ON TOPOLOGICAL SEQUENCE ENTROPY AND CHAOTIC MAPS ON INVERSE LIMIT SPACES

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Abstract. The aim of this paper is to prove the following results: a continuous map $f : [0, 1] \to [0, 1]$ is chaotic iff the shift map $\sigma_f : \lim\negthinspace \leftarrow (\{0, 1\}, f) \to \lim\negthinspace \leftarrow (\{0, 1\}, f)$ is chaotic. However, this result fails, in general, for arbitrary compact metric spaces. $\sigma_f : \lim\negthinspace \leftarrow (\{0, 1\}, f) \to \lim\negthinspace \leftarrow (\{0, 1\}, f)$ is chaotic iff there exists an increasing sequence of positive integers $A$ such that the topological sequence entropy $h_A(\sigma_f) > 0$. Finally, for any $A$ there exists a chaotic continuous map $f_A : [0, 1] \to [0, 1]$ such that $h_A(\sigma_{f_A}) = 0$.

1. Introduction

Let $(X, d)$ and $f : X \to X$ be a compact metric space and a continuous map respectively. Consider the space of sequences

$$\lim\negthinspace \leftarrow (X, f) = \{x = (x_0, x_1, \ldots, x_n, \ldots) : x_i \in X, f(x_i) = x_{i-1}, \text{ for } i = 1, 2, \ldots \}.$$ 

This set is called the inverse limit space associated to $X$ and $f$. Define a new metric $\tilde{d}$ on $\lim\negthinspace \leftarrow (X, f)$ as

$$\tilde{d}(x, y) = \sum_{i=0}^{\infty} \frac{d(x_i, y_i)}{2^i},$$

where $x = (x_0, x_1, \ldots, x_n, \ldots)$ and $y = (y_0, y_1, \ldots, y_n, \ldots)$. Then $(\lim\negthinspace \leftarrow (X, f), \tilde{d})$ is a compact metric space. Consider the natural projection $\pi : \lim\negthinspace \leftarrow (X, f) \to X$ defined by $\pi(x_0, x_1, \ldots, x_n, \ldots) = x_0$. Note that $\tilde{d}(\pi(x), \pi(y)) \leq d(\pi(x), \pi(y))$ for all $x, y \in \lim(X, f)$. The shift map is a homeomorphism $\sigma_f : \lim\negthinspace \leftarrow (X, f) \to \lim\negthinspace \leftarrow (X, f)$ defined by

$$\sigma_f(x) = \sigma_f(x_0, x_1, \ldots, x_n, \ldots) = (f(x_0), x_0, x_1, \ldots, x_n, \ldots).$$

It is clear that $\pi \circ \sigma_f = f \circ \pi$.

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Inverse limit spaces have been studied in the setting of dynamical systems in a large number of papers. In \cite{Li2006}, Shihai Li proved that some dynamical properties hold at the same time for \( f \) and \( \sigma_f \). In particular, he showed that \( f \) is chaotic in Devaney’s sense iff \( \sigma_f \) is also like that. He also proved a similar result for a suitable definition of \( w \)-chaos. In this paper, a similar result is studied in case of the Li-Yorke’s chaos. Recall briefly this definition of chaos.

A point \( p \in X \) is periodic if there exists a positive integer \( n \) such that \( f^n(p) = p \). The smallest positive integer satisfying this condition is called the period of \( p \). Denote by \( \text{Per}(f) \) the set of periodic points of \( f \). A point \( x \in X \) is said to be asymptotically periodic if there exists a \( p \in \text{Per}(f) \) such that

\[
\limsup_{n \to \infty} d(f^n(x), f^n(p)) = 0,
\]

\[
\liminf_{n \to \infty} d(f^n(x), f^n(p)) = 0,
\]

hold for all \( x, y \in D, x \neq y \). \( D \) is called a scrambled set of \( f \).

The Li-Yorke’s chaos on inverse limit spaces has been studied by Gu Rongbao in \cite{Gu2004}. In that paper the author attempts to prove that a continuous map \( f \) is chaotic iff the shift map \( \sigma_f \) is chaotic. However, in the proof he uses implicitly that \( f \) is surjective. As we will see later, this hypothesis on \( f \) cannot be removed in the following theorem essentially proved in \cite{Gu2004}.

