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CHARACTERIZATIONS OF SERIES IN BANACH SPACES

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ABSTRACT. In this paper we prove several new characterizations of weakly unconditionally Cauchy series in Banach spaces and in the dual space of a normed space. For a given series ζ, we consider the spaces $\mathcal{S}(\zeta)$, $\mathcal{S}_w(\zeta)$ and $\mathcal{S}_0(\zeta)$ of bounded sequences of real numbers $(a_i)_i$ such that the series $\sum_i a_i x_i$ is convergent, weakly convergent or $*$-weakly convergent, respectively. By means of these spaces we characterize conditionally and weakly unconditionally Cauchy series.

1. INTRODUCTION

The normed spaces of bounded sequences, convergent sequences, null sequences and eventually null sequences of real numbers, endowed with the sup norm, will be denoted, as usual, by ℓ_∞, c, c_0 and c_{00}, respectively.

Let us consider a real normed space X and $\sum_i x_i$ a series in X.

It is well known ([**1**], [**2**], [**3**] and [**5**]) that:

1. A weakly unconditionally Cauchy series in a Banach space can be characterized as a series $\sum_i x_i$ such that, for every null sequence $(t_i)_i$, $\sum_i t_i x_i$ is convergent.
2. In a normed space X, $\sum_{i=1}^\infty x_i$ is a weakly unconditionally Cauchy series if and only if the set

$$E = \left\{ \sum_{i=1}^n \alpha_i x_i : \ n \in \mathbb{N}, \ |\alpha_i| \le 1, \ i \in \{1, \dots, n\} \right\} \tag{1.1}$$

is bounded.
3. If X is a Banach space then the following conditions are equivalent:

(a) There exists a weakly unconditionally Cauchy series which is convergent, but is not unconditionally convergent, in X.

(b) There exists a weakly unconditionally Cauchy series which is weakly convergent, but is not convergent.

(c) There exists a weakly unconditionally Cauchy series which is not weakly convergent.

(d) The space X has a copy of c_0.

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Many studies have been made on the behaviour of a series of the form $\sum_i a_i x_i$, where $(a_i)_i$ is a bounded sequence of real numbers, for instance: the former characterization of weakly unconditionally Cauchy series, the characterization of unconditionally convergent series through the BM-convergence and the perfect convergence ([**3**] and [**4**]).

For any given series $\zeta = \sum_i x_i$ in X let us consider the sets:

1. $\mathcal{S} = \mathcal{S}(\zeta)$, of sequences $(a_i)_i \in \ell_\infty$ such that $\sum_i a_i x_i$ converges.
2. $\mathcal{S}_w = \mathcal{S}_w(\zeta)$, of sequences $(a_i)_i \in \ell_\infty$ such that $\sum_i a_i x_i$ is weakly convergent.
3. $\mathcal{S}_0 = \mathcal{S}_0(\zeta)$, of sequences $(a_i)_i \in \ell_\infty$ such that $\sum_i a_i x_i$ is $*$–weakly convergent (i.e. $\left(\sum_{i=1}^n a_i x_i\right)_n$ converges with respect to the topology $\sigma(X^{**}, X^*)$).

These sets, endowed with the sup norm, will be called the spaces of **convergence**, of **weak convergence** and of **weak-$*$ convergence** of the series ζ, respectively.

It is clear that if the Banach space X does not have a copy of c_0, then every weakly unconditionally Cauchy series is unconditionally convergent and we have $\mathcal{S} = \mathcal{S}_w = \mathcal{S}_0 = \ell_\infty$. Therefore, unless otherwise specified, we will suppose that X has a subspace isomorphic to c_0.

2. Characterizations of Weakly Unconditionally Cauchy Series

In this section we prove, for a Banach spaces, several characterizations of weakly unconditionally Cauchy series.

