AN OMEGA THEOREM ON
DIFFERENCES OF TWO SQUARES, II

M. KÜHLEITNER

Abstract. Let \( \rho(n) \) denote the number of pairs \((u, v) \in \mathbb{N} \times \mathbb{Z} \) with \( u^2 - v^2 = n \).
Due to a formula of Sierpinski, \( \rho(n) \) is closely related to the classical divisor function \( d(n) \). We establish a lower bound for the remainder term in the asymptotic expansion for the Dirichlet summatory function of \( \rho(n) \).

1. Introduction

As in part I of this paper [8], let \( \rho(n) \) denote the number of pairs \((u, v) \in \mathbb{N} \times \mathbb{Z} \) with \( u^2 - v^2 = n \). For the more general case where the square is replaced by a \( "k"\)-th power \( k \geq 2 \) see Krätzel [6], [7] and the recent paper of Nowak [9]. Due to an elementary formula of Sierpinski, our function \( \rho(n) \) is closely related to the classical divisor function \( d(n) \) by

\[
\rho(n) = d(n) - 2d\left(\frac{n}{2}\right) + 2d\left(\frac{n}{4}\right),
\]

where \( d(\cdot) = 0 \) for non-integers, due to Sierpinski.

For a large real variable \( x \), we consider the remainder term \( \theta(x) \) in the asymptotic formula

\[
T(x) = \sum_{n \leq x} \rho(n) = \frac{x}{2} \log x + (2\gamma - 1)\frac{x}{2} + \theta(x),
\]

where \( \gamma \) denotes throughout this paper the Euler-Mascheroni constant.

Upper bounds for \( \theta(x) \) can be readily established as a trivial generalization of the corresponding results for the Dirichlet divisor problem. It is known that

\[
D(x) = x \log x + (2\gamma - 1)x + \Delta(x)
\]

with

\[
\Delta(x) \ll x^{23/73}(\log x)^{461/146}.
\]

Received August 4, 1997.


Key words and phrases. Divisor problem, Dirichlet summatory function, asymptotic expansion.
M. KÜHLEITNER


Concerning lower estimates, the author proved in [8], on the basis of [1] and Hafner’s method [3], that

$$\theta(x) = \Omega_{+}\left(\frac{x \log x}{2} \left(\frac{(3+2\log 2)}{4} \exp\left(-A \frac{1}{\log \log \log x}\right)\right)^{1/4}\right).$$

The aim of the present article is an $\Omega_{-}$ result for $\theta(x)$, corresponding to that of Corrádi and Kátai [1] for the divisor problem.

**Theorem.**

$$T(x) = \frac{x}{2} \log x + (2\gamma - 1) \frac{x}{2} + \theta(x),$$

with

$$\theta(x) = \Omega_{-}\left(x^{1/4} \exp\left(c (\log \log x)^{1/4} (\log \log \log x)^{-3/4}\right)\right),$$

where $c$ is a positive absolute constant.

2. Notations and Lemmas

For large real $x$ we define $P_x$ as the set of all primes less than or equal to $x$, and $Q_x$ the set of all square-free integers composed only of primes from $P_x$. We write $|P_x|$ for the cardinality of $P_x$ and $M = 2^{|P_x|}$ for the cardinality of $Q_x$. We then have

$$|P_x| \leq \frac{x}{\log x} \quad \text{and} \quad M \ll \exp\left(c_1 \frac{x}{\log x}\right),$$

for some positive constant $c_1$. The largest integer in $Q_x$ is bounded by $e^{2x}$, since for $q \in Q_x$, we have

$$\log q \leq \sum_{p \leq x} \log p \leq 2x.$$

Let $S_x$ be the set of numbers defined by

$$S_x = \left\{ \mu = \sum_{q \in Q_x} r_q \sqrt{q} \quad \text{where} \quad r_q \in \{0, \pm 1\} \quad \text{and} \quad \text{at least two} \quad r_q \neq 0 \right\}.$$

Finally let

$$\eta(x) = \inf \left\{ |\sqrt{n} + 2\mu| \quad \text{with} \quad n \in \mathbb{N}_o \quad \text{and} \quad \mu \in S_x \right\},$$

and

$$q(x) = -\log (\eta(x)).$$

By a slight modification of the method used for the corresponding result in Gangadharan [2], one readily shows the following lemma.
Lemma 1. For \( x \to \infty \) we have
\[
x \ll q(x) \ll \exp \left( c_2 \frac{x}{\log x} \right),
\]
for some positive constant \( c_2 \).

