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CLOSED WALKS IN COSET GRAPHS AND VERTEX–TRANSITIVE NON–CAYLEY GRAPHS

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Abstract. We extend the main result of R. Jajcay and J. Širáň [Australasian J. Combin. 10 (1994), 105–114] to produce new classes of vertex-transitive non-Cayley graphs.

1. Introduction

The study of vertex-transitive graphs has a long and rich history in discrete mathematics. Prominent examples of vertex-transitive graphs are Cayley graphs which are important in both theory as well as aplications. Vertex-transitive graphs that are not Cayley graphs (for which we borrow te acronym VTNCG from [**12**]) have been an object of a systematic study since the early 80's. The research here was much influenced by the problem of finding the so called **non-Cayley numbers** [**3**], i.e., the numbers n for which there exists a VTNCG of order n.

A number of new construction of VTNCG's appeared in the 90's. They range from group-theoretical constructions (the basic references here are [**9**], [**10**]) to graph-theoretical ones (cf. [**12**], [**6**]). For the few classification results of vertex-transitive graphs we refer to [**8**], [**11**].

Recently, one of the direction of the reserch has focused on certain necessary combinatorial conditions for a graph to be Cayley [**1**], [**2**]. Based on this, new constructions of VTNCG's have been found [**3**], [**4**]; they can be viewed as a combination of the graph- and group-theoretical methods mentioned above.

The purpose of this paper is to prove two extension of the main theorem of [**3**] and to present new classes of VTNCG's arising from our results.

2. Terminology

Graphs considered in this paper are undirected, without loops and multiple edges; they may be finite or infinite but are always locally finite (i.e., every vertex has finite valency).

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Let Γ be a graph and let a, b be two adjacent vertices of Γ. An ordered pair (a, b) will be called an **arc**. Thus, any two adjacent vertices a, b of Γ give rise to two mutually reverse arcs, namely, (a, b) and (b, a). We can think of arcs as "edges with orientation".

Let G be a (finite or infinite) group and X a unit-free symmetric subset of G (i.e., $1 \notin X$ and $x^{-1} \in X$ whenever $x \in X$). The **Cayley graph** $C(G, X)$ has G as its vertex set, and $e = (a, b)$ is an arc of $C(G, X)$ if and only if there exists an element $x \in X$ such that $ax = b$. Because $x = a^{-1}b$ is uniquely determined, we have a function λ from the arc set of $C(G, X)$ onto the set X which assigns to every arc $e = (a, b)$ the element $\lambda(e) = a^{-1}b = x$ which we sometimes call a **label** of e. Observe that if there is an arc from a to b labelled x, then there also is an arc from b to a labelled x^{-1}.

Let G be a group, H a subgroup of G and X a symmetric unit-free subset of G. Let $H \cap X = \emptyset$. The vertex set of the **coset graph** $\mathrm{Cos}\,(G, H, X)$ is the set of all left cosets of H in G. In the coset graph, (aH, bH) is an arc if and only if there exists an element $x \in X$ such that $aHx \cap bH \neq \emptyset$ (or, equivalently, $a^{-1}b \in HxH = \{hxh' \; ; \; h, h' \in H\}$). It is easy to check that this definition is correct; i.e, it does not depend on the choice of cosets representatives and it produces graphs without loops and parallel edges. Observe that if $H = \{1\}$ then the coset graph reduces to a Cayley graph.

For an arc $e = (aH, bH)$ of the coset graph $Cos(G, H, X)$ let X_e denote the set of all $x \in X$ such that $a^{-1}b \in HxH$. If D is the arc set of the graph $\mathrm{Cos}\,(G, H, X)$, the labelling λ is now any mapping $D \longrightarrow X$ such that for each arc e $\lambda(e) \in X_e$.

