ON SOME GENERALIZATIONS OF $LC$–SPACES

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Abstract. The aim of this paper is to extend the notion of $LC$-spaces, i.e. spaces whose Lindelöf subsets are closed. We will consider four weaker forms of this concept and investigate their relationships with $LC$-spaces as well as among themselves. Accordingly, we continue the study of $LC$-spaces and related spaces.

1. Introduction

Lindelöf spaces have always played a highly expressive role in topology. They were introduced by Alexandroff and Urysohn back in 1929 and their name is due to Lindelöf’s proof in 1903 that from any collection of open sets covering an euclidean space one can extract a countable subcollection covering the space.

Special classes of Lindelöf spaces such as hereditarily Lindelöf spaces and maximal Lindelöf spaces have had considerable impact on General Topology. A class of spaces that occurs in the study of maximal Lindelöf spaces is the notion of $LC$-spaces, — a concept having much in common with $P$-spaces.

A topological space whose Lindelöf subsets are closed is called an $LC$-space by Mukherji and Sarkar and by Gauld, Mrsevic, Reilly and Vamanamurthy. $LC$-spaces are also known as $L$-closed spaces. They generalize $KC$-spaces ($=$ compact subsets are closed) and Hausdorff $P$-spaces ($=$ $F_\sigma$-sets are closed) and so $T_1$ and anticom pact ($=$ compact subsets are finite). Note that cid-spaces have been called weak $LC$-spaces by Mukherji and Sarkar.

In recent years there has been a significant interest in $LC$-spaces. Examples of $LC$-spaces that are not $P$-spaces can be found in and . By making use of an example of Kunen of a rigid Tychonoff Lindelöf $P$-space, Henrikson and Woods provided an example of a Lindelöf Tychonoff cid-space that is not an $LC$-space. Generalizations of some results from can be found in a paper by Ganster and Jankovic where several examples are given as well.

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Grant and Reilly [9] considered the situation when an LC-space is discrete and when a Hausdorff LC-space is countably compact. Very recently, Dontchev and Ganster [5] showed that the product of two LC-spaces need not be an LC-space, and Ganster, Kanibir and Reilly [7] pointed out that a locally LC-space need not be an LC-space.

In this paper we consider LC-spaces from a more generalized point of view. We introduce some new classes of spaces that contain properly the class of LC-spaces. More precisely, we consider a modified version of \(C\)-spaces (= compact sets have compact closures) as the spaces in which every Lindelöf subset has Lindelöf closure. We also investigate spaces where all Lindelöf \(F_{\sigma}\) sets are closed, and spaces where between each Lindelöf set and its closure there lies a Lindelöf \(F_{\sigma}\)-set. In the last section we consider (weakly) locally Lindelöf spaces and their relationships to generalized LC-spaces.

Our terminology is standard. The closure and the interior of a subset \(A\) of a space \((X, \tau)\) are denoted by \(\text{cl } A\) and \(\text{int } A\), respectively (or \(\text{cl}_\tau A\) and \(\text{int}_\tau A\) if there is a possibility of confusion). The set of all positive integers is denoted by \(\omega\).

2. Generalized LC-spaces

**Definition 1.** A topological space \((X, \tau)\) is called an LC-space [16, 8] if every Lindelöf subset of \(X\) is closed.

Note that LC-spaces are also known under the name \(L\)-closed [11]. We will now introduce the following four generalizations of LC-spaces.

**Definition 2.** A topological space \((X, \tau)\) is called

1. an \(L_1\)-space if every Lindelöf \(F_{\sigma}\)-set is closed,
2. an \(L_2\)-space if \(\text{cl } L\) is Lindelöf whenever \(L \subseteq X\) is Lindelöf,
3. an \(L_3\)-space if every Lindelöf subset \(L\) is an \(F_{\sigma}\)-set,
4. an \(L_4\)-space if, whenever \(L \subseteq X\) is Lindelöf, then there is a Lindelöf \(F_{\sigma}\)-set \(F\) with \(L \subseteq F \subseteq \text{cl } L\).

Our first result summarizes some immediate consequences of Definition 2.

