Acta Math. Univ. Comenianae
Vol. LXVIII, 1(1999), pp. 1–15

THE VECTOR INDIVIDUAL WEIGHTED ERGODIC THEOREM FOR BOUNDED BESICOVICH SEQUENCES

K. EL BERDAN

Abstract. In this paper we prove maximal ergodic theorem and a pointwise convergence theorem. Our result is to prove the convergence of

$$B_n(T,\alpha,f)=\frac{1}{n}\sum_{j=0}^{n-1}\alpha_j T^j f$$

for all $f\in L^1(\Omega,X)=L^1(X)$, where n tends to infinity, Ω is a σ-finite measure space, X is a reflexive Banach space, α_j is a bounded Besicovich sequence and T is a linear operator on $L^1(X)$ which is contracting in both $L^1(X)$ and $L^\infty(X)$.

Our result has the additional advantage as it is sufficiently general in order to extend the Beck and Schwartz random theorem.

We can also generalize this result to a multidimensional case.

Notations and Definitions

Denote by X a Banach space, (Ω,β,μ) a σ-finite measure space, $\|x\|_X$ the norm of a vector x in X.

- $L^1(\Omega,X)=L^1(X)=\Big\{f\colon\Omega\to X$, measurable and $\int_\Omega\|f(\omega)\|_X\,d\mu(\omega)<\infty\Big\}$ the space of integrable functions in the sense of Bochner which take values in X, and $L^\infty(\Omega,X)=L^\infty(X)=\Big\{f\colon\Omega\to X$, measurable and bounded a.e. (i.e $\sup_{\omega\in\Omega}\|f(\omega)\|_X<\infty)\Big\}$.
- $L^1=L^1(\Omega,R)$, $L^\infty=L^\infty(\Omega,R)$.
- For all $f\in L^1(X)$, $\|f\|_1=\int_\Omega\|f(\omega)\|_X\,d\mu(\omega)$ and $\|f\|_\infty=\sup_{\omega\in\Omega}\|f(\omega)\|_X$.
- For an operator T of $L^1(X)$ into itself: T is contracting in $L^1(X)$ iff $\|Tf\|_1\le\|f\|_1$ for all $f\in L^1(X)$, similarly, T is contracting in $L^\infty(X)$ iff for all $f\in L^\infty(X)$, $\|Tf\|_\infty\le\|f\|_\infty$.
- For $a>0$ and $f\in L^1(X)$, $f^{a-}=\frac{f}{\|f\|}\min\{\|f\|,a\}$, $f^{a+}=f-f^{a-}$, $f^*=\sup_n\|B_n(\alpha,T,f)\|$, $e^*(a,\alpha)=\{f^*>\alpha a\}$, $e(a)=\{\|f\|>a\}$ and for $A\subset\Omega$ we denote φ_A the indicator function of A.

Received July 7, 1995; revised March 5, 1998.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 27A35; Secondary 28A65.

We first define the term "Bounded Besicovich sequence". Let α_j be a sequence of complex numbers. We say that α_j is a bounded Besicovich sequence if

(i) there exists a positive real α such that $|\alpha_j| < \alpha$ for every $j \in N$,

(ii) for $\varepsilon > 0$ there exists a trigonometric polynomial φ_ε such that

$$\lim_n \frac{1}{n+1} \sum_{j=0}^{n} |\alpha_j - \varphi_\varepsilon(j)| < \varepsilon\,.$$

Introduction

In [**5**] J. Olsen proved an individual weighted ergodic theorem for bounded Besicovich sequences. He proved the a.e. convergence of

$$B_n(T, \alpha, f) = \frac{1}{n+1} \sum_{j=0}^{n} \alpha_j T^j f$$

where T is linear operator on $L^1 = L^1(\Omega, R)$ which is contracting in $L^1 = L^1(\Omega, R)$ and in $L^\infty = L^\infty(\Omega, R)$, and α_j is a bounded Besicovich sequence.

In [**2**] R. V. Chacon proved a maximal ergodic lemma for operators which act in the space of functions taking their values in a Banach space, and he used this result to obtain a vector valued ergodic theorem as a generalization of the Dunford and Shwartz theorem.

In this paper we intend to generalize the individual ergodic theorem of Olsen to operators acting in $L^1(X)$.

In his proof Olsen used the dominated operator of T-linear modulus (see [**5**, Lemma 2.1]) but in our case we have shown in [**4**] that there exists a vector operator without linear modulus contracting in L^1.

We prove that the method of Chacon can be adapted to this situation, then our result is the convergence μ-a.e. of $B_n(T, \alpha, f)$ where T is a linear operator on $L^1(X)$ which is contracting in both $L^1(X)$ and $L^\infty(X)$.

In the second part, we generalize these results to a multidimensional case: If α_j is a bounded Besicovich sequence and $\lambda_d = i_1 s_1 + \cdots + i_d s_d$ where $s_j \in N$ for $j = 1, \ldots, d$ then the limit of

$$B_n(T, d, \alpha, f) = \frac{1}{(n+1)^d} \sum_{i_1=0}^{n} \cdots \sum_{i_d}^{n} \alpha_{\lambda_d} T^{\lambda_d} f$$

exists a.e. for all $f \in L^1(X)$.