**Theorem 1.1.** Suppose \( f \) that is surjective. Then it is chaotic in the sense of Li-Yorke if and only if the map \( \sigma_f \) is chaotic in the sense of Li-Yorke.

When continuous maps \( f : [0, 1] \to [0, 1] \) are concerned, the Li-Yorke’s chaos is connected with the notion of topological sequence entropy. Let us recall the definition (see \cite{Barreira2010}). Let \( A = \{a_i\}_{i=1}^\infty \) be an increasing sequence of positive integers. Given \( \epsilon > 0 \), we say that \( E \subset X \) is an \((A, \epsilon, n, f)\)-separated set if for any \( x, y \in E \) with \( x \neq y \) there exists \( 1 \leq k \leq n \) such that \( d(f^{\alpha_k}(x), f^{\alpha_k}(y)) > \epsilon \). Denote by \( s_n(A, \epsilon, f) \) the cardinality of any maximal \((A, \epsilon, n, f)\)-separated set. The topological sequence entropy of \( f \) is given by

\[
h_A(f) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s_n(A, \epsilon, f).
\]

In general,

\[
h_A(f) \geq h_A(\sigma_f)
\]

for every \( A \). When \( f \) is surjective we obtain the equality (see \cite{Barreira2010})

\[
h_A(f) = h_A(\sigma_f).
\]

The connection between the Li-Yorke’s chaos and the topological sequence entropy is established in the following result (see \cite{Barreira2010} and \cite{Barreira2010}.)
Theorem 1.2. Let $c, d \in \mathbb{R}$ and let $f : [c, d] \to [c, d]$ be continuous. Then

(a) $f$ is chaotic if and only if there exists an increasing sequence of positive integers $A$ such that $h_A(f) > 0$.

(b) For any increasing sequence $A$ there exists a chaotic map $f_A : [c, d] \to [c, d]$ such that $h_A(f_A) = 0$.

Theorem 2.2 does not hold in general for continuous maps on arbitrary compact metric spaces as it can be seen in \textcolor{red}{[3]}. In that paper, on $[0, 1] \times [0, 1]$ a chaotic map $f$ with $\sup_A h_A(f) = 0$ and a non-chaotic map $g$ with $\sup_A h_A(g) > 0$ are constructed.

The aim of this paper is to prove the following results: $f : [0, 1] \to [0, 1]$ is chaotic iff $\sigma_f$ is chaotic. Theorem 1.2 holds for maps $\sigma_f : \lim([0, 1], f) \to \lim([0, 1], f)$. Moreover, an example of a chaotic map $f$ for which $\sigma_f$ is not chaotic is given.

2. Positive Results for One-Dimensional Maps

Let $f : [0, 1] \to [0, 1]$ be continuous. Consider $[a, b] = \bigcap_{n \geq 0} f^n[a, b]$. Then $f|_{[a, b]} : [a, b] \to [a, b]$ is obviously surjective.

Proposition 2.1. Under the above conditions $f$ is chaotic iff $f|_{[a, b]}$ is chaotic.

Proof. It is clear that if $f|_{[a, b]}$ is chaotic then $f$ is chaotic. Suppose that $f$ is chaotic and let $D$ be a scrambled set of $f$. It is easy to see that $f^n(D)$ is also a scrambled set of $f$. Let $D_n = f^n(D) \cap [a, b]$. If $\{f^n(x) : n \geq 0\} \cap [a, b] = \emptyset$, then $x$ is asymptotically periodic and then $x \notin D_n$ for all $n \in \mathbb{N}$. So, it must exist a positive integer $n_0$ such that $D_{n_0}$ is uncountable. Then, $f|_{[a, b]}$ is chaotic.

Theorem 2.2. Let $f : [0, 1] \to [0, 1]$ be continuous. Then:

(a) $f$ is chaotic if and only if $\sigma_f$ is chaotic.