**Theorem 2.1.**

(i) If \((X, \tau)\) is an LC-space then \((X, \tau)\) is an \(L_i\)-space, \(i = 1, 2, 3, 4\).

(ii) \((X, \tau)\) is an LC-space if and only if it is an \(L_1\)-space and an \(L_3\)-space.

(iii) Every Lindelöf space is an \(L_2\)-space, and every \(L_2\)-space having a dense Lindelöf subset is Lindelöf.

(iv) Every space which is \(L_1\) and \(L_4\) is an \(L_2\)-space.

(v) Every \(L_2\)-space is an \(L_4\)-space, and every \(L_3\)-space is an \(L_4\)-space.

(vi) Every \(L_3\)-space is \(T_1\), and every \(T_1\) \(L_1\)-space is cid.
(vii) The property $L_3$ is hereditary, and the properties $L_1$, $L_2$ and $L_4$ are hereditary on $F_\sigma$-sets.

(viii) Every $P$-space is an $L_1$-space.

In 1979, Bankston \cite{2} introduced the so-called anti-operator on a topological space. A space $(X, \tau)$ is said to be anti-Lindelöf if each Lindelöf subset of $X$ is countable. Recall also that $(X, \tau)$ is called a $Q$-set space if each subset of $(X, \tau)$ is an $F_\sigma$-set. The proof of the following result is straightforward and hence omitted.

**Proposition 2.2.** (i) Every $T_1$ anti-Lindelöf space is an $L_3$-space. Hence every $T_1$, anti-Lindelöf $L_1$-space is an $LC$-space.

(ii) Every $Q$-set space is an $L_3$-space, hence so is every strongly $\sigma$-discrete and every regular submaximal space with countable Souslin number.

Note that, although $Q$-set spaces are $L_3$-spaces they need not be $LC$-spaces as the set of all integers with the cofinite topology shows. Our next result provides a condition under which $L_3$-spaces are $Q$-set spaces.

**Proposition 2.3.** Every hereditarily Lindelöf $L_3$-space is a $Q$-set space.

We now provide some examples to show that among the $L_i$-spaces there are no more implications than those listed in Theorem 2.1.

**Example 2.4.** Let $R$ be the set of reals and let $\tau$ be the usual topology. Then $(R, \tau)$ is a hereditarily Lindelöf $L_2$-space and thus an $L_4$-space, but neither an $LC$-space nor an $L_1$-space nor an $L_3$-space.

**Example 2.5.** Let $R$ be the set of reals and let $\tau$ be the rational sequence topology on $R$ (see , Example 65\cite{19}, Example 65). Then $(R, \tau)$ is a separable non-Lindelöf space where each point has a countable neighbourhood. Hence $(R, \tau)$ is anti-Lindelöf and so an $L_3$-space and also an $L_4$-space. Clearly $(R, \tau)$ is neither an $L_2$-space nor an $L_1$-space.

**Example 2.6** (see \cite{20}). Let $X$ be the set of reals and let $\tau$ be the density topology on $X$. It is consistent with the axioms of set theory that the only hereditarily Lindelöf subspaces are the countable ones\cite{20}. Since the density topology is perfect, all Lindelöf subsets are hereditarily Lindelöf and hence countable. Since $(X, \tau)$ is a cid-space it is thus an $LC$-space. Additionally we note that $(X, \tau)$ is neither Lindelöf nor separable.

Our next task is to show that there exists a Hausdorff $L_1$-space that is not an $LC$-space. For this we need some preparation. Let $R$ be the set of reals and let $\tau$ be the usual topology on $R$. $B \subseteq R$ is called a Bernstein set if $B$ and $R - B$ intersect every uncountable closed subset of $(R, \tau)$. $L \subseteq R$ is said to be a Lusin set if $L$ is uncountable and the intersection of $L$ with any meager subset of $(R, \tau)$ is at most countable. For the convenience of the reader let us mention some basic facts about these notions (see e.g. \cite{18}).
Proposition 2.7. (i) There exists a Bernstein subset of \((R, \tau)\), and if \(B\) is a Bernstein set then \(R - B\) is also a Bernstein set.