I. One Dimensional Case

We now state Chacon's result [**2**]:

Theorem 1.1 (Chacon)**.** *Let T be a linear operator on $L^1(X)$ contracting in $L^1(X)$ and in $L^\infty(X)$.*

(i) If $a > 0$, $e^*(a) = \left\{ \omega \in \Omega;\ \ \sup_n \left\| \frac{1}{n+1} \sum_{j=0}^{n} T^j f(\omega) \right\|_X > a \right\}$

$$\int_{e^*(a)} \left(a - \left\|f^{a-}(\omega)\right\|_X\right) d\mu(\omega) \leq \int_\Omega \left\|f^{a+}(\omega)\right\|_X d\mu(\omega).$$

(ii) For $f \in L^1(X)$, the limit of $A_n(T,f)(\omega) = \frac{1}{n} \sum_{j=0}^{n-1} T^j f(\omega)$ exists strongly for every $\omega \in \Omega$ as n tends to infinity.

(iii) If $1 < p < \infty$, then there exists a function $f^{**} \in L^p(X)$ such that

$$\left\|A_n(T)f\right\|_X \leq \left\|f^{**}\right\|_X \quad (a.e.\ n \geq 0).$$

Main Result

We now state and prove our main result.

Theorem 1.2. *Let X be a reflexive Banach space, T be a linear operator on $L^1(X)$ contracting in $L^1(X)$ and in $L^\infty(X)$, α_j be a bounded Besicovich sequence then:*

(i) For $f \in L^1(X)$, the limit of $B_n(T,\alpha,f)(\omega) = \frac{1}{n} \sum_{j=0}^{n-1} \alpha_j T^j f(\omega)$ exists strongly for every $\omega \in \Omega$ as n tends to infinity.

(ii) If $1 < p < \infty$, $f \in L^p(X)$ the average $B_n(T,\alpha,f)$ converges a.e. and

$$\left\| \sup_n \left\|B_n(T,\alpha,f)\right\|_X \right\|_p \leq \left(\frac{p}{p-1}\right)^{1/p} \|f\|_p .$$

Before proving Theorem 1.2, let us remark that if $\alpha_j = 1$ for every j, we have

$$e^*(a,1) = e^*(a) = \left\{ \sup_n \left\| \frac{1}{n+1} \sum_{k=0}^{n} T^k f \right\| > a \right\} .$$

We will have to prove that Chacon's lemma is valid also for the averages $B_n(T,\alpha,f)$.

Lemma 1.3. *If $f \in L^1(X)$ and $a > 0$, we have then*

$$\int_{e^*(a,\alpha)} \left(a - \left\|f^{a-}(\omega)\right\|_X\right) d\mu(\omega) \leq \int_\Omega \left\|f^{a+}(\omega)\right\|_X d\mu(\omega)\,.$$

Proof of Lemma 1.3. We can suppose that $\alpha_j > 0$ for all j in N. As in [**2**] we define:

$$f_0 = f^{a+}\,,$$
$$f_{i+1} = Tf_i - \frac{Tf_i}{\left\|Tf_i\right\|_X} \min\left\{\left\|Tf_i\right\|_X, a - \left\|f^{a-}\right\|_X - \sum_{k=0}^{i}\left\|Tf_k - f_{k+1}\right\|_X\right\}$$

and

$$d_0 = 0\,,$$
$$d_{i+1} = \frac{Tf_i}{\left\|Tf_i\right\|_X} \min\left\{\left\|Tf_i\right\|_X, a - \left\|f^{a-}\right\|_X - \sum_{k=0}^{i}\left\|Tf_k - f_{k+1}\right\|_X\right\}\,.$$

By [**2**] these sequences satisfy the following relations:

(1) $\left\|f^{a-}(\omega)\right\|_X + \sum_{k=0}^{i}\left\|d_k(\omega)\right\|_X \leq a$ for every i in N and for every ω in Ω,

(2) $\left\|Tf_i(\omega)\right\|_X = \left\|f_{i+1}(\omega)\right\|_X + \left\|d_{i+1}(\omega)\right\|_X$ for all ω in Ω,

(3) $T^i f = T^i f^{a-} + f_i + \sum_{k=0}^{i} T^{i-k} d_k$ for every i in N,

(4) $\sum_{i=0}^{n}\Big[T^i f^{a-} + \sum_{k=0}^{i} T^{i-k} d_k\Big] = \sum_{i=0}^{n} T^i\Big(f^{a-} + \sum_{k=0}^{n-i} d_k\Big)$ for every n in N,

(5) if for $\omega \in \Omega$ $f_{i+1}(\omega) \neq 0$ then $a = \left\|f^{a-}(\omega)\right\|_X + \sum_{k=0}^{i+1}\left\|d_k(\omega)\right\|_X$.