Let us consider the linear map $\sigma_3\colon \mathcal{S}_0 \to X^{**}$ given by $\sigma_3((a_i)_i) = x^{**}$, where x^{**} is the $*$-weak ($\sigma(X^{**}, X^*)$) sum of the series $\sum_i a_i x_i$. We denote $\sigma_2 = \sigma_3|_{\mathcal{S}_w}$, $\sigma_1 = \sigma_3|_{\mathcal{S}}$ and $\sigma_{00} = \sigma_3|_{c_{00}}$. Clearly the images of these three maps are contained in X.

Theorem 1. *Let $\sum_i x_i$ be a series in a Banach space X. The following statements are equivalent: 1. $\mathcal{S}_0 = \ell_\infty$.*

2. The space $\mathcal{S}_0$ is complete.

3. The series $\sum_i x_i$ is a weakly unconditionally Cauchy series.

4. The map σ_3 is continuous.

5. The map σ_2 is continuous.

6. The map σ_1 is continuous.

7. The map σ_{00} is continuous.

Moreover, if any of these statements is verified then:

$$||\sigma_{00}|| = ||\sigma_0|| = ||\sigma_1|| = ||\sigma_2|| = ||\sigma_3|| = M,$$

where $\sigma_0\colon c_0 \to X$ is the continuous linear map defined by $\sigma_0\left((a_i)_i\right) = \sum_i a_i x_i$ and $M = \sup\left\{\|\sum_{i=1}^n \alpha_i x_i\| : |\alpha_i| \le 1,\, i \in \{1, \dots, n\},\, n \in \mathbb{N}\right\}$.

Proof.

(1) $\Rightarrow$ (2) This is obvious.

(2) $\Rightarrow$ (3) Since $\mathcal{S}_0$ is complete, it is clear that $c_0 \subseteq \mathcal{S}_0$. Hence, for every $(a_i)_i$ in c_0 and x^* in X^*, the real series $\sum_i a_i x^*(x_i)$ is convergent. Therefore the series $\sum_i x^*(x_i)$ is unconditionally convergent and $\sum_i x_i$ is a weakly unconditionally Cauchy series.

(3) $\Rightarrow$ (4) Since $\sum_i x_i$ is a weakly unconditionally Cauchy series, (1.1) implies that $M = \sup\{\|\sum_{i=1}^n \alpha_i x_i\| : |\alpha_i| \le 1,\, i \in \{1,\dots,n\},\, n \in \mathbb{N}\}$ is well defined. Therefore, for $(a_i)_i$ in $B_{\mathcal{S}_0}$ and $n \in \mathbb{N}$, we have $\sum_{i=1}^n a_i x_i \in B_{X^{**}}(0,M)$. Since the sequence $\left(\sum_{i=1}^n a_i x_i\right)_n$ is $*$–weakly convergent to $\sigma_3\left((a_i)_i\right) = \sum_i a_i x_i$ and $B_{X^{**}}(0,M)$ is $*$-weakly closed, we have $\sigma_3\left((a_i)_i\right) \in B_{X^{**}}(0,M)$. Hence $||\sigma_3|| \le M$.

(4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) These are obvious.

(7) $\Rightarrow$ (3) Since $\sigma_3\left(B_{c_{00}}\right) = E$, it follows that E is bounded, hence $\sum_i x_i$ is a weakly unconditionally Cauchy series.

(3) $\Rightarrow$ (1) Let us suppose that $(a_i)_i \in \ell_\infty$. We have $\sum_{k=1}^n \frac{a_k}{||(a_i)_i||} x_k \in E$, for every $n \in \mathbb{N}$, and there exists $r > 0$ such that $s_n = \sum_{k=1}^n a_k x_k \in B_{X^{**}}(0,r)$, for every $n \in \mathbb{N}$. Since $\sum_i a_i x_i$ is a weakly unconditionally Cauchy series we have $\sum_i |x^*(a_i x_i)| < \infty$, for $x^* \in X^*$. Therefore $(s_n)_n$ is a $*$-weakly convergent sequence.