A **walk of length** k in a graph is a an alternating sequence $W = v_0, e_0, v_1, e_1, \ldots, v_{k-1}, e_{k-1}, v_k$ where v_i are vertices and e_i is an arc from v_i to v_{i+1}. We say that the walk is **closed** if $v_0 = v_k$; in this case we say that the walk is **based** at v_0. If $\Gamma = C(G, X)$ then we will describe the walks starting at the vertex 1 using arcs only. For example, the walk $v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k$ such that $v_0 = 1$, $\lambda(e_0) = x_0$, $v_1 = x_0$, $\lambda(e_1) = x_1, \ldots, \lambda(e_{k-1}) = x_{k-1}$, $v_k = x_0 x_1 \ldots x_{k-1}$, will be written as $(x_0, x_1, \ldots, x_{k-1})$. In the case when $\Gamma = \mathrm{Cos}\,(G, H, X)$ with labelling λ, the walk $a_1H, e_1, a_2H, e_2, a_3H, e_3, \ldots, e_k, a_kH$ will just be denoted by $(a_1H, x_1, a_2H, x_2, a_3H, x_3, \ldots, x_k, a_kH)$ where $x_i = \lambda(e_i)$. Again, note that this type of encoding walks depends on the choice of the labels λ.

Let $\mathrm{Aut}\,(\Gamma)$ be the group of all automorphisms of the graph Γ. We say that Γ is **vertex transitive** if for arbitrary two vertices a and b there exists an automorphism $\pi \in Aut(\Gamma)$ such that $\pi(a) = b$.

It is well known (see, e.g., [**3**]) that **a graph Γ is vertex-transitive if and only if it is isomorphic to some coset graph** $\mathrm{Cos}\,(G, H, X)$.

A necessary condition for a graph to be isomorphic to a Cayley graph $C(G, X)$ was proved in [**1**].

Lemma 1. *Let $\Gamma = C(G, X)$ be a locally finite Cayley graph and p be a prime. Then the number of closed walks of length p, based at any fixed vertex of Γ, is congruent* $(\mathrm{mod}\ p)$ *to the number of elements in X for which $x^p = 1$.*

3. Walks in Coset Graphs

In this section we shall investigate the structure of closed walks in coset graphs. Throughout we will suppose that G is a group, H is a finite subgroup of G and X is a unit-free symmetric subset of G, (i.e., $1 \notin X$ and $x^{-1} \in X$ for each $x \in X$). We begin with a few elementary facts (see also [**5**], [**3**]).

Lemma 2. *Let $\Gamma = \mathrm{Cos}\,(G, H, X)$ be a coset graph such that $XHX \cap H = \{1\}$. Then*

(1) *For each $x \in X$, the number of left cosets in HxH is equal to $|H|$.*
(2) *Let $h, g \in H$ and $x \in X$; then $h \neq g$ if and only if $hxH \neq gxH$.*
(3) *Every arc of Γ has a uniquely determined label $x \in X$ i.e., $|X_e| = 1$ for each arc e.*
(4) *The valency of Γ is equal to $|X||H|$.*

Proof. (1) The number of left cosets in HxH is equal to $[H : H \cap xHx^{-1}] = [H : \{1\}] = |H|$.

(2) The sufficiency is obvious. For the necessity, let $h, g \in H$, $h \neq g$. If $hxH = gxH$ then $x^{-1}g^{-1}hx \in H$. But we also have $x^{-1}g^{-1}hx \in XHX$, which implies $x^{-1}g^{-1}hx = 1$, and so $h = g$, a contradiction.

(3) Suppose that there exists an arc from aH to bH with two labels $x, y \in X$, $x \neq y$. Then $Ha^{-1}bH = HxH$ and $Ha^{-1}bH = HyH$, and so $HxH = HyH$. It follows that there exist elements h_1, h_2, k_1, $k_2 \in H$ such that $h_1xh_2 = k_1yk_2$, or equivalently $xh_2k_2^{-1}y^{-1} = h_1^{-1}k_1 \in H$. But since $xh_2k_2^{-1}y^{-1} \in XHX$, we have $xh_2k_2^{-1}y^{-1} = 1$. Rearranging terms we obtain $y^{-1}x = k_2h_2^{-1} \in H \cap XHX$, which implies $1 = y^{-1}x$, and $x = y$, a contradiction.