(ii) If \(B\) is a Bernstein set then \(B\) is a dense Baire subspace. In particular, for any open set \(U\), \(U \cap B\) is uncountable.

(iii) Under the continuum hypothesis (CH), every subset of \(R\) of 2\(^{nd}\) category contains a Lusin set.

Now let \(R = B_1 \cup B_2\) be the disjoint union of two Bernstein sets \(B_1\) and \(B_2\), and let \(\{G_n : n \in \omega\}\) be a countable base for \((R, \tau)\). For each \(n \in \omega\) choose a Lusin set \(L_n \subseteq B_1 \cap G_n\) and let \(L = \bigcup\{L_n : n \in \omega\}\). Then \(L\) is also a Luzin set such that for any nonempty open set \(U\) in \((R, \tau)\), \(U \cap L\) is uncountable. Let \(X = L \cup B_2\). We will define a new topology \(\sigma\) on \(X\) in the following way: a basic neighbourhood of \(x \in L\) is a set \(W_x\) containing \(x\) and having the form \(W_x = (U \cap L) - C\) where \(U\) is open in \((R, \tau)\) and \(C\) is countable. A basic neighbourhood of \(x \in B_2\) has the form \(\{x\} \cup ((V \cap L) - C)\) where \(V\) is an open set in \((R, \tau)\) containing \(x\) and \(C\) is countable. Note that every countable subset of \((X, \sigma)\) is closed. Since the Countable Complement Extension Topology \(\tau^*\) on \(R\) is hereditarily Lindelöf (see Example 63\[19\], Example 63) and \(\tau^*|_{L} = \sigma|_{L}\), \(L\) is a Lindelöf subspace of \((X, \sigma)\) which is not closed. Hence \((X, \sigma)\) is not an LC-space.

Let \(A\) be closed and Lindelöf in \((X, \sigma)\). Then \(A \cap B_2\) is countable since \(B_2\) is closed and discrete in \((X, \sigma)\). Suppose that \(A\) is uncountable. Then \(A \cap L\) is not meager in \((R, \tau)\). If \((A \cap L)^*\) denotes the set of condensation points of \(A \cap L\) then \((A \cap L)^*\) is closed in \((R, \tau)\), \((A \cap L) - (A \cap L)^*\) is countable and so \((A \cap L) \cap (A \cap L)^*\) is not nowhere dense in \((R, \tau)\), i.e. there exists a nonempty open set \(W\) in \((R, \tau)\) such that \(W \subseteq (A \cap L)^*\). Let \(x \in W \cap B_2\) and suppose that \(x \notin \text{cl}_\sigma A\). Then there exists an open set \(V\) in \((R, \tau)\) containing \(x\) and a countable set \(C\) such that \(V \subseteq W\) and \(\{x\} \cup ((V \cap L) - C) \cap (A \cap L)\) is empty, i.e. \(V \cap A \cap L\) is countable, which is a contradiction to \(x \in (A \cap L)^*\). So we have \(W \cap B_2 \subseteq \text{cl}_\sigma A \cap B_2 = A \cap B_2\). Hence \(W \cap B_2\) is countable. On the other hand, since \(B_2\) is a dense Baire subspace of \((R, \tau)\), \(W \cap B_2\) has to be uncountable, a contradiction. So we have shown that any set which is closed and Lindelöf in \((X, \sigma)\) has to be countable, and consequently every Lindelöf \(F_\sigma\)-set in \((X, \sigma)\) is countable and thus closed in \((X, \sigma)\). So \((X, \sigma)\) is an \(L_1\)-space and we have shown

Example 2.8. Under (CH) there exists a Hausdorff \(L_1\)-space \((X, \sigma)\) which is not an LC-space. Also, one easily checks that \((X, \sigma)\) is not an \(L_4\)-space, hence neither an \(L_2\)-space nor an \(L_3\)-space.

Example 2.9. The set of reals with the “right ray” topology is a \(P\)-space and thus an \(L_1\)-space. The rationals form a Lindelöf non-closed subset and so this space is not an LC-space.