It is clear that if $(a_i)_i \in \mathcal{S}_0$ is not the null sequence then we have $\sum_{k=1}^n \frac{a_k}{||(a_i)_i||} \in E$, for every $n \in \mathbb{N}$. Therefore, for $n \in \mathbb{N}$ and $x^* \in B_{X^*}$, $\|\frac{1}{||(a_i)_i||} x^*(s_n)\| \le M$. It follows that

$$||\sigma_3((a_i)_i)|| \le M||(a_i)_i||. \tag{2.2}$$

Therefore for every $(a_i)_i \in \mathcal{S}_0$ we have $||\sigma_3|| \le M$. As a consequence we have that

$$||\sigma_{00}|| \le ||\sigma_0|| \le ||\sigma_1|| \le ||\sigma_2|| \le ||\sigma_3|| \le M.$$

On the other side, for every $\epsilon > 0$ there exist real numbers $\alpha_1, \dots \alpha_n$ such that $|\alpha_k| \le 1$, for $1 \le k \le n$, and $M - \epsilon < ||\sum_{k=1}^n \alpha_k x_k||$. Let us consider the sequence $(a_i)_i \in c_{00}$ defined by $a_i = \alpha_i$ if $i \in \{1,\dots,n\}$ and $a_i = 0$ if $i > n$. We have $M - \epsilon < ||\sigma_{00}||$ and we can conclude $||\sigma_{00}|| = ||\sigma_0|| = ||\sigma_1|| = ||\sigma_2|| = ||\sigma_3|| = M$.□

Let us consider a series $\sum_i x_i^*$ in the dual X^* of a normed space X and let us denote by $\mathcal{S}_{*w}(\zeta)$ the set of bounded sequences of real numbers $(a_i)_i$ such that the series $\sum_i a_i x_i^*$ is $*$-weakly convergent; i.e. it converges with respect to the topology $\sigma(X^*, X)$. This space, with the sup norm, is a normed space that we will be called the **space of $*$-weak convergence** of the series $\sum_i x_i^*$. Let us also consider the linear map $\sigma_* : \mathcal{S}_{*w} \to X^*$, defined by $\sigma_*((a_i)_i) = x^*$, where x^* is the $*$-weak sum of $\sum_i a_i x_i^*$.

Corollary 2. *Let X be a normed space and let $\zeta = \sum_i x_i^*$ be a series in X^*. Then, ζ is weakly unconditionally Cauchy series if and only if the linear map $\sigma_*\colon \mathcal{S}_{*w} \to X^*$ is continuous.*

Proof. It is easy to check that $\mathcal{S}_0 \subseteq \mathcal{S}_{*w}$. If ζ is a weakly unconditionally Cauchy series then Theorem 1 implies that $\mathcal{S}_0 = \mathcal{S}_{*w} = \ell_\infty$. Therefore $\sigma_* = \sigma_3$ is continuous.

Conversely, if $\sigma_*\colon \mathcal{S}_{*w} \to X^*$ is continuous then the restriction $\sigma_*|_{\mathcal{S}} : \mathcal{S} \to X^*$ is also continuous. Since $A = \{(\alpha_i)_i \in c_{00} : |\alpha_i| \le 1\} \subseteq \mathcal{S}$ is a bounded set and $\sigma_*|_{\mathcal{S}}(A) = E$, it follows that E is bounded. Hence $\sum_i x_i^*$ is a weakly unconditionally Cauchy series. □

Remark 3. It is clear that if $\sum_i x_i^*$ is a weakly unconditionally Cauchy series in X^* then the ranges of the continuous linear maps σ_{00}, σ_0, σ_1, σ_2 and σ_* are contained in X^* and $||\sigma_{00}|| = ||\sigma_0|| = ||\sigma_1|| = ||\sigma_2|| = ||\sigma_*|| = M$, where

$$M = \sup\left\{ \left\| \sum_{i=1}^{n} \alpha_i x_i^* \right\| : |\alpha_i| \le 1, \, i \in \{1, \ldots, n\}, \, n \in \mathbb{N} \right\}.$$

We also note that if ζ is not a weakly unconditionally Cauchy series then $\mathcal{S}_{*w} \not\subseteq \mathcal{S}_0$.