Multiply equality (3) by α_j to obtain

$$\alpha_j T^i f = \alpha_i T^i f^{a-} + \alpha_i f_i + \sum_{k=0}^{i} \alpha_i T^{i-k} d_k\,.$$

To prove the following equality

$$(*) \qquad \sum_{i=0}^{n}\Big[\alpha_i T^i f^{a-} + \sum_{k=0}^{i} \alpha_i T^{i-k} d_k\Big] = \sum_{i=0}^{n} T^i\Big(\alpha_i f^{a-} + \sum_{k=0}^{n-i} \alpha_{i+k} d_k\Big)$$

we shall argue by induction on n. It is true for $n=0$. Suppose now that it is valid up to n and let us prove it for $n+1$:

$$\begin{aligned}
&\sum_{i=0}^{n+1}\Big[\alpha_i T^i f^{a-} + \sum_{k=0}^{i}\alpha_i T^{i-k} d_k\Big] \\
&\quad = \sum_{i=0}^{n}\alpha_i\Big(T^i f^{a-} + \sum_{k=0}^{i} T^{i-k} d_k\Big) + \alpha_{n+1}\Big(T^{n+1} f^{a-} + \sum_{k=0}^{n+1} T^{n+1-k} d_k\Big) \\
&\quad = \underbrace{\sum_{i=0}^{n} T^i\Big(\alpha_i f^{a-} + \sum_{k=0}^{n-i}\alpha_{i+k} d_k\Big)}_{\mu_n} + \underbrace{\alpha_{n+1}\Big(T^{n+1} f^{a-} + \sum_{k=0}^{n+1} T^{n+1-k} d_k\Big)}_{\lambda_n}.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\sum_{i=0}^{n+1} T^i\Big(\alpha_i f^{a-} + \sum_{k=0}^{n+1-i}\alpha_i d_k\Big) = \sum_{i=0}^{n} T^i\Big(\alpha_i f + \sum_{k=0}^{n-i}\alpha_{i+k} d_k + \alpha_{n+1} d_{n+1-i}\Big) \\
&\quad = \sum_{i=0}^{n} T^i\Big(\alpha_i f + \sum_{k=0}^{n-i}\alpha_{i+k} d_k\Big) + T^{n+1} f^{a-} + \alpha_{n+1}\sum_{i=0}^{n+1} T^i d_{n+1-i} \\
&\quad = \underbrace{\sum_{i=0}^{n} T^i\left(\alpha_i f^{a-} + \sum_{k=0}^{n-i}\alpha_{i+k} d_k\right)}_{\mu_n} + \underbrace{\alpha_{n+1}\left(T^{n+1} f^{a-} + \sum_{k=0}^{n+1} T^{n+1-k} d_k\right)}_{\lambda_n}.
\end{aligned}$$

It follows that $(*)$ holds for every n in N. Thus

$$\begin{aligned}
\sum_{i=0}^{n+1}\alpha_i T^i f &= \sum_{i=0}^{n} T^i\Big(\alpha_i f^{a-} + \sum_{k=0}^{n-i}\alpha_{i+k} d_k\Big) + \sum_{i=0}^{n+1}\alpha_i f_i \\
&= \Big(\alpha_0 f^{a-} + \sum_{k=0}^{n}\alpha_k d_k\Big) + \sum_{i=1}^{n} T^i\Big(\alpha_i f^{a-} + \sum_{k=0}^{n-i}\alpha_{i+k} d_k\Big) + \sum_{i=0}^{n+1}\alpha_i f_i
\end{aligned}$$

Now, we shall show that for all $\omega \in e^*(\alpha, a) = \Big\{\omega \in \Omega;\ \sup_n \Big\|\frac{1}{n}\sum_{j=0}^{n}\alpha_j T^j f(\omega)\Big\|_X > a\Big\}$ we have

$$a = \Big\|f^{a-}(\omega)\Big\|_X + \sum_{k=0}^{\infty}\Big\|d_k(\omega)\Big\|_X.$$

Let $\omega \in e^*(\alpha, a)$ then there exists $n = n(\omega)$ such that:

$$\begin{aligned}
\alpha a n + \alpha a \le \Big\|\sum_{i=1}^{n-1}\alpha_i T^i f(\omega)\Big\| &\le \Big(\alpha_0\big\|f^{a-}(\omega)\big\|_X + \sum_{k=0}^{n}\alpha_k\big\|d_k(\omega)\big\|_X\Big) \\
&\quad + \sum_{i=1}^{n}\Big\|T^i\Big(\alpha_i f^{a-} + \sum_{k=0}^{n-i}\alpha_{i+k} d_k\Big)(\omega)\Big\|_X + \sum_{i=0}^{n}\alpha_i\big\|f_i(\omega)\big\|_X.
\end{aligned}$$

By (1) we have

$$\alpha_i\big\|f^{a-}(\omega)\big\|_X+\sum_{k=0}^{n}\alpha_{i+k}\big\|d_k(\omega)\big\|_X\leq\alpha_i\big\|f^{a-}(\omega)\big\|_X+\sum_{k=0}^{n}\alpha_i\big\|d_k(\omega)\big\|_X$$
$$=\alpha_i\Big[\big\|f^{a-}(\omega)\big\|_X+\sum_{k=0}^{n}\big\|d_k(\omega)\big\|_X\Big]$$

and, as the operator T is contracting in $L^\infty(X)$, then

$$\sum_{i=1}^{n}\Big\|T^i(\alpha_i f^{a-}+\sum_{k=0}^{n-i}\alpha_{i+k}d_k(\omega)\Big\|_X\leq\sum_{i=1}^{n}\Big\|T^i(\alpha_i f^{a-}+\sum_{k=0}^{n-i}\alpha_{i+k}d_k)\Big\|_\infty$$
$$\leq\sum_{i=1}^{n}\Big\|\Big(\alpha_i f^{a-}+\sum_{k=0}^{n-i}\alpha_{i+k}d_k\Big)\Big\|_\infty=\sum_{i=1}^{n}\sup_{\omega\in\Omega}\Big\|\Big(\alpha_i f^{a-}+\sum_{k=0}^{n-i}\alpha_{i+k}d_k\Big)(\omega)\Big\|_X$$
$$\leq\sum_{i=1}^{n}\sup_{\omega\in\Omega}\Big[\alpha_i\big\|f^{a-}(\omega)\big\|_X+\sum_{k=0}^{n-i}\alpha_{i+k}\big\|d_k(\omega)\big\|_X$$
$$\leq\alpha a\Big]$$