Lemma 2. There exists a positive constant \( c_3 \) such that
\[
\sum_{q \in Q_x} \frac{d(q)}{q^{3/4}} \gg \exp \left( c_3 \frac{x^{1/4}}{\log x} \right).
\]

Proof. By the definition of \( Q_x \), we have
\[
\sum_{q \in Q_x} \frac{d(q)}{q^{3/4}} = \prod_{p \leq x} (1 + 2^{1/4}) = \exp \left( \sum_{p \leq x} \log \left( 1 + 2^{1/4} \right) \right)
\geq \exp \left( \sum_{p \leq x} p^{-3/4} + O(1) \right) \gg \exp \left( c_3 \frac{x^{1/4}}{\log x} \right). \quad \square
\]

As in Gangadharan [2] define for real \( z \),
\[
V(z) = 2\cos \left( \frac{z}{2} \right)^2 = 1 + \frac{e^{iz} + e^{-iz}}{2},
\]
and
\[
T_x(u) = \prod_{q \in Q_x} V \left( u \sqrt{q} - \frac{5\pi}{4} \right).
\]

Lemma 3. We have
\begin{enumerate}
\item \( 0 \leq T_x(u) \leq 2^M \), for all \( u \),
\item \( T_x'(u) \ll M 2^M e^z \), for all \( u \),
\item \( T_x(u) = T_0 + T_{1,x} + T_{2,x} + T_{3,x} \) where,
\[
T_0 = 1,
T_{1,x} = \frac{e^{5\pi i/4}}{2} \sum_{q \in Q_x} e^{-iu\sqrt{q}}
T_{3,x} = \sum_{\mu \in S_x} h_{\mu} e^{iu\mu},
\]
\end{enumerate}
\[T_{2,x} \text{ is the complex conjugate of } T_{1,x} \text{ and } \vert h_{\mu} \vert \leq 1/4.\]

Proof. The proof of Lemma 3 is straightforward by the definition of \( V(z) \) and \( T_x(u) \).
3. Proof of the Theorem

We start with the well known Voronoi identity for
\[
\Delta_1(x) \overset{\text{def}}{=} \int_0^x \Delta(t) \, dt = \frac{x}{4} + \frac{x^{3/2}}{2\sqrt{2}\pi^2} \sum_{n=1}^\infty \frac{d(n)}{n^{5/4}} \sin \left(4\pi \sqrt{nx} - \frac{\pi}{4}\right) + O(1).
\]
Inserting this in
\[
\theta(x) = \Delta(x) - 2 \Delta \left(\frac{x}{2}\right) + 2 \Delta \left(\frac{x}{4}\right),
\]
and substituting \(T = 4\pi \sqrt{x}\), we get
\[
E_1(T) \overset{\text{def}}{=} \int_0^T E(t) \, t \, dt = T^{3/2} \sum_{n=1}^\infty \frac{d(n)}{n^{5/4}} \left(\sin (T\sqrt{n} - \pi/4) - 2^{5/4} \sin (T\sqrt{n/2} - \pi/4) + 2^{3/2} \sin (T\sqrt{n/4} - \pi/4)\right),
\]
with
\[
E(t) = 2\pi \sqrt{2\pi} \left(\theta(t^2/16\pi^2) - 1/4\right).
\]
Define
\[
P(x) = \exp \left(a \frac{x}{\log x}\right)
\]
such that
\[
q(x) \leq P(x) \quad \text{and} \quad M^2 \leq P(x),
\]
and let
\[
\sigma_x = \exp (-2P(x)).
\]
Next define for fixed \(x\),
\[
\gamma_x = \sup_{u>0} \frac{-2\pi \sqrt{2\pi} \theta(u^2/16\pi^2)}{u^{1/2+1/P(x)}}.
\]
We may assume that \(\gamma_x < \infty\), otherwise more than Theorem 1 would be true. Thus
\[
(2) \quad \gamma_x u^{1/2+1/P(x)} + A + E(u) \geq 0,
\]
for all \(u\), where \(A = 2\pi \sqrt{2\pi}/4\).
Let
\[
J_x = \sigma_x^{5/2} \int_0^\infty \left(\gamma_x u^{1/2+1/P(x)} + A + E(u)\right) u \exp (-\sigma_x u) T_x(u) \, du.
\]
The next lemma provides an asymptotic expansion for \(J_x\).
Lemma 4. For $x \to \infty$,