(b) $\sigma_f : \lim([0, 1], f) \to \lim([0, 1], f)$ is chaotic if and only if there exists an increasing sequence of positive integers $A$ such that $h_A(\sigma_f) > 0$.

(c) For any increasing sequence of positive integers $A$ there exists a chaotic map $f_A : [0, 1] \to [0, 1]$ such that $h_A(\sigma_A) = 0$.

Proof. It is clear that

$$\lim([0, 1], f) = \{(x_0, x_1, \ldots, x_n, \ldots) : x_i \in [a, b], f(x_i) = x_{i-1}\} = \lim([a, b], f).$$

First of all we prove (a). Assume that $f$ is chaotic. By Proposition 2.1, $f|_{[a, b]}$ is also chaotic. Applying Theorem 1.1 it follows that $\sigma_f$ is chaotic. Conversely, suppose that $\sigma_f$ is chaotic. Applying Theorem 1.1 it follows that $f|_{[a, b]}$ is chaotic. Proposition 2.1 proves that $f$ is chaotic.

Part (b). If $\sigma_f$ is chaotic, then it follows by (a) that $f$ is chaotic. Hence, by Proposition 2.1, $f|_{[a, b]}$ is chaotic. Applying Theorem 2.2(a), there exists an
increasing sequence of positive integers such that \( h_A(f|_{[a,b]}) > 0 \). Since \( f|_{[a,b]} \) is surjective, by (2), \( h_A(\sigma_f) = h_A(f|_{[a,b]}) > 0 \). Now suppose that \( \sigma_f \) is non-chaotic. Assertion (a) states that \( f \) is non-chaotic. Applying Theorem 2 and (1), we conclude that \( h_A(\sigma_f) \leq h_A(f) = 0 \) for any increasing sequence of positive integers \( A \).

Part (c). Let \( A \) be an arbitrary sequence of positive integers. By Theorem 5(b), there exists a chaotic map \( f_A : [0,1] \to [0,1] \) such that \( h_A(f_A) = 0 \). Since \( f_A \) is chaotic, by (a), \( \sigma_{f_A} \) is also chaotic. By (2), \( h_A(\sigma_{f_A}) \leq h_A(f_A) = 0 \), and the proof ends. \( \square \)

3. A Counterexample

As usual, \( Z \) will stand for the set of integers, while if \( Z \subset \mathbb{Z} \) then \( \mathbb{Z}^n \) (resp. \( \mathbb{Z}^\infty \)) will denote the set of finite sequences of length \( n \) (resp. infinite sequences) of elements from \( Z \). If \( \theta \in \mathbb{Z}^n \) or \( \alpha \in \mathbb{Z}^\infty \) then we will often describe them through their components as \( (\theta_1, \theta_2, \ldots, \theta_n) \) or \( (\alpha_i)_{i=1}^\infty \), respectively. The shift map \( \sigma : \mathbb{Z}^\infty \to \mathbb{Z}^\infty \) is defined by \( \sigma((\alpha_i)_{i=1}^\infty) = (\alpha_{i+1})_{i=1}^\infty \). If \( \theta \in \mathbb{Z}^n \) and \( \vartheta \in \mathbb{Z}^m \) (with \( m \leq \infty \)) then \( \theta + \vartheta \in \mathbb{Z}^{n+m} \) (where \( n+\infty \) means \( \infty \)) will denote the sequence \( \lambda \) defined by \( \lambda_i = \theta_i \) if \( 1 \leq i \leq n \) and \( \lambda_i = \vartheta_{i-n} \) if \( i > n \). In what follows we will denote \( 0 = (0,0,\ldots,0,\ldots) \) and \( 1 = (1,1,\ldots,1,\ldots) \), while if \( \alpha \in \mathbb{Z}^\infty \), then \( \alpha_{|n} \in \mathbb{Z}^n \) is defined by \( \alpha_{|n} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \).