(4) It is sufficient to prove that the valency of the vertex H is equal to $|X||H|$, because Γ is regular. The vertex H is adjacent to all vertices determined by left cosets from HxH for all $x \in X$. It follows that the valency of H is $\sum_{x\in X}[H : H \cap xHx^{-1}] = \sum_{x\in X}[H : 1] = |X||H|$. □

We note that if H is an invariant subgroup of G such that $H \neq \{1\}$ then $XHX \cap H \neq \{1\}$. Indeed, suppose that $XHX \cap H = \{1\}$ and consider $h \in H$, $1 \neq h$. Then it follows from Lemma 2, part (3) that $xH \neq hxH$. But H is invariant, and so there exists $l \in H$ such that $hx = xl$, which implies $hxH = xlH = xH$, a contradiction.

Sometimes we will use the notation $(a_iH, x_i)_p$ for the walk $(a_0H, x_0, a_1H, x_1, \ldots, a_{p-1}H, x_{p-1}, a_0H)$. If $a_0 = 1$ then we say that this walk is H-**based**.

Let $\mathcal{S}$ be the set of all sequences of the form $(a_0H, x_0, a_1H, x_1, a_2H, \dots, a_{p-1}H, x_{p-1})$ such that $a_0 = 1$ and $a_i^{-1}a_{i+1} \in Hx_iH$ for each i (mod p). Let $\theta\colon \mathcal{S} \longrightarrow \mathcal{S}$ be a mapping which sends the sequence $(a_iH, x_i)_p$ to $(b_iH, y_i)_p$ where $b_i = a_1^{-1}a_{i+1}$ and $y_i = x_{i+1}$, for all i (mod p). It is easy to check that $b_0 = 1$ and $b_i^{-1}b_{i+1} \in Hy_iH$, so θ is a well defined permutation on the set $\mathcal{S}$. Also it is clear that each sequence from $\mathcal{S}$ induces a closed H-based walk in the coset graph. An easy check show that θ^2 sends the sequence $(a_iH, x_i)_p$ to $(b_iH, y_i)_p$ where $b_i = a_2^{-1}a_{i+2}H$ and $y_i = x_{i+2}$. If we continue we obtain that θ^j sends the sequence $(a_iH, x_i)_p$ to $(b_iH, y_i)_p$ where $b_i = a_j^{-1}a_{j+i}H$ and $y_i = x_{i+j}$. Also it is clear that θ^p is the identity mapping on $\mathcal{S}$.

Let $\alpha = (a_0H, x_0, a_1H, x_1, a_2H, \dots, a_{p-1}H, x_{p-1}, a_0H)$ be a walk such that $a_k = 1$ for some $k \in \{0, \dots, p-1\}$. Then the coresponding H-based walk $(a_kH, x_k, \dots, a_{p-1}H, x_{p-1}, a_0H, x_0, \dots, a_{k-1}H, x_{k-1})$ will be denoted $[\alpha]$.

The basic observation is now the following: If p is prime, then the orbits of θ in $\mathcal{S}$ have length either 1 or p.

Lemma 3. *Let* $\Gamma = \operatorname{Cos}(G, H, X)$ *be a coset graph such that* $XHX \cap H = \{1\}$ *and let* p *be a prime number. Let* $\alpha = (a_iH, x_i)_p$ *and* $\beta = (b_iH, y_i)_p$ *be two sequences from* $\mathcal{S}$ *such that* $\beta = \theta^j(\alpha)$ *for some* $j, 1 \le j \le p-1$ *(i.e.,* $b_i = a_j^{-1}a_{i+j}$ *and* $y_i = x_{i+j}$*. All indices are to be read* mod p*). Then the walks* α *and* β *are identical* H*-based closed walks in* Γ *if and only if there exist* $z \in X$ *and* $c \in G$ *such that* $x_i = z$ *and* $a_iH = c^iH$ *for each* i (mod p)*.*

Proof. First we prove the sufficiency. If $x_i = z$ and $a_iH = c^iH$ for each i (mod p) then $\alpha = (c^iH, x)_p$ and $\beta = (c^iH, x)_p$ because $b_iH = a_j^{-1}a_{i+j}H = c^{-j}c^{i+j}H = c^iH$ and $y_i = x_{i+j} = x$.