3. Spaces of Convergence and Weak Convergence of a Weakly Unconditionally Cauchy Series

When $\sum_i x_i$ is a weakly unconditionally Cauchy series we have $\mathcal{S}_0(\sum_i x_i) = \ell_\infty$; nevertheless the spaces $\mathcal{S}$ and $\mathcal{S}_w$ let us obtain some information on the series: these spaces allow us to characterize when a series is conditionally convergent and weakly unconditionally Cauchy.

Proposition 4. *Let X be a Banach space and let $\sum_i x_i$ be a series in X. The series $\sum_i x_i$ is conditionally convergent and weakly unconditionally Cauchy if and only if $c \subsetneq \mathcal{S} \subseteq \mathcal{S}_w \subsetneq \ell_\infty$.*

Proof. Let $(a_i)_i$ be a sequence in c that is convergent to $a \in \mathbb{R}$. It is clear that the series $\sum_i a_i x_i$ is convergent.

Let $\sum_k x_{i_k}$ be a non trivial absolutely convergent subseries of $\sum_i x_i$. Let $(b_i)_i$ be the sequence in $\ell_\infty \setminus c$ defined by

$$b_i = \begin{cases} 1, & \text{if } i = i_k \text{ for some } k \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

The series $\sum_i b_i x_i$ is convergent. Since there exists some subseries $\sum_j x_{i_j}$ of the series $\sum_i x_i$ which is not weakly convergent, we have that $\mathcal{S}_w \subsetneq \ell_\infty$.

Conversely, since $c \subseteq \mathcal{S}$, it follows that $\sum_{i=1}^\infty x_i$ is a convergent and weakly unconditionally Cauchy series. It is obvious that $\sum_i x_i$ is not an unconditionally convergent series. □

If $\sum_i x_i$ is a weakly unconditionally Cauchy series which is not convergent (this series may be weakly convergent or not) then $(x_i)_i$ has either a subsequence that converges to 0 or does not have any convergent subsequence.

Example 5. *1. Let $\sum_i x^{(i)}$ be the series in c_0 defined by*

$$x^{(2i-1)} = e^{(2i-1)}, \quad x^{(2i)} = \frac{1}{2i} e^{(2i)}, \quad i \geq 1,$$

where the sequence $\left(e^{(i)}\right)_i$ is the c_0-basis. It is obvious that $\sum_i x^{(i)}$ is not a weakly convergent series, but it is weakly unconditionally Cauchy and also that the subsequence $\left(x^{(2i)}\right)_i$ is convergent to 0.

2. Let $\sum_i x^{(i)}$ be the series in c_0 defined by

$$x^{(1)} = \frac{1}{2} e^{(1)}, \quad x^{(i)} = \frac{1}{2} e^{(i)} - e^{(i-1)}, \text{ for } i > 1.$$

Clearly $\sum_i x^{(i)}$ is a weakly unconditionally Cauchy series which is not weakly convergent. Moreover, the sequence $\left(x^{(i)}\right)_i$ does not have convergent subsequences.

3. Let $\sum_i x^{(i)}$ be the series in c_0 defined by

$$\begin{aligned} x^{(1)} &= e^{(1)}, \\ x^{(2)} &= \frac{1}{2} e^{(2)}, \\ x^{(2i+1)} &= -e^{(2i-1)} + e^{(2i+1)}, \quad \text{for } i \geq 1, \\ x^{(2i)} &= -\frac{1}{2i-2} e^{(2i-2)} + \frac{1}{2i} e^{(2i)}, \quad \text{for } i \geq 2. \end{aligned}$$

Clearly $\sum_i x^{(i)}$ is a weakly unconditionally Cauchy series which is weakly convergent to 0 and is not convergent. Nevertheless, $\left(x^{(2i)}\right)_i$ is a subsequence of $\left(x^{(i)}\right)_i$ that converges to 0.

