete Integrability of dh*d in the Nilpotent Case

Suppose, in this section, that (M, h) is a Nijenhuis-manifold, h is nilpotent and decomposes TM in s cyclic subspaces. Using the notations of section 1 we have:

**Proposition 2.3.** The subspaces ker h^r; r = 1, . . . , p − 1 are involutive if and only if ∀i, j = 1, . . . , s such that j ≥ i, we have; \([v^α_i, v^β_j] \in ker h^h\) for α = 1, . . . , q_i, β = 1, . . . , q_j.

**Proof.** The condition is sufficient. Let r ∈ {1, . . . , p − 1}. ker h^r is involutive, \(X := h^{q_i-r}(v^i_1), Y := h^{q_j-r}(v^j_1)\) where \(r'' \geq r\) and \(r' \geq r\), be two elements of ker h^r. We suppose that i, j are arranged in such a way i ≤ j. If \(r'' \geq r' \geq r\) we have:
\[ 0 = h^0[v^1_i, h^{q_i-r'''}(v^j_1)] \]
\[ = -\sum_{u=1}^{q_i-1} [h^{q_i-u}(v^i_1), h^{q_i-r'''+u}(v^j_1)] + \sum_{u=0}^{q_j-1} h[h^{q_i-u-1}(v^i_1), h^{q_j-r'''+u}(v^j_1)] \]
\[ = -\sum_{u=1}^{r''-1} [h^{q_i-u}(v^i_1), h^{q_j-r'''+u}(v^j_1)] + \sum_{u=0}^{r'-1} h[h^{q_i-u-1}(v^i_1), h^{q_j-r'''+u}(v^j_1)] \]
\[ = -\sum_{u=1}^{r'-1} [h^{q_j-r'}(v^i_1), h^{q_j-r'''}(v^j_1)] + \sum_{u=0}^{r'-'-1} h[h^{q_j-r'-1}(v^i_1), h^{q_j-r'''}(v^j_1)] = h^r[X, Y]. \]

We deduce that \([X, Y] \in ker h^r\) and consequently \([X, Y] \in ker h^r\) because \(r' \leq r\). Similarly, if \(r' \geq r'' \geq r\), then
\[ 0 = h^q[v^{r''-r'+1}_1, v^{q_j-r'''+1}_j] = h^q[h^{r''-r'+1}_1, h^{q_j-r'''}(v^j_1)] = h^{r''}[X, Y]. \]
Consequently $[X, Y] \in \ker h^r$. Therefore $\ker h^r$ is involutive for every natural integer $r$. Conversely, let $v_i^\alpha \in \ker h^{q_i}, \ v_j^\beta \in \ker h^{q_j}, \ i \leq j$. We deduce that $\ker h^{q_j} \subseteq \ker h^{q_i}$ since $q_i \geq q_j$ but $\ker h^{q_j}$ is involutive, then $[v_i^\alpha, v_j^\beta] \in \ker h^{q_j}$.

**Theorem 2.1.** Suppose that $h$ is nilpotent of order $p$ ($p \geq 2$), analytic, $\{h, h\} = 0$ and such that $\dim(\text{Im} \ h^{p-1}) \geq \dim(\ker h) - 1$. Fix $x_0 \in M$. Then there exists a neighborhood $U$ of $x_0$ such that any $x \in U$ admits a “complete system” of conservation laws.

**Proof.** $\dim(\text{Im} \ h^{p-1}) \geq \dim(\ker h) - 1$ implies that $\dim V_i = q_i = p$ for $i = 1, \ldots, s - 1$. In the other hand, all the cyclic subspaces but the last are of the same dimension. In this case the order of nilpotence of $h$ is equal to $p$, which implies that the square bracket of two arbitrary vector fields, at point $x_0$ is an element of $\ker h^{q_s}_{x_0} = T_{x_0} M$. Then, $\forall i, j = 1, \ldots, s$ such that $j \geq i$, we have: $[v_i^\alpha, v_j^\beta] \in \ker h^{q_j}$ for $\alpha = 1, \ldots, q_i, \ \beta = 1, \ldots, q_j$. In particular case, if $j = i = s$ the two vectors $v_s^\alpha, v_s^\beta$ are in the cyclic subspace $V_s$, so the bracket of the two vectors is an element of $V_s$ then $[v_s^\alpha, v_s^\beta] \in \ker h^{q_s}$. This allows us to apply the previous proposition and say that the operator $dh^*d$ is completely integrable. □

**Corollary 2.1.** If $h$ is nilpotent of order $p$ ($p \geq 2$), analytic, $\{h, h\} = 0$ and such that $h$ is 2-cyclic, then the operator $dh^*d$ is completely integrable.

**Proof.** It’s particular case of the previous theorem. In fact $s - 1 = 1$ and $\dim V_1 = p$. □

**References**


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