Question. Does there exist a Hausdorff \(L_1\)-space which fails to be an LC-space without any set-theoretic assumptions?
3. Characterizations

In this section we will provide characterizations of $L_1$-spaces and additional characterizations of $LC$-spaces.

In 1984, Gauld, Mrsevic, Reilly and Vamanamurthy introduced the co-Lindelöf topology of a given space $(X, \tau)$. They showed that $l(\tau) = \{\emptyset\} \cup \{G \in \tau : X - G \text{ is Lindelöf in } (X, \tau)\}$ is a topology on $X$ with $l(\tau) \subseteq \tau$, called the co-Lindelöf topology of $(X, \tau)$.

**Theorem 3.1.** For a space $(X, \tau)$ the following are equivalent:

1. $(X, \tau)$ is an $L_1$-space,
2. $(X, l(\tau))$ is a $P$-space.

*Proof. (1) ⇒ (2):* For each $n \in \omega$, let $A_n$ be closed in $(X, l(\tau))$ and let $A = \bigcup\{A_n : n \in \omega\}$. If $A = X$ we are done. Otherwise each $A_n$ is closed and Lindelöf in $(X, \tau)$ and thus $A$ is closed and Lindelöf in $(X, \tau)$. Hence $A$ is closed in $(X, l(\tau))$.

*Proof. (2) ⇒ (1):* For each $n \in \omega$, let $A_n$ be closed and Lindelöf in $(X, \tau)$ and let $A = \bigcup\{A_n : n \in \omega\}$. Then each $A_n$ is closed in $(X, l(\tau))$ and so $A$ is also closed in $(X, l(\tau))$. Hence $A$ is closed in $(X, \tau)$ and so $(X, \tau)$ is an $L_1$-space.

**Theorem 3.2.** For a Hausdorff space $(X, \tau)$ the following are equivalent:

1. $(X, \tau)$ is an $LC$-space,
2. $(X, \tau)$ is an $L_1$-space and an $L_2$-space.

*Proof. (1) ⇒ (2):* This is obvious.

*Proof. (2) ⇒ (1):* Let $L$ be a Lindelöf subset of $(X, \tau)$ and let $x \notin L$. Since $(X, \tau)$ is Hausdorff, for each $y \in L$ there exist an open set $V_y$ containing $y$ with $x \notin \text{cl}V_y$. Clearly $\{V_y : y \in L\}$ is a cover of $L$ and so there exists a countable set $C \subseteq L$ such that $L \subseteq \bigcup\{V_y : y \in C\} \subseteq \bigcup\{\text{cl}V_y : y \in C\}$. For each $y \in C$, $L \cap \text{cl}V_y$ is Lindelöf and so $\text{cl}(L \cap \text{cl}V_y)$ is Lindelöf since $(X, \tau)$ is an $L_2$-space. Furthermore, if $W = \bigcup\{\text{cl}(L \cap \text{cl}V_y) : y \in C\}$ then $W$ is a Lindelöf $F_\sigma$-set and, since $(X, \tau)$ is an $L_1$-space, $W$ is a closed Lindelöf set not containing $x$. Thus $x \notin \text{cl}L$. This shows that $L$ is closed in $(X, \tau)$.

**Remark 3.3.** Note that the Hausdorff condition cannot be removed as the following example shows. Let $X$ be any countably infinite set with a distinguished point $p$ and let $\tau$ be the point excluded topology on $X$ (see e.g. [4]), i.e. $\tau = \{\emptyset\} \cup \{U \subseteq X : p \notin U\}$.

Clearly $(X, \tau)$ is a countable, thus hereditarily Lindelöf, non-Hausdorff space but not an $LC$-space since it is not discrete. If $A$ is a nonempty $F_\sigma$-set then $p \in A$ and so $(X, \tau)$ is an $L_1$-space. Since $(X, \tau)$ is Lindelöf it is also an $L_2$-space.

**Theorem 3.4.** For a Hausdorff space $(X, \tau)$ the following are equivalent:

1. $(X, \tau)$ is an $LC$-space.
2. Every locally countable family of Lindelöf sets is closure preserving.
(3) Every countable family of Lindelöf sets is closure preserving.