4. Let $\sum_i x^{(i)}$ be the series in c_0 defined by

$$x^{(1)} = e^{(1)} \quad \text{and} \quad x^{(i)} = -e^{(i-1)} + e^{(i)}, \text{for } i > 1.$$

Clearly $\sum_i x^{(i)}$ is a weakly unconditionally Cauchy series which is weakly convergent to 0 and is not convergent. Nevertheless, $\left(x^{(i)}\right)_i$ does not have any convergent subsequence.

Remark 6. If X is a Banach space and $\sum_i x_i$ is a weakly unconditionally Cauchy series which does not converge in X then we can characterize the existence of basic subsequences of $(x_i)_i$ that are equivalent to the c_0-basis $\left(e^{(i)}\right)_i$.

Let $(x_{i_k})_k$ be a basic subsequence of $(x_i)_i$ equivalent to $\left(e^{(i)}\right)_i$. Let $[x_{i_k}]$ be the closed linear span of the $(x_{i_k})_k$. There exists an isomorphism

$$T \colon [x_{i_k}] \subsetneq X \to c_0$$

such that $T(x_{i_k}) = e^{(k)} \in c_0$, for every $k \in \mathbb{N}$. Thus

$$1 = \left\|e^{(k)}\right\| = \|T(x_{i_k})\| \leq \|T\| \, \|x_{i_k}\| .$$

Therefore $\|x_{i_k}\| \geq \frac{1}{\|T\|}$, for every $k \in \mathbb{N}$. Hence $\inf\{\|x_{i_k}\| : k \in \mathbb{N}\} > 0$. This proves that $(x_i)_i$ is not convergent to 0.

Conversely, if $\lim_{i\to\infty} x_i \neq 0$, it is obvious that there exists an infinite subset $N \subsetneq \mathbb{N}$, such that $\inf_{i\in N} \|x_i\| = \delta > 0$. Since $\sum_{i\in N} x_i$ is a weakly unconditionally Cauchy series, then $x_i \xrightarrow{w} 0$ as $i \to \infty$ and $i \in M$ (i. e. $(x_i)_i$ is weakly convergent to 0). It follows ([**3**]) that there exists a basic subsequence $(x_{i_k})_k$ of $(x_i)_{i\in M}$.

It is clear that $\sum_k x_{i_k}$ is a weakly unconditionally Cauchy subseries and that $\inf_{k\in\mathbb{N}} \|x_{i_k}\| > 0$. This proves ([**2**]) that $(x_{i_k})_k$ is equivalent to $\left(e^{(k)}\right)_k \subseteq c_0$.

Therefore, $(x_i)_i$ **has a basic subsequence equivalent to the** c_0**-basis** $\left(e^{(i)}\right)_i$ **if and only if** $\lim_{i\to\infty} x_i \neq 0$**.**

Proposition 7. *Let X be a Banach space and let $\zeta = \sum_i x_i$ be a weakly unconditionally Cauchy series in X which is not convergent.*

1. *If there exists a subsequence $(x_{i_k})_k$ of $(x_i)_i$ such that $\lim_{k\to\infty} x_{i_k} = 0$ then*

$$c_0 \subsetneq \mathcal{S}(\zeta) \subsetneq c_0 \cup (\ell_\infty \setminus c).$$

   Moreover, if $\sum_i x_i$ is also weakly convergent then $c \subsetneq \mathcal{S}_w(\zeta) \subsetneq \ell_\infty$ and if $\sum_i x_i$ is not weakly convergent then $c_0 \subsetneq \mathcal{S}_w(\zeta) \subsetneq c_0 \cup (\ell_\infty \setminus c)$.
2. *If $(x_i)_i$ does not have a convergent subsequence then $\mathcal{S}(\zeta) = c_0$. Moreover, if the series $\sum_i x_i$ is weakly convergent then $c \subseteq \mathcal{S}_w(\zeta) \subsetneq \ell_\infty$.*

Proof. 1. Since $\lim_{k\to\infty} x_{i_k} = 0$, then there exists an absolutely convergent subseries $\sum_r x_{i_r}$ of $\sum_i x_i$, such that $\|x_{i_r}\| < \dfrac{1}{2^r}$ for every $i_r \in \mathbb{N}$. It is clear that there exists $(b_i)_i \in \ell_\infty \setminus c$ such that $\sum_i b_i x_i = \sum_r x_{i_r}$.