Necessity. If α and β are identical then $x_0 = y_0 = x_{j+0}$, $x_1 = y_1 = x_{j+1}, \dots$, $x_{p-j} = y_{p-j} = x_0$, $x_{p-j+1} = y_{p-j+1} = x_1, \dots, x_{p-1} = y_{p-1} = x_{j-1}$. Therefore $x_0 = x_1 = \ldots = x_{p-1} = y_0 = y_1 = \ldots = y_{p-1} =: x$, because p is prime.

The following relations hold:

$$
\begin{aligned}
a_0H &= b_0H = a_j^{-1}a_{j+0}H\\
a_1H &= b_1H = a_j^{-1}a_{j+1}H\\
&\dots\\
a_{p-1}H &= b_{p-1} = a_i^{-1}a_{j+p-1}H
\end{aligned}
$$

Because $a_jH = a_j^{-1}a_{2j}H$, we have $a_j^2H = a_{2j}H$. Substituting this into the equality $a_{2j}H = a_i^{-1}a_{3j}H$ we obtain $a_j^2H = a_{2j}H = a_j^{-1}a_{3j}H$ and so $a_j^3H = a_{3j}H$. Continuing this way we subsequently obtain:

$$
\begin{aligned}
a_jH &= a_jH\\
a_{2j}H &= a_j^2H\\
&\dots
\end{aligned}
$$

$$a_{(p-1)j}H = a_j^{p-1}H$$
$$a_0H = a_j^pH \qquad (a_0 = 1)$$

Because p is prime we have $\{0, j, 2j, \dots, (p-1)j\} = \{0, 1, 2, \dots, p-1\}$ and so $a_1H = a_{lj}H = a_j^lH$ for some l. Then our walks α, β are of the form $(H, x, a_j^lH, x, a_j^{2l}H, \dots, a_j^{(p-1)l}H, x, H)$. Finelly setting $a_i^l = a$ then our walks can be written as $(H, x, aH, x, a^2H, \dots, a^{p-1}H, x, H)$. The fact that $a^pH = H$ follows easily. □

Now we introduce a set M which plays a substantial role in our next theorem. Let $V = \{a \in G : a \in HxH \text{ for some } x \in X,\ a^p \in H\}$. Let $\sim$ be an equivalence relation on V such that $a \sim b \iff aH = bH$ *and* $a^2H = b^2H$. Finally, let $M = V/\sim$.

Theorem 4. *Let* $\Gamma = \mathrm{Cos}\,(G, H, X)$ *be a coset graph where* H *is a finite subgroup of* G *and* X *is a finite symmetric unit-free subset of* G *such that* $XHX \cap H = 1$. *Let* p *be a prime number.*

Then the number of closed walks of length p*, based at any fixed vertex of* Γ*, is congruent* (mod p) *to the number of elements in* M.

Moreover, $|M| = \sum_{x \in X} |\{v \in H : (xv)^p \in H\}||H|$.

Proof. It is sufficient to consider walks based at the vertex H, because Γ is vertex transitive. We prove the claim in the following three steps:

(a) The number of closed walks of the form $(a_iH, x_i)_p$ where $x_i \neq x_j$ for some pair $i, j \in \{0, 1, \dots, p-1\}$, is divisible by p.

(b) The number of closed walks of the form $(a^iH, x)_p$ such that $a^pH = H$ is congruent (mod p) to the number of elements in M.

(c) The number of closed walks of the form $(a_iH, x)_p$ which are not from part (b) is divisible by p.

Let $H = \{h_1, h_2, \dots, h_n\}$.

Proof of (a). In this case we deal with a subset $\mathcal{S}' \subset \mathcal{S}$ formed by sequences $(a_iH, x_i)_p$ where $x_j \neq x_k$ for some $j \neq k$. On this subset each orbit of θ has length p and the orbits are disjoint.

Proof of (b). Let $\mathcal{S}''$ be the set of all elements α of $\mathcal{S}$ for which $\alpha\theta^j = \alpha$ for some $1 \leq j \leq p-1$. Lemma 3 implies that $\mathcal{S}'' = \mathcal{S} \cap \{(a^iH, x)_p : x \in X, a \in G\}$.

Choose any walk $W = (a^iH, x)_p$. Then $a = hxl$, $a^2 = hxlhxl, a^3 = hxlhxlhxl$, $\dots, a^{p-1} = (hxl)^{p-1}$. Let us denote $lh =: v$. Then W has the form $(H, x, hxH, x, hxvxH, x, hxvxvxH, \dots, h(xv)^{p-1}H, x, H)$.