$$J_x = e^2 \Gamma\left(\frac{5}{2}\right) \gamma_x - \frac{1}{4} \Gamma\left(\frac{5}{2}\right) \sum_{q \in Q_x} \frac{d(q)}{q^{1/4}} + o(\gamma_x) + o(1).$$

Proof. Do deal with the first two terms of $J_x$, we observe that, for $r = 1$ or $r = \frac{3}{2} + \frac{1}{P(x)}$,

$$\int_{0}^{\infty} u^r \exp (-\sigma_x u) T_x(u) \, du = \Gamma(1 + r) \sigma_x^{-(1+r)}
+ \sum_{i=1,2,3} \int_{0}^{\infty} u^r \exp (-\sigma_x u) T_{i,x}(u) \, du$$

where $1 \leq r \leq \frac{3}{2} + \frac{1}{P(x)}$.

The part of $T_{1,x}$ contributes exactly,

$$\frac{e^{5\pi i/4}}{2} \Gamma(1 + r) \sum_{q \in Q_x} \frac{1}{(\sigma_x + i\sqrt{q})^{1+r}} \ll \sum_{q \in Q_x} q^{-(1+r)/2}
\ll \sum_{q \in Q_x} 1 \ll M \ll \sqrt{P(x)} = o(\sigma_x^{-5/2}).$$

The contribution of $T_{2,x} = \overline{T_{1,x}}$ is obviously no more than this. Finally $T_{3,x}$ contributes

$$\sum_{\mu \in S_x} \frac{h_{\mu}}{(\sigma_x + i\mu)^{1+r}} \ll 3^M \eta(x)^{-(1+r)}
\ll \exp(M \ln 3 + (1 + r)(-\log \eta(x)) \ll \exp(3P(x)) = o(\sigma_x^{-5/2}).$$

Next we deal with the contribution of $E(u)$ to $J_x$. Our first step is to integrate by parts to introduce $E_1(u)$ in the integral. Thus,

$$I \overset{\text{def}}{=} \int_{0}^{\infty} E(u)u \exp (-\sigma_x u) T_x(u) \, du = - \int_{0}^{\infty} E_1(u) \frac{d}{du} \left(\exp (-\sigma_x u) T_x(u)\right) \, du,$$

since $E_1(u) \ll u^{3/2}$ for large $u$ and $E_1(0) = 0$. Inserting the series representation for $E_1(u)$ and integrating term by term, noting that the series converges absolutely for every $u$ and uniformly on compact sets, we get

$$I = - \sum_{n=1}^{\infty} \frac{d(n)}{n^{1/4}} \text{Im} \left( e^{-\pi i/4} I_n \right) + O\left( \int_{0}^{\infty} \left| \frac{d}{du} \left(\exp (-\sigma_x u) T_x(u)\right) \right| \, du \right)
+ O\left( \int_{0}^{\infty} u^{1/2} \exp (-\sigma_x u) |T_x(u)| \, du \right),$$
since
\[ u^{3/2} \frac{d}{du} \left( \exp(-\sigma_x u) T_x(u) \right) = \frac{d}{du} \left( u^{3/2} \exp(-\sigma_x u) T_x(u) \right) - \frac{3}{2} u^{1/2} \exp(-\sigma_x u) T_x(u), \]
and
\[ I_n \stackrel{\text{def}}{=} \int_0^\infty \left( e^{iu\sqrt{n}} - 2^{5/4} e^{iu\sqrt{n}/2} + 2^{3/2} e^{iu\sqrt{n}/4} \right) \frac{d}{du} \left( u^{3/2} \exp(-\sigma_x u) T_x(u) \right) du. \]