Let us suppose that $(a_i)_i \in c \setminus c_0$. Let $a \neq 0$ be such that $\lim_{i\to\infty} a_i = a$. We have $\sum_{i=1}^n (a_i - a)x_i = \sum_{i=1}^n a_i x_i - a\sum_{i=1}^n x_i$. Therefore, the series $\sum_i a_i x_i$ does not converge and $(a_i)_i \notin \mathcal{S}$.

For any given absolutely convergent subseries $\sum_r x_{i_r}$ of $\sum_i x_i$, let $N = \{i_1, \ldots, i_r, \ldots\}$ and let us consider the set $M = \mathbb{N} \setminus N$. We have that $\sum_{i\in M} x_i$ does not converge and $\mathcal{S} \subsetneq c_0 \cup (\ell_\infty \setminus c)$.

Now, let us suppose that $\sum_i x_i$ is a weakly convergent series. For any given $(a_i)_i \in c$ we have that the series $\sum_i a_i x_i$ is also weakly convergent. Hence $c \subsetneq \mathcal{S}_w$. It is obvious that $\mathcal{S}_w \subsetneq \ell_\infty$.

On the other hand, let us suppose that the series $\sum_i x_i$ is not weakly convergent. Then, for every sequence $(a_i)_i \in c \setminus c_0$, the series $\sum_i a_i x_i$ can not be weakly convergent. Hence $(c \setminus c_0) \cap \mathcal{S}_w = \emptyset$ and $c_0 \subsetneq \mathcal{S}_w \subsetneq c_0 \cup (\ell_\infty \setminus c)$.

2. Let us prove that $\mathcal{S} \subseteq c_0$. Since there is not any subsequence of $(x_i)_i$ that is convergent to 0 we may suppose that $x_i \neq 0$ and therefore $\inf_{i\in\mathbb{N}} \|x_i\| > 0$. If the

series $\sum_i a_i x_i$ is convergent we have that $|a_i| \leq \frac{1}{\inf_{i\in\mathbb{N}} \|x_i\|} \|a_i x_i\| \longrightarrow 0$, because $0 \leq |a_i| \inf_{i\in\mathbb{N}} \|x_i\| \leq |a_i| \|x_i\| = \|a_i x_i\|$.

Therefore $\lim_{i\to\infty} a_i = 0$.

If $\sum_i x_i$ is a weakly convergent series then it is clear that $c \subseteq \mathcal{S}_w \subsetneq \ell_\infty$. □

Remark 8. We observe that:

1. We have $\mathcal{S}(\zeta) = c_0$ if and only if $\zeta = \sum_i x_i$ is a divergent weakly unconditionally Cauchy series and $(x_i)_i$ does not have any convergent subsequence.

2. If $\sum_i x_i$ is a weakly convergent series such that there exists some non trivial weakly convergent subseries $\sum_k x_{i_k}$ and furthermore $(x_i)_i$ does not have any convergent subsequence, then it is obvious that $c \subsetneq \mathcal{S}_w$.

3. Let $(x_i)_i$ be a sequence without convergent subsequences. If the series $\sum_i x_i$ is not weakly convergent but it has some non trivial weakly convergent subseries, then $c_0 \subsetneq \mathcal{S}_w \subsetneq c_0 \cup (\ell_\infty \setminus c)$.

Remark 9. If X is a Banach space and $\zeta = \sum_i x_i$ is a series in X we have that:

1. The space $\mathcal{S}(\zeta)$ is separable if and only if $(x_i)_i$ does not have any subsequence that converges to 0.