whence

$$(n+a)\alpha a\leq\alpha\Big[\big\|f^{a-}(\omega)\big\|_X+\sum_{k=0}^{n}\big\|d_k(\omega)\big\|_X\Big]+na\alpha+\sum_{i=1}^{n-1}\alpha_i\big\|f_i(\omega)\big\|_X$$
$$\leq\alpha\Big\{\Big[\big\|f^{a-}(\omega)\big\|_X+\sum_{k=0}^{n}\big\|d_k(\omega)\big\|_X\Big]+na+\sum_{i=1}^{n-1}\alpha_i\big\|f_i(\omega)\big\|_X\Big\}$$

and so

$$a\leq\Big[\big\|f^{a-}(\omega)\big\|_X+\sum_{k=0}^{n}\big\|d_k(\omega)\big\|_X\Big]+\sum_{i=1}^{n-1}\big\|f_i(\omega)\big\|_X$$

by relation (5) we have

$$a=\big\|f^{a-}(\omega)\big\|_X+\sum_{k=0}^{\infty}\big\|d_k(\omega)\big\|_X$$

for all $\omega\in e^*(\alpha,a)$.

The rest of the proof can be obtained in the same way as in Chacon's method, in fact, using (6) and knowing that $\|T\|_1\leq 1$ we deduce

$$\int_{e^*(a,\alpha)}\big(a-\big\|f^{a-}(\omega)\big\|_X\big)\,d\mu\leq\sum_{k=0}^{\infty}\big(\big\|Tf_k\big\|_1-\big\|f_{k+1}\big\|_1\big)\leq\sum_{k=0}^{\infty}\big(\big\|f_k\big\|_1-\big\|f_{k+1}\big\|_1\big)$$
$$\leq\big\|f_0\big\|_1=\int_\Omega\big\|f_0(\omega)\big\|\,d\mu(\omega)=\int_\Omega\big\|f^{a+}(\omega)\big\|\,d\mu(\omega).$$

Proof of Theorem 1.2. Since the Lemma 1.3 gives us maximal weak inequality for averages $B_n(T,\alpha,f)$ it suffices to prove the convergence for f belonging to a set which is dense every where in $L^1(X)$. We know that $L^\infty(X)$ is such a set, so for $f \in L^\infty(X)$ we have:

$$\frac{1}{n+1}\sum_{i=0}^{n}\alpha_i T^i f = \frac{1}{n+1}\sum_{i=0}^{n}\varphi_\varepsilon(i)T^i f + \frac{1}{n+1}\sum_{i=0}^{n}\big[\alpha_i - \varphi_\varepsilon\big]T^i f.$$

Let θ be a complex number. The operator $Uf = e^{i\theta}Tf$ is contracting in both $L^1(X)$ and $L^\infty(X)$ and the theorem follows in the case $\alpha_n = e^{in\theta}$ from Chacon's theorem. The linearity of convergence gives that

$$\lim_n \frac{1}{n+1}\sum_{i=0}^{n}\varphi_\varepsilon(i)T^i f$$

exists and is finite a.e. for any trigonometric polynomial φ_ε, and $f \in L^\infty(X)$.

In fact we have for this operators a strong inequality in $L^\infty(X)$:

$$\left\|\sup_n\left\|\frac{1}{n+1}\sum_{i=0}^{n}\varphi_\varepsilon(i)T^i f\right\|_X\right\|_\infty \le k_\varepsilon\|f\|_\infty$$

and by the definition b) we also have

$$\limsup_n \frac{1}{n+1}\sum_{i=0}^{n}|\alpha_i - \varphi_\varepsilon(i)|\big\|T^i f\big\|_X \le \varepsilon\|f\|_\infty.$$

By Lemma 1.3 we have

$$a\mu(e^*(a,\alpha)) \le \int_{e(a)}\|f(\omega)\|_X\,d\mu$$

and so, using the rearrangement formula we get:

$$\begin{aligned}
\big\|f^*\big\|_p^P &= \int_\Omega [f^*]^p\,d\mu = p\alpha^p\int_\Omega\int_0^{f^*/\alpha}\lambda 1_{e^*(\lambda,\alpha)}\,d\lambda\,d\mu(\omega)\\
&= p\alpha^p\int_\Omega\int_0^\infty \lambda^{p-1}\mu\big[e^*(\lambda,\alpha)\big]\,d\lambda\\
&\le p\alpha^p\int_\Omega\int_0^\infty \lambda^{p-2}\big\|f(\omega)\big\|_X\,d\lambda\,d\mu(\omega)\\
&= p\alpha^p\int_\Omega\int_0^\infty \lambda^{P-2}1_{e(\lambda)}\,d\lambda\,d\mu(\omega)\\
&= p\alpha^p\int_\Omega\big\|f(\omega)\big\|_X\Big[\int_0^{\|f(\omega)\|}\lambda^{p-2}\,d\lambda\Big]\,d\mu(\omega)\\
&= \alpha^p\frac{p}{p-1}\int_\Omega\big\|f(\omega)\big\|_X^p\,d\mu(\omega).
\end{aligned}$$

□

Remark 1.4. We notice that Lemma 1.3 remains true for any bounded sequence even if it is not a Besicovich one.