This section is devoted to construct a chaotic map \( f \) on a compact metric space for which \( \sigma_f \) is non-chaotic. In order to do this we will need some information concerning so-called weakly unimodal maps of type \( 2^{\infty} \). Recall briefly the definition. We say that a continuous map \( f : [0,1] \to [0,1] \) is weakly unimodal if \( f(0) = f(1) = 0 \), it is non-constant and there is \( c \in (0,1) \) such that \( f|_{(0,c)} \) and \( f|_{(c,1)} \) are monotone. The map \( f \) is said to be of type \( 2^{\infty} \) if it has periodic points of period \( 2^n \) for any \( n \geq 0 \) but no other periods.

Weakly unimodal maps of type \( 2^{\infty} \) (briefly, \( w \)-maps) were studied in 4. In that paper it was proved that for any \( w \)-map \( f \) it is possible to construct a family \( \{K_{\alpha}(f)\}_{\alpha \in \mathbb{Z}^\infty} \) (or simply \( \{K_{\alpha}\}_{\alpha \in \mathbb{Z}^\infty} \) if there is no ambiguity on \( f \)) of pairwise disjoint (possibly degenerate) compact subintervals of \( [0,1] \) satisfying the following key properties (P1)–(P4):

(P1) The interval \( K_0 \) contains all absolute maxima of \( f \).
(P2) Define in \( \mathbb{Z}^\infty \) the following total ordering: if \( \alpha, \beta \in \mathbb{Z}^\infty \), \( \alpha \neq \beta \) and \( k \) is the first integer such that \( \alpha_k \neq \beta_k \), then \( \alpha < \beta \) if either \( \text{Card } \{1 \leq i < k : \alpha_i \leq 0\} \) is even and \( \alpha_k < \beta_k \) or \( \text{Card } \{1 \leq i < k : \alpha_i \leq 0\} \) is odd and \( \beta_k < \alpha_k \). Then \( \alpha < \beta \) if and only if \( K_\alpha < K_\beta \) (that is, \( x < y \) for all \( x \in K_\alpha, y \in K_\beta \)).
(P3) Let \( \alpha \in \mathbb{Z}^\infty \), \( \alpha \neq 0 \), and let \( k \) be the first integer such that \( \alpha_k \neq 0 \). Define \( \beta \in \mathbb{Z}^\infty \) by \( \beta_i = 1 \) for \( 1 \leq i \leq k-1, \beta_k = 1 - |\alpha_k| \) and \( \beta_i = \alpha_i \) for \( i > k \).
Then $f(K_\alpha) = K_\beta$ and $f(K_0) \subset K_1$.

For any $n$ and $\alpha \in \mathbb{Z}^\infty$, let $K_{\alpha|_n}(f)$ (or just $K_{\alpha|_n}$) be the least interval including all intervals $K_\beta$, $\beta \in \mathbb{Z}^\infty$, such that $\alpha|_n = \beta|_n$. Then

(P4) For any $\alpha \in \mathbb{Z}^\infty$, $K_\alpha = \bigcap_{n=1}^\infty K_{\alpha|_n}$.

Additionally, for any fixed $n$ it can be easily checked that the intervals $K_\theta$, $\theta \in \mathbb{Z}^n$, are open and pairwise disjoint and (after replacing $\infty$ by $n$, $0$ by $(0, \ldots, 0)$ and $1$ by $(1, 1, \ldots, 1)$), they also satisfy (P1)–(P3). Observe that if $\theta \in \{-1, 0, 1\}^n$ and we put $|\theta| := (|\theta_1|, |\theta_2|, \ldots, |\theta_n|)$ then $f^{|\theta|}(K_\theta) \subset K_{|\theta|}$; in particular, $f^{|\theta|}(K_0) \subset K_{|\theta|}$ if $\theta \in \{0, 1\}^n$.

In the rest of this section $\tilde{f}$ will denote a fixed w-map with the additional property that $\alpha \in \mathbb{Z}^\infty$ implies $K_\alpha(\tilde{f})$ is non-degenerate if and only if there is an $n \geq 0$ such that $\sigma^n(\alpha) = 0$. An example of such a map is constructed in [4]; it is possible to show that the stunted tent map $\tilde{f}(x) = \min\{1 - |2x - 1|, \mu\}$ ($\mu \approx 0.8249 \ldots$) from [7] is also a w-map with this property.