2. The space $\mathcal{S}(\zeta)$ has a copy of ℓ_∞ if and only if $(x_i)_i$ has some subsequence that converges to 0.

3. If ζ does not converge and $(x_i)_i$ does not have any subsequence that converges to 0 then the convergence of $\sum_i a_i x_i$ implies that $(a_i)_i \in c_0$. Therefore $\mathcal{S}(\zeta)$ is dense in c_0.

For any given a series ζ in a Banach space X, it is clear that $\mathcal{S} \subseteq \mathcal{S}_w$. But, when is $\mathcal{S} \subsetneq \mathcal{S}_w$?

There are some situations in which $\mathcal{S} \subsetneq \mathcal{S}_w$ holds. For instance, if X has Schur property then $\mathcal{S} = \mathcal{S}_w$. The reciprocal result is also true; i.e. if $\mathcal{S}(\zeta) = \mathcal{S}_w(\zeta)$ for any series ζ, then X satisfies Schur property. In fact, let $(z_i)_i$ be a sequence in X which is weakly convergent to $z \in X$ and let us consider the series $\sum_i x_i$ such that $x_i = z_i - z_{i+1}$, for every $i \in \mathbb{N}$. We have that $\sum_i x^*(x_i) = x^*(z_1) - x^*(z)$, for every $x^* \in X^*$. Therefore $1_\mathbb{N} \in \mathcal{S}_w(\sum_i^\infty x_i)$, where $1_\mathbb{N}$ is the constant sequence whose terms are equal to one. Hence $\sum_i x_i = z_1 - z$ and $\lim_i z_i = z$.

Nevertheless, the general problem about the relation between $\mathcal{S}$ and $\mathcal{S}_w$ is not yet solved. The following proposition is a partial result.

Proposition 10. *Let $\sum_i x_i$ be a convergent and weakly unconditionally Cauchy series in a Banach space X. Then, there exists a permutation Π such that $\mathcal{S}\left(\sum_i x_{\Pi(i)}\right) \subsetneq \mathcal{S}_w\left(\sum_i x_{\Pi(i)}\right)$.*

Proof. Let $(a_i)_i \in \mathcal{S}$ be such that $\sum_i a_i x_i$ is a convergent and weakly unconditionally Cauchy series. There exists a permutation Π such that $(a_{\Pi(i)})_i \notin \mathcal{S}\left(\sum_i x_{\Pi(i)}\right)$.

Let $y = \sum_i a_i x_i$. Since $\sum_i x^*(a_i x_i)$ is a real absolutely convergent series, for every $x^* \in X^*$, we have that y is the weak sum of the series $\sum_i a_{\Pi(i)} x_{\Pi(i)}$. Hence $(a_{\Pi(i)})_i \in \mathcal{S}_w \left(\sum_i x_{\Pi(i)}\right)$. □

The analysis we have given above can be extended, in a natural way, to series defined in the dual X^* of a normed space X. It is obvious that in X^* we have that $\mathcal{S}_w \subseteq \mathcal{S}_{*w}$.

With the same arguments as before, it is evident that X has Grothendieck property if and only if $\mathcal{S}_w(\gamma) = \mathcal{S}_{*w}(\gamma)$ for every series $\gamma = \sum_{n=1}^{\infty} x_n^*$ in X^*.

It is well known that if X is a Grothendieck space then any weakly unconditionally Cauchy series is unconditionally convergent. This result can also be deduced from Theorem 1, because if $\sum_i x_i^*$ is a weakly unconditionally Cauchy series, we have that $\ell_\infty = \mathcal{S}_{*w} = \mathcal{S}_w$ and then $\sum_i x_i^*$ is unconditionally convergent. Besides that, we can deduce that X^* does not have a copy of ℓ_∞.

In spite of these and others situations, the relation between $\mathcal{S}_w$ and $\mathcal{S}_{*w}$ is also an open problem.

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