Let us consider some examples to which Theorem 1.2 is applied:

Examples 1.5.

1. Let $X = \mathbb{R} \times \mathbb{R}$ (reflexive Banach space) with norm $\|(x, y)\| = \|x| + |y|$. $\Omega = \{1, 2\}$ a probability space, $\mu(1) = \mu(2) = \frac{1}{2}$; $L^1(\{1, 2\}, \mathbb{R} \times \mathbb{R})$ being a Banach space of dimension 4. Notice that for

$$T = \begin{pmatrix} a & b & a' & b' \\ c & d & c' & d' \\ e & f & e' & f' \\ g & h & g' & h' \end{pmatrix}$$

T will be contracting on $L^1(\{1, 2\}, \mathbb{R} \times \mathbb{R})$ if the sum of the absolute values of the terms in each column is less than 1.

It is easy to show that the operator T is contracting on $L^\infty(\{1, 2\}, \mathbb{R} \times \mathbb{R})$ if the terms of the matrix T satisfy

$$\begin{cases} |a| + |c| + |a'| + |c'| \leq 1 \\ |a| + |c| + |b'| + |d'| \leq 1 \\ |b| + |d| + |a'| + |c'| \leq 1 \\ |b| + |d| + |b'| + |d'| \leq 1 \end{cases} \quad \text{and} \quad \begin{cases} |e| + |g| + |e'| + |g'| \leq 1 \\ |e| + |g| + |f'| + |h'| \leq 1 \\ |f| + |h| + |e'| + |g'| \leq 1 \\ |f| + |h| + |f'| + |h'| \leq 1 \end{cases}$$

Let T be the linear operator on $L^1(\{1, 2\}, \mathbb{R} \times \mathbb{R})$ represented by a square matrix of order 4 defined by

$$T = \begin{pmatrix} 2/9 & 0 & 3/8 & 3/7 \\ 1/4 & 1/9 & 2/9 & 2/9 \\ 0 & 1/7 & 2/7 & 1/4 \\ 2/5 & 1/9 & 2/9 & 3/10 \end{pmatrix}$$

T is contracting on both $L^1(\{1, 2\}, \mathbb{R} \times \mathbb{R})$ and $L^\infty(\{1, 2\}, \mathbb{R} \times \mathbb{R})$.

Consider also the sequence $\alpha_n = e^{i\theta n}$ where θ a complex number, we have by Theorem 1.2 the convergence of the sequence

$$\frac{1}{n+1} \sum_{k=0}^{n} \alpha_k T^k f = \frac{1}{n+1} \sum_{k=0}^{n} e^{i\theta k} \begin{pmatrix} 2/9 & 0 & 3/8 & 3/7 \\ 1/4 & 1/9 & 2/9 & 2/9 \\ 0 & 1/7 & 2/7 & 1/4 \\ 2/5 & 1/9 & 2/9 & 3/10 \end{pmatrix}^k \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}$$

for all $f = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \varphi_{\{1\}} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \varphi_{\{2\}} \in L^1(\{1, 2\}, \mathbb{R} \times \mathbb{R})$.

2. Let X be a reflexive Banach space, Ω a probability space, φ is a transformation from Ω to Ω such that for all $f \in L^1(X)$,

$$\int_\Omega \left\| f \circ \varphi(\omega) \right\| d\mu(\omega) \leq \int_\Omega \left\| f(\omega) \right\| d\mu(\omega)$$

(or φ is a measure preserving transformation) and let $(\alpha_i)_{i \in N}$ be any Besicovich bounded sequence then

$$\lim_n \frac{1}{n+1} \sum_{i=0}^{n} \alpha_i (f \circ \varphi^i)$$

exists a.e. for every $f \in L^1(X)$.

Applications. The general result of Theorem 1.2 can be applied to give a generalization of the vector valued random ergodic theorem of Beck and Schwartz [**1**].

Theorem 1.6. *Let be defined on Ω a strongly measurable function U_ω with values in the Banach space $B(X)$ of bounded linear operators on a space X. Suppose that $\|U_\omega\| \leq 1$ for all $\omega \in \Omega$. Let φ be a measure preserving transformation in (Ω, β, μ) and $(\alpha_i)_{i \in N}$ be a Besicovich bounded sequence, then for $f \in L^1(X)$, the limit*

$$\lim_n \frac{1}{n+1} \sum_{k=0}^{n} \alpha_k U_\omega U_{\phi(\omega)} \dots U_{\phi^{k-1}(\omega)} f(\varphi^k(\omega))$$

exists for almost all $\omega \in \Omega$.