Each element of the set $V = \{a \in G : \exists_{x \in X},\ a \in HxH,\ a^p \in H\}$ determines a walk of the form $(a^iH, x)_p$. It may happen that different elements from V define the same walk; our aim is to identify all such occasions. Let

$$Q = (a^i H, x)_p = (H, x, hxH, x, hxvxH, x, hxvxvxH, \dots, h(xv)^{p-1}H, x, H),$$
$$Q' = (b^i H, x)_p = (H, y, lyH, y, lyuyH, y, lyuyuyH, \dots, l(yu)^{p-1}H, y, H)$$

We claim that the walks Q and Q' are identical if and only if $aH = bH$ and $a^2H = b^2H$.

The necessity is evident, and we prove the sufficiency. If $aH = bH$ then $hxH = lyH$ and so $y^{-1}l^{-1}hx \in H$. But $y^{-1}l^{-1}hx \in XHX$ which implies $y^{-1}l^{-1}hx = 1$, and therefore $xy^{-1} = h^{-1}l \in H$. Since $xy^{-1} \in XHX$ we have $1 = xy^{-1} = h^{-1}l$ then $x = y$ and $h = l$. Because $a^2H = b^2H$, we obtain $hxvxH = lyuyH = hxuxH$ and so $x^{-1}u^{-1}vx \in H$. But $x^{-1}u^{-1}vx \in XHX$ thus $vx = ux$ and $u = v$. Then for all i we have $a^iH = b^iH$.

The equivalence relation $\sim$ on V defined by $a \sim b \iff aH = bH$ and $a^2H = b^2H$ has the following property: if $a \sim b$ then the walks $(a^iH, x)_p$, $(b^iH, x)_p$ are identical. Then the number of walks in part (b) is equal to the cardinality of the set $V/\sim$.

Now we prove that $|M| = \sum_{x\in X} |v \in H : (xv)^p \in H||H|$. Let us consider the walks with all arcs labeled x. Let $(H, x, hxH, x, hxvxH, x, hxvxvxH, \dots, h(xv)^{p-1}H, x, H)$, and $(H, x, lxH, x, lxuxH, x, lxuxuxH, \dots, l(xu)^{p-1}H, x, H)$ be two such walks. If $u \neq v$ then these walks are different. Indeed, if they are the same then $hxH = lxH$ which implies $l = h$ and $x^{-1}l^{-1}hx = 1$. We also suppose that $hxvxH = lxuxH$, thus $x^{-1}u^{-1}x^{-1}l^{-1}hxvx = x^{-1}u^{-1}vx$. But $x^{-1}u^{-1}vx \in XHX$ and so we have $x^{-1}u^{-1}vx = 1$ and $u = v$.

Notice that $(H, x, hxH, x, hxvxH, x, hxvxvxH, \dots, h(xv)^{p-1}H, x, H)$ is a walk from part (b) if and only if $(xv)^p \in H$. The elements $x \in X$ and $v \in G$ determine the following n different walks

$$(H, x, h_1xH, x, h_1xvxH, x, h_1xvxvxH, \dots, h_1(xv)^{p-1}H, x, H)$$
$$(H, x, h_2xH, x, h_2xvxH, x, h_2xvxvxH, \dots, h_2(xv)^{p-1}H, x, H)$$
$$\dots$$
$$(H, x, h_nxH, x, h_nxvxH, x, h_nxvxvxH, \dots, h_n(xv)^{p-1}H, x, H).$$

The number of walks with all arcs labeled x is equal to $|v \in H : (xv)^p \in H||H|$. But if two walks $(H, x, hxH, x, hxvxH, x, hxvxvxH, \dots, h(xv)^{p-1}H, x, H)$ and $(H, y, hyH, y, hyuyH, y, hyuyuyH, \dots, h(yu)^{p-1}H, y, H)$ have different first arcs $(x \neq y)$ then they are distinct. It follows that the number of walks in part (b) is $\sum_{x\in X} |\{v \in H : (xv)^p \in H\}||H|$.