Estimating the contributions of the error terms, we see that
\[ \int_0^\infty \left| \frac{d}{du} \left( \exp(-\sigma_x u) T_x(u) \right) \right| du \leq \int_0^\infty \left| T_x (u)' - \sigma_x T_x (u) \right| \exp(-\sigma_x u) du \leq 4^M \sigma_x^{-1} + 2^M \ll \exp \left( c \sqrt{P(x)} \right) \left( 1 + \exp(2P(x)) \right) = o(\sigma_x^{-5/2}); \]
and
\[ \int_0^\infty u^{1/2} \exp(-\sigma_x u) |T_x(u)| du \ll 2^M \int_0^\infty u^{1/2} \exp(-\sigma_x u) du \ll 2^M \sigma_x^{-3/2} \ll \exp \left( c \sqrt{P(x)} + 3P(x) \right) = o(\sigma_x^{-5/2}). \]

We integrate \( I_n \) by parts once more and expand \( T_x(u) \) as in (3) of Lemma 3, to get
\[ I_n = -i \sum_{k=0,\ldots,3} \int_0^\infty \left( \sqrt{n} e^{iu\sqrt{n}} - 2^{5/4} \sqrt{\frac{n}{2}} e^{iu\sqrt{n}/2} + 2^{3/2} \sqrt{\frac{n}{4}} e^{iu\sqrt{n}/4} \right) \]
\[ \times u^{3/2} \exp(-\sigma_x u) T_{1,x}(u) du = I_0(n) + I_1(n) + I_2(n) + I_3(n), \]
for short. We shall show that the main term of \( I_n \) comes from \( I_1(n) \). In fact, the contribution of \( I_0(n) \) is
\[ \ll \sqrt{n} |\sigma_x - i \sqrt{n}|^{-5/2} \ll n^{-3/4}, \]
that of \( I_2(n) \) is
\[ \ll \sqrt{n} \sum_{q \in Q_x} |\sigma_x - i(\sqrt{n} + \sqrt{q})|^{-5/2} \ll Mn^{-3/4}. \]
The contribution of $I_3(n)$ is bounded by

\[
I_3(n) \ll \sqrt{n} \sum_{\mu \in S_x} |\sigma_x - i(\sqrt{n} - \mu)|^{-5/2}
\]

\[
\ll \begin{cases} 
  \sqrt{n} 3^M (\eta(x))^{-5/2}, & \text{if } n \leq 2 \max\{|\mu| : \mu \in S_x\} \\
  n^{-3/4} 3^M, & \text{else}.
\end{cases}
\]

This $\max\{|\mu| : \mu \in S_x\}$ is bounded by $Me^{cx}$ for some positive constant $c$. Hence the total contribution to $I$ is bounded by

\[
\ll \sum_{n \leq 2Me^{cx}} \frac{d(n)}{n^{5/4}} \sqrt{n} 3^M \exp \left(-5 \log \frac{\eta(x)}{2}\right) + O \left(3^M \sigma_x^{-5/4} \sum_{n > 2Me^{cx}} \frac{d(n)}{n^2}\right)
\]

\[
\ll 3^M \sigma_x^{-5/4} \sum_{n \leq 2Me^{cx}} n^{-3/4+\epsilon} + O(3^M \sigma_x^{-5/4})
\]

\[
\ll 3^M \sigma_x^{-5/4} (Me^{cx})^{1/4+\epsilon}
\]

\[
= o(\sigma_x^{-5/2}).
\]