Proof. For $f \in L^1(X)$ we define

$$Uf(\omega) = U_\omega(f(\varphi(\omega)))\,.$$

Then it can be easily seen that it satisfies the conditions of Theorem 1.2 and hence the condition follows at once from Theorem 1.2. □

II. A Multidimensional Case

Obtaining an extension of Theorem 1.2 to distinct several operators $T_1, \dots, T_d$ which more general means difficult. But if $T_1 = T^{s_1}, \dots . \ T_d = T^{s_d}$ where T is an linear operator on $L^1(X)$ and $s_k \in N$ for $k = 1, \dots, d$, then Theorem 1.2 can be extended to this case. Let

$$B_n(T, d, \alpha, f) = \frac{1}{(n+1)^d} \sum_{i_1=0}^{n} \cdots \sum_{i_d=0}^{n} \alpha_{\lambda_d} T_1^{i_1} \dots T_d^{i_d} f$$

where $\lambda_d = i_1 s_1 + \dots i_d s_d$ and α_j be a bounded Besicovich sequence. Let $f_d^* = \sup_n \left\| B_n(T, d, \alpha, f) \right\|_X$ and $e_d^*(a, \alpha) = \left\{ f_d^* > \alpha a \right\}$.

Theorem 2.1. *Let X be a reflexive Banach space, T be a linear operator on $L^1(X)$ contracting in $L^1(X)$ and in $L^\infty(X)$, α_j be a bounded Besicovich sequence. Then for $\lambda_d = i_1 s_1 + \cdots + i_d s_d$, $s = s_1 + \ldots, s_d$:*

(i) For $f \in L^1(X)$, et $a > 0$ we have

$$\int_{e_d^*(a,\alpha)} \left(a - \left\|f^{a-}(\omega)\right\|_X\right) d\mu(\omega) \leq s \int_\Omega \left\|f^{a+}(\omega)\right|_X d\mu(\omega).$$

(ii) For $f \in L^1(X)$, the limit of $B_n(T,d,\alpha,f)(\omega)$ exists strongly for every $\omega \in \Omega$ as n tends to infinity.

(iii) If $1 < p < \infty$, $f \in L^p(X)$ and $\alpha = \sup_k \left|\alpha_k\right|$, the average $B_n(T,d,\alpha,f)$ converges a.e. and

$$\left\|\sup_n \left\|B_n(T,d,\alpha,f)\right\|_X\right\|_p \leq \alpha \left(\frac{p}{p-1}\right)^{1/p} \|f\|_p .$$

Proof. We will prove the theorem in the case where $d = 2$, and $s_1 = s_2 = 1$ only, for the sake of simplicity. A similar proof to that used in this case gives the general result in the d-dimensional case ($d > 2$).

We now study the following averages:

$$B_n(T,2,\alpha,f)(\omega) = \frac{1}{(n+1)^2} \sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_{i+j} T^{i+j} f.$$

By the relation (3) in the proof of the Theorem 1.2 we can write

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_{i+j} T^{i+j} f = \sum_{j=0}^{n} \sum_{k=0}^{n} \alpha_{j+k} \Big[T^{j+k} f^{a-} + \sum_{m=0}^{j+k} T^{j+k-m} d_m\Big] + \sum_{j=0}^{n} \sum_{k=0}^{n} \alpha_{j+k} f^{j+k} .$$

We prove an analogous equality to $(*)$ as in the proof of Theorem 1.2

(7)

$$\sum_{j=0}^{n} \sum_{k=0}^{n} \alpha_{j+k} \Big[T^{j+k} f^{a-} + \sum_{m=0}^{j+k} T^{j+k-m} d_m\Big] = \sum_{j=0}^{n} \underbrace{\sum_{t=j}^{n+j} \alpha_t \Big(T^t f^{a-} + \sum_{m=0}^{t} T^{t-m} d_m\Big)}_{1}$$

$$= \sum_{j=0}^{n} \underbrace{\Big\{ \sum_{t=0}^{n+j} \alpha_t \big(T^t f^{a-} + \sum_{m=0}^{t} T^{t-m} d_m\big) - \sum_{t=0}^{j-1} \alpha_t \Big(T^t f^{a-} + \sum_{m=0}^{t} T^{t-m} d_m\Big) \Big\}}_{1}$$

$$
\begin{aligned}
&= \sum_{j=0}^{n}\Big\{\sum_{t=0}^{n+j} T^t\Big(\alpha_t f^{a-} + \underbrace{\sum_{m=0}^{n+j-t}\alpha_{t+m}d_m}_{2}\Big) \\
&\qquad - \sum_{t=0}^{j-1} T^t\Big(\alpha_t f^{a-} + \sum_{m=0}^{j-1-t}\alpha_{t+m}d_m\Big)\Big\} \qquad \text{by } (*) \\
&= \sum_{j=0}^{n}\Big\{\sum_{t=0}^{n+j} T^t\Big[\alpha_t f^{a-} + \underbrace{\sum_{m=0}^{j-t-1}\alpha_{t+m}d_m + \sum_{m=j-t}^{n+j-t}\alpha_{t+m}d_m}_{2}\Big] \\
&\qquad - \sum_{t=0}^{j-1} T^t\Big(\alpha_t f^{a-} + \sum_{m=0}^{j-1-t}\alpha_{t+m}d_m\Big)\Big\} \\
&= \sum_{j=0}^{n}\Big\{\sum_{t=0}^{n+j} T^t\Big(\alpha_t f^{a-} + \sum_{m=0}^{j-t-1}\alpha_{t+m}d_m\Big) + \underbrace{\sum_{t=0}^{n+j}\Big[\sum_{m=j-t}^{n+j-t}\alpha_{t+m}d_m\Big]}_{3} \\
&\qquad - \sum_{t=0}^{j-1} T^t\Big(\alpha_t f^{a-} + \sum_{m=0}^{j-1-t}\alpha_{t+m}d_m\Big)\Big\} \\
&= \sum_{j=0}^{n}\sum_{t=j}^{n+j} T^t\Big(\alpha_t f^{a-} + \sum_{m=0}^{j-t-1}\alpha_{t+m}d_m\Big) + \underbrace{\sum_{t=0}^{n+j} T^t\Big(\sum_{m=j-t}^{n+j-t}\alpha_{t+m}d_m\Big)}_{3}\Big\}.
\end{aligned}
$$