Proof. (1) ⇒ (2): Let \( \{ L_i : i \in I \} \) be a locally countable family of Lindelöf subsets, i.e. each \( x \in X \) has a neighbourhood \( U_x \) intersecting at most countably many sets \( L_i \). Since each \( L_i \) is closed we need to show that \( L = \bigcup \{ L_i : i \in I \} \) is closed. Let \( x \in X - L \) and let \( U_x \) be a neighbourhood of \( x \) such that \( I_0 = \{ i \in I : U_x \cap L_i \text{ is nonempty} \} \) is at most countable. This implies that \( \bigcup \{ L_i : i \in I_0 \} \) is Lindelöf and thus closed. Hence there is a neighbourhood \( V_x \) of \( x \) with \( V_x \cap ( \bigcup \{ L_i : i \in I_0 \} ) \) is empty. So we have \( V_x \cap ( \bigcup \{ L_i : i \in I \} ) = \emptyset \). This shows that \( L \) is closed.

(2) ⇒ (3): This is obvious.

(3) ⇒ (1): Let \( L \) be a Lindelöf subset and let \( x \notin L \). Since \((X, \tau)\) is Hausdorff, for each \( y \in L \) there exists an open neighbourhood \( V_y \) of \( y \) with \( x \notin \text{cl} V_y \). Choose a countable set \( C \subseteq Y \) with \( L \subseteq \bigcup \{ V_y : y \in C \} \). Since \( \{ L \cap \text{cl} V_y : y \in C \} \) is a countable family of Lindelöf sets, we have \( \text{cl} L \subseteq \bigcup \{ \text{cl}(L \cap \text{cl} V_y) : y \in C \} \) and so \( x \notin \text{cl} L \). Hence \( L \) is closed.

Note, however that the Sierpinski space \((X, \tau)\) where \( X = \{0, 1\} \) and \( \tau = \{\emptyset, \{0\}, X\} \) is non-Hausdorff and every countable family of Lindelöf sets is closure preserving. Obviously, \((X, \tau)\) fails to be an LC-space.

Theorem 3.5. For a space \((X, \tau)\) the following are equivalent:

1. \((X, \tau)\) is an \( L_1 \)-space.
2. Every locally countable family of closed Lindelöf sets is closure preserving.
3. Every countable family of closed Lindelöf sets is closure preserving.

Proof. (1) ⇒ (2): This is very similar to the proof of (1) ⇒ (2) in Theorem 3.4.

(2) ⇒ (3): This is obvious.

(3) ⇒ (1): Let \( L \) be a Lindelöf \( F_\sigma \)-set, i.e. \( L = \bigcup \{ L_n : n \in \omega \} \) where each \( L_n \) is closed. By assumption, \( \text{cl} L = \bigcup \{ \text{cl} L_n : n \in \omega \} = \bigcup \{ L_n : n \in \omega \} = L \). Thus \( L \) is closed.

\[ \Box \]

4. Local Lindelöfness and Generalized LC-spaces

In this section we consider locally Lindelöf spaces and their relationships to generalized LC-spaces.

Definition 3. A topological space \((X, \tau)\) is called locally Lindelöf (resp. weakly locally Lindelöf) if each point has a closed Lindelöf (resp. Lindelöf) neighbourhood.

Note that a weakly locally Lindelöf space need not be a locally Lindelöf space as any uncountable point generated space \[ \Box \] shows.

Our first result is immediate and so its proof is omitted.
Proposition 4.1. (i) Every weakly locally Lindelöf $L_2$-space is locally Lindelöf, and so every weakly locally Lindelöf space which is $L_1$ and $L_4$ is locally Lindelöf.

(ii) Every $F_{\sigma}$-subspace of a (weakly) locally Lindelöf space is (weakly) locally Lindelöf.

(iii) Every locally Lindelöf $Q$-set space is hereditarily locally Lindelöf.

Theorem 4.2. Every locally Lindelöf space $(X,\tau)$ is an $L_1$-space if and only if it is a $P$-space.