Proof of (c). Let $\mathcal{S}'''$ be the set of all elements α of $\mathcal{S}$ for which there exists $x \in X$ and $a_i \in G$ $i = 1, \dots, p-1$ such that $\alpha = (a_iH, x)_p$ and $\alpha\theta^j \neq \alpha$ for some $1 \leq j \leq p-1$. Every orbit of θ on $\mathcal{S}'''$ has p elements and the orbits are disjoint. Thus $|\mathcal{S}'''|$ is divisible by p. □

4. Vertex-transitive Non-Cayley Graphs

In this section we prove two generalizations of the following principal result of [**3**].

Theorem 5. ([**3**]) *Let G be a group, let H be a finite subgroup of G, and let X be a finite symmetric unit-free subset of G such that $XHX \cap H = \{1\}$. Further, suppose that there are at least $|X| + 1$ distinct ordered pairs $(x, h) \in X \times H$ such that $(xh)^p = 1$ for some fixed prime $p > |X||H|^2$. Then the coset graph $\Gamma = \mathrm{Cos}\,(G, H, X)$ is a vertex-transitive non-Cayley graph.*

In the first generalization of Theorem 5 we relax the condition $(xh)^p = 1$.

Theorem 6. *Let G be a group, let H be a finite subgroup of G, and let X be a finite symmetric unit-free subset of G such that $XHX \cap H = \{1\}$. Further, suppose that there are at least $|X| + 1$ distinct ordered pairs $(x, h) \in X \times H$ such that $(xh)^p \in H$ for some fixed prime $p > |X||H|^2$. Then the coset graph $\Gamma = \mathrm{Cos}\,(G, H, X)$ is a vertex-transitive non-Cayley graph.*

Proof. Let M be the set from Theorem 4; we have $|M| = \sum_{x \in X} |\{h \in H : (xh)^p \in H\}||H| = |\{(x, h) : x \in X,\, h \in H,\, (xh)^p \in H\}||H|$. From our assumptuions it follows that $(|X|+1)|H| \le |M| \le |X||H|^2 < p$. Theorem 4 implies that the number of closed walks in $\Gamma = \mathrm{Cos}\,(G, H, X)$ is congruent $\pmod{p}$ to the number $|M|$, where $|M|$ is at least $(|X| + 1)|H|$ $(p > (|X| + 1)|H|)$. The valency of Γ is $|X||H|$. If Γ is a **Cayley** graph $\Gamma = C(K, L)$ then edges in this **Cayley** graph are labeled by $|X||H|$ distinct labels. Then $|\{k \in K : k^p = 1\}| \le |X||H|$. But by Lemma 1, the number of closed walks in $\Gamma = C(K, L)$ is congruent $\pmod{p}$ to the number $|\{k \in K : k^p = 1\}|$ where $|\{k \in K : k^p = 1\}| \le |X||H|$, a contradiction. □

In the second generalization of Theorem 5 we will not require the existence of $|X| + 1$ ordered pairs but just $|X|$, assuming that $|X||H|$ is odd.

Theorem 7. *Let G be a group, let H be a finite subgroup of G, and let X be a finite symmetric unit-free subset of G such that $XHX \cap H = \{1\}$. Let $|H||X|$ be an odd number. Further, suppose that there are at least $|X|$ distinct ordered pairs $(x, h) \in X \times H$ such that $(xh)^p \in H$ for some fixed prime $p > |X||H|^2$. Then the coset graph $\Gamma = \mathrm{Cos}\,(G, H, X)$ is a vertex-transitive non-Cayley graph.*

Proof. The proof is similar to the preceding one. The number of closed walks in $\Gamma = \mathrm{Cos}\,(G, H, X)$ is congruent $\pmod{p}$ to a number i, where i is at least $|X||H|$.

If Γ is a **Cayley** graph $\Gamma = C(K, L)$ then edges in this **Cayley** graph are labeled by $|X||H|$ distinct labels. Because $|L| = |X||H|$ is an odd number and L is a symmetric unit-free subset then there exists an edge labelled with $l \in L$ such that $l^{-1} = l$. But $l^p = l \ne 1$. Then the number of closed walks in $\Gamma = C(K, L)$ is congruent $\pmod{p}$ to a number z where $z \le (|X| - 1)|H| < |X||H|$, a contradiction. □

5. Examples

Our first two examples are generalizations of Example 1 in [**3**].