Let

$$
\chi_n(\omega) = \sum_{j=0}^{n}\sum_{k=0}^{n}\alpha_{j+k}\Big[T^{j+k}f^{a-} + \sum_{m=0}^{j+k}T^{j+k-m}d_m\Big](\omega)
$$

and

$$
f_C^* = \sup_n \big\|C_n(T,\alpha,f)\big\|_X
$$

where

$$
C_n(T,\alpha,f) = \frac{1}{(2n+1)^2}\sum_{j=0}^{n}\sum_{k=0}^{n}\alpha_{j+k}T^{j+k}f
$$

and $e_C^*(a,\alpha) = \{f_C^* > a\alpha\}$. Fix $\omega \in e_C^*(a,\alpha)$ there exists $n = n(\omega)$ such that

$$
(2n+1)^2\alpha a \le \Big\|\sum_{j=0}^{n}\sum_{k=0}^{n}\alpha_{j+k}T^{j+k}f(\omega)\Big\|_X \le \big\|\chi_n(\omega)\big\|_X + \sum_{j=0}^{n}\sum_{k=0}^{n}\alpha_{j+k}\big\|f_{j+k}\big\|_X.
$$

But

$$\|\chi_n(\omega)\|_X \leq \Big[\alpha_0\|f^{a-}(\omega)\|_X + \sum_{k=0}^{n}\alpha_k\|d_k(\omega)\|_X\Big]$$
$$+\sum_{j=0}^{n}\sum_{t=j}^{n+j}\Big\|T^t\Big(\alpha_t f^{a-}+\sum_{m=0}^{j-t-1}\alpha_{t+m}d_m\Big)(\omega)\Big\|_X+\sum_{j=0}^{n}\sum_{t=0}^{n+j}\Big\|T^t\Big(\sum_{m=j-t}^{n+j-t}\alpha_{t+m}d_m\Big)(\omega)\Big\|_X.$$

We know that T is contracting in $L^\infty(X)$, using (1) we obtain:

$$\|\chi_n(\omega)\|_X \leq \Big[\alpha_0\|f^{a-}(\omega)\|_X + \sum_{k=0}^{n}\alpha_k\|d_k(\omega)\|_X\Big] + \sum_{j=0}^{n}\sum_{t=j}^{n+j}a\alpha + \sum_{j=0}^{n}\sum_{t=0}^{n+j}a\alpha$$
$$\leq \Big[\alpha_0\|f^{a-}(\omega)\|_X + \sum_{k=0}^{n}\alpha_k\|d_k(\omega)\|_X\Big] + \sum_{j=0}^{n}(n-j-j)\alpha a + \sum_{j=0}^{n}(n+j)\alpha a\,.$$

By (3) we can write

$$(2n+1)^2\alpha a \leq \Big[\alpha_0\|f^{a-}(\omega)\|_X + \sum_{k=0}^{n}\alpha_k\|d_k(\omega)\|_X\Big]$$
$$+ n(n+1)\alpha a + 2n^2\alpha a + \sum_{j=0}^{n}\sum_{k=0}^{n}\alpha_{j+k}\|f_{j+k}\|_X$$
$$\leq \alpha\Big[\|f^{a-}(\omega)\|_X + \sum_{k=0}^{n}\|d_k(\omega)\|_X\Big] + (3n^2+n)\alpha a + \alpha\sum_{j=0}^{n}\sum_{k=0}^{n}\|f_{j+k}\|_X.$$

As in the case $d=1$ we have by the relations (3), (4) and (5)

$$a \leq \Big[\|f^{a-}(\omega)\|_X + \sum_{k=0}^{n}\|d_K(\omega)\|_X\Big] + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\|f_{j+k}(\omega)\|_X.$$

By the (5) we see for all $\omega \in e_C^*(a,\alpha)$

$$a = \|f^{a-}(\omega)\|_X + \sum_{k=0}^{\infty}\|d_k(\omega)\|_X.$$

This implies

$$\int_{e_C^*(a,\alpha)}\big(a - \|f^{a-}(\omega)\|_X\big)\,d\mu(\omega) \leq \sum_{k=0}^{\infty}\|d_k\|_1 \leq \|f_0\|_1 = \int_\Omega \|f_0(\omega)\|\,d\mu(\omega).$$