$\text{Bd} \ (Z)$, $\text{Cl} \ (Z)$ and $\text{Int} \ (Z)$ will respectively denote the boundary, the closure and the interior of $Z$.

Now, we are ready to construct our counterexample. Consider

$$X = \bigcup_{\alpha \in \{-1, 0, 1\}^\infty} \text{Bd} \ (K_\alpha).$$

Let us emphasize that $\text{Bd} \ (K_\alpha)$ consists of both endpoints of $K_\alpha$ if it is non-degenerate and of its only point if it is degenerate. Let $f = \tilde{f}|_X : X \to X$ be the restriction of the above-mentioned w-map $\tilde{f}$ to the set $X$. The following lemma shows that the above choices make sense.

**Lemma 3.1.** $X$ is a compact set and $f : X \to X$ is a well-defined continuous map.

**Proof.** Since

$$X = \left( \bigcap_{n=1}^\infty \bigcup_{\theta \in \{-1, 0, 1\}^n} \text{Cl} \ (K_\theta) \right) \setminus \bigcup_{\alpha \in \{-1, 0, 1\}^\infty} \text{Int} \ (K_\alpha),$$

by (P2) and (P4), $X$ is compact.

Recall that if $0 \neq \alpha \in \{-1, 0, 1\}^\infty$ then $\tilde{f}$ carries the interval $K_\alpha$ onto $K_\beta$ with $\beta$ defined as in (P3) (and hence belonging to $\{-1, 0, 1\}^\infty$). Moreover, $\tilde{f}$ is monotone on $K_\alpha$ because of (P1). So it maps the endpoints of $K_\alpha$ onto the endpoints of $K_\beta$. Similarly, since $K_1$ is degenerate both endpoints of $K_0$ are mapped onto its only point. The conclusion is that $f(X) \subset X$ and the map $f : X \to X$ is well-defined (and it is clearly continuous).

Let $X_1 = \bigcup_{\alpha \in \{0, 1\}} \text{Bd} \ (K_\alpha)$. Note that, by (P3), $\bigcap_{n\geq0} f^n(X) = X_1$. Let us see that $f$ is chaotic while $f|_{X_1}$ is non-chaotic.
Theorem 3.2. \( f|_{X_1} \) is non–chaotic and hence \( \sigma_f \) is non–chaotic.

Proof. Let \( x, y \in X_1 \) with \( x \in K_\alpha \) and \( y \in K_\beta \) for some \( \alpha \neq \beta \), \( \alpha, \beta \in \{0,1\}^\infty \).
We will see that there exists a positive real number \( M \) satisfying
\[
\liminf_{i \to \infty} |f^i(x) - f^i(y)| \geq M,
\]
and hence \( x \) and \( y \) cannot belong to the same scrambled set \( D \). This proves that \( \text{Card}(D) \leq 2 \) for each scrambled set \( D \) of \( f|_{X_1} \) and so \( f|_{X_1} \) is non–chaotic.

Let \( j \) be the first positive integer satisfying \( \alpha_j \neq \beta_j \). Suppose, for example, that \( \alpha_j = 0 \) and \( \beta_j = 1 \). For any \( \theta \in \{0,1\}^{j-1} \) consider the closed interval \( A_{\theta_0} \) satisfying \( K_{\theta_{j+1}} < A_{\theta_0} < K_{\theta_0} \) or \( K_{\theta_0} < A_{\theta_0} < K_{\theta_{j+1}} \), and let \( M = \min \{|A_{\theta_0}| : \theta \in \{0,1\}^{j-1}\} > 0 \). By (P3) and (P2), \( f^i(x) \in K_{\theta_{j+1}} \) and \( f^i(y) \in K_{\theta_0} \) or viceversa for all \( i \in \mathbb{N} \) and for all \( \theta \in \{0,1\}^{j-1} \). This shows that
\[
|f^i(x) - f^i(y)| \geq M
\]
which concludes the proof. \( \square \)

For any \( \alpha \in \{-1,0,1\}^\infty \) let \( \tau(\alpha|n) = \sum_{i=1}^{n} |\alpha_i|2^{i-1} \) for all \( n \in \mathbb{N} \).