Example 1. Let $G = \langle x, y | x^2 = y^r = 1,\ (xy)^p = y^k \rangle$. Assume that G contains no relation of type $xy^i x = y^j$. Let $r \geq 3$ and let $p > r^2$ be a prime. Then the graph $\mathrm{Cos}\,(G, \langle y \rangle, \{X\})$ satisfies the conditions of Theorem 6. Indeed, if $H = \langle y \rangle$ and $X = \{x\}$ then HXH generates G, $XHX \cap H = \{1\}$. We also have that $(xy)^p \in H$ and $(xy^{-1})^p \in H$. Then the graph $\mathrm{Cos}\,(G, \langle y \rangle, \{x\})$ is a vertex transitive non-Cayley graph.

Example 2. Let $G = \left\langle x, y | x^3 = y^r = 1,\ (xy)^p = y^k \right\rangle$. Assume that G contains no relation of type $xy^i x = y^j$ and $xy^i x^{-1} = y^j$. Let $r \geq 3$ be an odd number and let $p > r^2$ be a prime. Theorem 7 implies that $\mathrm{Cos}\,(G, \langle y \rangle, \{x, x^{-1}\})$ is a vertex transitive non-Cayley graph.

Comparing with [**3**], our Examples 1 and 2 are more general because in [**3**] it was required that $(xy)^p = 1$. Allowing $(xy)^p = y^k$, $k > 0$ we obtain new and interesting classes of VTNCG's. The fact that they are indeed non-Cayley does not follow from the main theorem of [**3**] (which shows that our generalized theorems can be useful).

Our last example introduces a new construction of VTNCG's which can be obtained by the methods of [**3**]; however, we think it may be worth presenting.

Example 3. Let S_p be the symmetric group on p elements where p is a prime number. Consider a p-cycle $C = (1, \ldots, p)$ and a 3-cycle $D = (1, 1+x, 1+2x)$ where $(p, x) = 1$. Let $H := \langle D \rangle$ and $X := \{C, C^{-1}\}$. The cycles C and D generate the alternating group A_n. It can be checked that $C^p = id$, $(C^{-1})^p = id$, $CD = (1, \ldots, p)(1, 1{+}x, 1{+}2x) = (1, 2, \ldots, 1{+}2x, 2{+}2x, \ldots, p, 1{+}x, \ldots, 2x)$ and so $(CD)^p = \mathrm{id}$. An easy computation shows that the following 12 permutations are not in H:

$$\begin{aligned}
CDC^{-1} &= (1, 2, \ldots, x-1, 2x, 1+x, 2+x, \ldots, 2x-1, p, 1+2x, \\
&\qquad 2+2x, \ldots, p-1, x), \\
CD^{-1}C^{-1} &= (1, 2, \ldots, x-1, p, 1+x, 2+x, \ldots, 2x-1, x, 1+2x, \\
&\qquad 2+2x, \ldots, p-1, 2x), \\
CDC &= (3, 4, \ldots, 1+x, 2+2x, 3+x, 4+x, \ldots, 1+2x, 2, 3+2x, \\
&\qquad 4+2x, \ldots, 1, 2+x), \\
CD^{-1}C &= (3, 4, \ldots, 1+x, 2, 3+x, 4+x, \ldots, 1+2x, 2+x, 3+2x, \\
&\qquad 4+2x, \ldots, 1, 2+2x), \\
C^{-1}D^{-1}C &= (CDC^{-1})^{-1}, \\
CDC^{-1} &= (CD^{-1}C^{-1})^{-1},
\end{aligned}$$

$$C^{-1}D^{-1}C^{-1} = (CDC)^{-1},$$
$$C^{-1}DC^{-1} = (CD^{-1}C)^{-1},$$

$CC, CC^{-1}, C^{-1}C, C^{-1}C^{-1}$. From this it follows that $XHX \cap H = \mathrm{id}$. Theorem 6 now implies that the graph $\mathrm{Cos}\,(A_p, \langle D\rangle, \{C, C^{-1}\})$ is a vertex transitive non-Cayley graph.

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