On the other hand we can write

$$B_n(T,\alpha,f) = \frac{(2n+1)^2}{(n+1)^2}C_n(T,\alpha,f)$$

which gives $f_B^* \le 2f_C^*$ hence $e_B^*(a,\alpha) \subseteq e_C^*(a/2,\alpha)$. For $b = a/2$ we have

$$\int_{e_B^*(a,\alpha)} \big(a - \|f^{a-}(\omega)\|_X\big)\, d\mu(\omega) \le 2\int_{e_C^*(a,\alpha)} \big(b - \|f^{b-}(\omega)\|_X\big)\, d\mu(\omega)$$
$$\le 2\int_\Omega \|f^{b+}(\omega)\|_X\, d\mu(\omega) \le 2\int_\Omega \|f^{a+}(\omega)\|_X\, d\mu(\omega)$$

so

$$a\mu\big[e_B^*(a,\alpha)\big] \le a\mu\big[e_C^*(a/2,\alpha)\big] \le 2\int_\Omega \|f(\omega)\|\, d\mu(\omega). \tag{8}$$

Now, we shall prove that averages

$$B(n_1,n_2,T,\alpha,f) = \frac{1}{n_1n_2}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} \varphi^1(i)\varphi^2(j)T^{i+j}f$$

converge on a dense set in $L^1(X)$. We will need the following lemma:

Lemma 2.2. *Let T be a linear operator on $L^1(X)$ which is contracting in both $L^1(X)$ and $L^\infty(X)$, then for $f \in \mathrm{Inv}\,(T) + \mathrm{Im}\,(I-T) \cap L^\infty(X)$ the limit*

$$\lim \frac{1}{n_1n_2}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} T^{i+j}f$$

exists as n_1 and n_2 tend to infinity.

Proof. Let $f = g + (h - Th)$ with $Tg = g$, $g \in L^1(X)$ and $h \in L^\infty(X)$. Then

$$\frac{1}{n_1n_2}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} T^{i+j}f = g + \frac{1}{n_1n_2}\sum_{i=0}^{n_1} T^i\Big[\sum_{j=0}^{n_2}(T^jh - T^{j+1}f)\Big]$$
$$= g + \frac{1}{n_1n_2}\sum_{i=0}^{n_1}\big[T^ih - T^{n_2-1}f\big]$$
$$= g + \frac{1}{n_1}\Big[\frac{1}{n_2}\sum_{i=0}^{n_1} T^ih\Big] - \frac{1}{n_1}\Big[\frac{1}{n_2}\sum_{i=0}^{n_1} T^{n_2-1+i}h\Big].$$

But $\|T\|_\infty \le 1$ hence

$$\Big\|\frac{1}{n_2}\Big[\frac{1}{n_1}\sum_{i=0}^{n_1} T^ih\Big]\Big\|_\infty \le \frac{1}{n_2}\|h\|_\infty \xrightarrow{n_2\to\infty} 0$$

and

$$\Big[\frac{1}{n_2}\Big[\frac{1}{n_1}\sum_{i=0}^{n_1} T^{n_2-1+i}h\Big]\Big\|_\infty \le \frac{1}{n_2}\|h\|_\infty \xrightarrow{n_2\to\infty} 0.$$

Let $U_k f = e^{i\theta_k} T_k f$, $k = 1, 2$. U_k is a linear operator satisfying the conditions of Lemma 2.2 and so the theorem holds in the case $\phi^k(n) = e^{\theta_k n}$.

This implies that $\frac{1}{n_1 n_2} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \varphi^1(i)\varphi^2(j) T^{i+j} f$ converges a.e. on $\mathrm{Inv}\,(T) + \mathrm{Im}\,(I - T) \cap L^1(X)$ which is dense, by Kakutani-Yoshida theorem in $L^1(X)$ (X a reflexive Banach space). From Theorem 2.1(i) and the linearity of convergence of sequences we obtain that

$$\lim_{n_1, n_2 \to \infty} \frac{1}{n_1 n_2} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \varphi^1(i)\varphi^2(j) T^{i+j} f$$

exists and is finite a.e. for any trigonometric polynomial ϕ^k, $k = 1, 2$, and $f \in L^1(X)$.

The general case ($d > 2$) is similar to the real case studied by Olsen [**5**].

The second assertion (ii) follows from the maximal equality (3) and the rearrangement formula used in part I.

Let $\alpha_j = 1$, and $s_k = 1$ for all j in N and $k = 1, \ldots, d$ the average $B_n(T, d, \alpha, f)$ becomes

$$B_n(T, d, \alpha, f) = \frac{1}{(n+1)^d} \sum_{i_1=0}^{n} \cdots \sum_{i_d=0}^{n} T^{i_1 + \cdots + i_d} f = \Big(\frac{1}{n+1} \sum_{j=0}^{n} T^j \Big)^d f.$$

Using Theorem 2.1, we deduce the following:

Corollary 2.3. *Let X be a reflexive Banach space, T be a linear operator on $L^1(X)$ contracting in $L^1(X)$ and in $L^\infty(X)$, then for $d \in N$ and $f \in L^1(X)$*

$$\lim_n \Big(\frac{1}{n+1} \sum_{j=0}^{n} T^j \Big)^d f$$

exists a.e.

Acknowledgement. The author wishes to express his thank to Professors Sylvie Delabriere, Louis Sucheston, Jean Paul Thouvenot and the referee for kind advice.

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K. El Berdan, Lebanese University, Faculty of Sciences I, Departement of Mathematics, Hadeth-Beirut, (Mazraa P. O. Box: 14 - 6 573), Lebanon; *e-mail*: kberdan@inco.com.lb