Therefore,

\[
I = -\frac{1}{2} \sum_{n=1}^{\infty} \frac{d(n)}{n^{5/4}} \frac{i}{Q_x} \sum_{q \in Q_x} \int_0^\infty \left(\sqrt{n} e^{iu(\sqrt{n} - \sqrt{q})} - 2^{5/4} \sqrt{n} e^{iu(\sqrt{n}/2 - \sqrt{q})}\right) + O(\sigma_x^{-5/2})
\]

\[
\ll \frac{2^{3/2}}{\sqrt{4}} e^{iu(\sqrt{n}/4 - \sqrt{q})} u^{3/2} \exp (-\sigma_x u) du + o(\sigma_x^{-5/2})
\]

\[
= -\frac{1}{2} \sum_{q \in Q_x} \left(\frac{d(q)}{q^{5/4}} - 2^{5/4} \frac{d(2q)}{(2q)^{5/4}} + 2^{3/2} \frac{d(4q)}{(4q)^{5/4}}\right) \int_0^\infty \sqrt{q} u^{3/2} \exp (-\sigma_x u) du
\]

\[
+ O \left(\sum_{n=1}^{\infty} \frac{d(n)}{n^{5/4}} \sum_{q \in Q_x} \int_0^\infty \sqrt{n} e^{iu(\sqrt{n}/4 - \sqrt{q})} u^{3/2} \exp (-\sigma_x u) du\right) \cdot du.
\]

For this last error term we get a bound exactly as above for $I_3(n)$ with $M$ replacing the factor $3^M$, since

\[
\sqrt{n} - \sqrt{q} \gg (\sqrt{n} + \sqrt{q})^{-1} \gg e^{-x} \gg \exp (-P(x)),
\]

for $n \leq 2 \max\{q : q \in Q_x\} \gg 2e^{2x}$ and $n \neq q$.

We get,

\[
I = -\frac{1}{2} \Gamma \left(\frac{5}{2}\right) \sigma_x^{-5/2} \left(\sum_{q \in Q_x} (d(q) - d(2q) + \frac{1}{2} d(4q)) q^{-3/4} + o(\sigma_x^{-5/2})\right)
\]

\[
= -\frac{1}{4} \Gamma \left(\frac{5}{2}\right) \sigma_x^{-5/2} \sum_{q \in Q_x} d(q) q^{-3/4} + o(\sigma_x^{-5/2}),
\]
since 
\[ d(q) - d(2q) + \frac{1}{2} d(4q) = \frac{1}{2} d(q). \]
This completes the proof of Lemma 4. □

Since \( \sigma_x > 0 \) and \( J_x > 0 \) by (2), we have
\[ \exp \left( c \frac{x^{1/4}}{\log x} \right) \ll \sum_{q \in \mathbb{Q}_x} d(q)q^{-3/4} \ll \gamma_x, \]
by Lemma 2 and the last assertion by Lemma 4.

Thus by the definition of \( \gamma_x \) there is a sequence \( u_x \) which tends to infinity with \( x \), such that
\[ -\theta(u_x^2) \gg u_x^{1/2} \exp \left( \frac{\log u_x}{P(x)} + c \frac{x^{1/4}}{\log x} \right), \]
since \( \theta(u) \) is bounded for bounded \( u \), which follows for small \( u \) from
\[ \theta(u) = -\frac{u}{2} \log u - (2\gamma - 1) \frac{u}{2}, \]
and is obvious for the other values of \( u \).

Consider first the values of \( u_x \) for which
\[ \log \frac{u_x}{P(x)} \leq c \frac{x^{1/4}}{\log x}. \]

Taking logarithms on both sides, we have
\[ \log \log u_x \ll \frac{x}{\log x}. \]

Since \( y^{1/4}(\log y)^{-3/4} \) is an increasing function of \( y \) for sufficiently large \( y \), we have from (3)
\[ \frac{(\log \log u_x)^{1/4}}{(\log \log \log u_x)^{3/4}} \ll \frac{x^{1/4}}{\log x}, \]
from which the desired estimate follows.

Consider now those values of \( x \) for which
\[ c \frac{x^{1/4}}{\log x} \leq \log \frac{u_x}{P(x)}. \]

We may assume that
\[ \frac{(\log \log u_x)^{1/4}}{(\log \log \log u_x)^{3/4}} \gg \frac{\log u_x}{P(x)}, \]
otherwise the estimate holds obviously. Taking logarithms on both sides gives
\[ \log \log u_x \ll \frac{x}{\log x}, \]
from which the estimate follows as above. This proves the theorem. □
References


M. Kühleitner, Institut für Mathematik, Universität für Bodenkultur, Gregor Mendel Straße 33, A-1180 Wien, Austria; e-mail: kleitner@mail.boku.ac.at