Theorem 3.3. \( f : X \to X \) is chaotic.

Proof. Define on \( \{-1,1\}^\infty \) the following relation: \( \alpha \sim \beta \) if and only if there exists a positive integer \( k \) such that \( \sigma^k(\alpha) = \sigma^k(\beta) \). Obviously \( \sim \) is an equivalence relation. Moreover, for \( \alpha \in \{-1,1\}^\infty \) the class of \( \alpha \) is given by
\[
[\alpha] = \bigcup_{k=0}^{\infty} \{\sigma^{-k}(\sigma^k(\alpha))\} \cap \{-1,1\}^\infty.
\]
Since \( \{-1,1\}^\infty \) is uncountable and \( [\alpha] \) is countable for all \( \alpha \in \{-1,1\}^\infty \), the set containing all the equivalence classes \( \{-1,1\}^\infty / \sim \) is uncountable.

Let \( \mathcal{A} \) be a set containing one and only one representative \( \alpha \in [\alpha] \) for all \( [\alpha] \in \{-1,1\}^\infty / \sim \), and let \( D \) be the set containing exactly one \( x \in X \cap K_\alpha \) for all \( \alpha \in \mathcal{A} \). We claim that \( D \) is a scrambled set for \( f \). In order to see this take \( x, y \in D \), \( x \in K_\alpha \) and \( y \in K_\beta \) with \( \alpha, \beta \in \mathcal{A} \), \( \alpha \neq \beta \). Then there exists an increasing sequence of positive integers \( \{k_i\}_{i=0}^{\infty} \) satisfying \( \alpha_{k_i} \neq \beta_{k_i} \) and \( \alpha_j = \beta_j \) if \( j \neq k_i \) for all \( i \in \mathbb{N} \). Note that \( \tau(\alpha|n) = \tau(\beta|n) \) for all \( n \in \mathbb{N} \). Suppose, for example, that \( \alpha_{k_i} = 1 \) and \( \beta_{k_i} = -1 \) for some \( i \). Then, by (P3),
\[
f^{\tau(\alpha|k_i)}(x) \in K_{(0,0,\ldots,0,1)\ast\sigma^{k_i}(\alpha)} \quad \text{and} \quad f^{\tau(\alpha|k_i)}(y) \in K_{(0,0,\ldots,0,-1)\ast\sigma^{k_i}(\beta)}.
\]
By (P2), \( |f^{\tau(\alpha|k_i)}(x) - f^{\tau(\alpha|k_i)}(y)| \geq |K_0| \) and then
\[
\limsup_{n \to \infty} |f^n(x) - f^n(y)| \geq \limsup_{i \to \infty} |f^{\tau(\alpha|k_i)}(x) - f^{\tau(\alpha|k_i)}(y)| \geq |K_0|.
\]
Let now $j \neq k_i$ for all $i \in \mathbb{N}$, and suppose that $\alpha_j = \beta_j = 1$ (the case $\alpha_j = -1$ is analogous). Then $f^{\tau(\alpha_j)}(y), f^{\tau(\alpha_j)}(x) \in K_{(0,0,\ldots,0,1)}$. For any $n \in \mathbb{N}$ let $K^+_{0,n}$ and $K^-_{0,n}$ be the right and left side components of $K_0 \setminus K_{0,n}$. Applying (P4), for any $\varepsilon > 0$ there exists a positive integer $n_\varepsilon$ such that $\max\{|K^+_{0,n_\varepsilon}|, |K^-_{0,n_\varepsilon}|\} < \varepsilon$ for all $n \geq n_\varepsilon$. By (P2) and (P3), $K_{(0,0,\ldots,0,1)} \subset K^-_{0,j}$ or $K_{(0,0,\ldots,0,1)} \subset K^+_{0,j}$, and so for $j \geq n_\varepsilon$ we conclude that $|f^{\tau(\alpha_j)}(y) - f^{\tau(\alpha_j)}(x)| < \varepsilon$. This proves that

$$\liminf_{n \to \infty} |f^n(y) - f^n(x)| = 0,$$

and the proof concludes. \hfill \Box

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**References**


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