Proof. We already know that a $P$-space is an $L_1$-space. Now let $F$ be an $F_{\sigma}$-set in $(X,\tau)$. If $x \notin F$ choose a closed Lindelöf neighbourhood $U$ of $x$. Then $U \cap F$ is a Lindelöf $F_\sigma$-set in $(X,\tau)$ and so closed, since $(X,\tau)$ is an $L_1$-space. Hence $U - (U \cap F)$ is a neighbourhood of $x$ disjoint from $F$. This shows that $F$ is closed and so $(X,\tau)$ is a $P$-space. □

We note that if we replace the ‘$L_1$-condition’ in Theorem 4.2 by any of the other ‘$L_1$-conditions’ then the space $(X,\tau)$ need not even be a cid-space. If we take a non-discrete, countable, zero-dimensional Hausdorff space $(X,\tau)$ then $(X,\tau)$ clearly is an $L_i$-space for $i = 2, 3, 4$ since it is countable and thus hereditarily Lindelöf. Since $(X,\tau)$ is not discrete it is not a cid-space and hence not a $P$-space. A space $(X,\tau)$ satisfying the hypothesis above is, for example, the space Seq $(\xi)$ discussed in [14].

Corollary 4.3. Every Hausdorff, locally Lindelöf $L_1$-space $(X,\tau)$ is an $LC$-space.

Corollary 4.4. Every weakly locally Lindelöf $LC$-space $(X,\tau)$ is a $P$-space.

Recall that a space $(X,\tau)$ is said to be a weak $P$-space if any countable union of regular closed sets is closed. One can show easily that $(X,\tau)$ is a weak $P$-space if and only if for every countable family $\{U_n : n \in \omega\}$ of open sets, $\text{cl}(\bigcup \{U_n : n \in \omega\}) = \bigcup \{\text{cl} U_n : n \in \omega\}$.

Theorem 4.5. Let $(X,\tau)$ be a weak $P$-space. Then the following are equivalent:

1. $(X,\tau)$ is locally Lindelöf,
2. $(X,\tau)$ is a weakly locally Lindelöf $L_2$-space.

Proof. (1) $\Rightarrow$ (2): Let $L$ be a Lindelöf subset of $(X,\tau)$. Each point of $L$ has an open neighbourhood $U_x$ such that $\text{cl} U_x$ is Lindelöf. Pick a countable subset $C$ of $L$ such that $L \subseteq \bigcup \{U_x : x \in C\}$. Since $(X,\tau)$ is a weak $P$-space we have $\text{cl} L \subseteq \bigcup \{\text{cl} U_x : x \in C\} = W$. Since $W$ is Lindelöf we conclude that $\text{cl} L$ is Lindelöf and so $(X,\tau)$ is an $L_2$-space.

(2) $\Rightarrow$ (1): This is Proposition 4.1. □

Recall that a subset $A$ of a space $(X,\tau)$ is called locally closed if $A$ is the intersection of an open set and a closed set, or, equivalently, if each point $x \in A$ has a neighbourhood $V$ such that $A \cap V$ is a closed subset of $V$. $(X,\tau)$ is said to be submaximal if every dense set is open or, equivalently, if every subset of $(X,\tau)$ is locally closed.
Theorem 4.6. Let \((X, \tau)\) be an LC-space and let \(A\) be a weakly locally Lindelöf subspace. Then \(A\) is locally closed in \((X, \tau)\).

Proof. Choose \(x \in A\). Then \(x\) has a neighbourhood \(V\) in \((X, \tau)\) such that \(A \cap V\) is Lindelöf in \(A\) and thus also in \(X\). So \(A \cap V\) is closed in \((X, \tau)\) and thus also in \(V\). This shows that \(A\) is locally closed in \((X, \tau)\). \(\square\)

Corollary 4.7. Every hereditarily weakly locally Lindelöf LC-space is submaximal.

Remark 4.8. It is well known that maximally connected spaces are submaximal. A hereditarily weakly locally Lindelöf LC-space is not necessarily maximally connected, in fact such a space might be hereditarily disconnected as the One-point-Lindelöfication of an uncountable discrete space shows.

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References

21. Wilansky A., Between $T_1$ and $T_1$, Amer. Math. Monthly 74 (1967), 261–266.

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