ON THE MULTIPLICITY OF \((X^a - Y^b, X^c - Y^d)\)

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Abstract. In this paper an explicite formula for the computation of the multiplicity of ideal \((X^a - Y^b, X^c - Y^d)\) is given.

Let \(K[X, Y]\) be a polynomial ring over a field \(K\), \(A = K[X, Y]_\langle X,Y \rangle\) be a local ring with the maximal ideal \(M = (X, Y) \cdot A\). For an \(M\)-primary ideal \(Q\) in \(A\) we denote by \(P(n) := \ell(A/Q^{n+1})\) the Hilbert-Samuel function, where \(\ell(A/Q^{n+1})\) is the length of \(A\)-module \(A/Q^{n+1}\). The function \(P(n)\) is for \(n \gg 0\) a polynomial in \(n\) of degree 2 which can be written as \(P(n) = e_0(Q) n^2 + e_1(Q)n + e_2(Q)\). The coefficient \(e_0(Q)\) is called the multiplicity of \(Q\). It is well-known, that \(e_0(Q)\) is a positive integer (for more details see [3]).

In this short note we give a formula for the computation of the multiplicity for certain class of \(M\)-primary ideals in \(A\). It is a third article of the series beginning with [1], [2]. Our main result is the following theorem.

**Theorem.** Let \(Q = (X^a - Y^b, X^c - Y^d) \leq A\) be a \(M\)-primary ideal in \(A = K[X, Y]_\langle X,Y \rangle\) (\(a, b, c, d\) are positive integers). Then

\[e_0(Q) = \min\{ad, bc\}.

To prove the Theorem we need the following lemma.

**Lemma.** Let \(Q = (X^a - Y^b, X^c - Y^d) \cdot A\) be a \(M\)-primary ideal in the local ring \(A = K[X, Y]_\langle X,Y \rangle\) (\(a, b, c, d\) are positive integers).

(a) if \(b \leq a\) and \(c \leq d\) then \(e_0(Q, A) = bc\).

(b) if \(a \leq b\) and \(d \leq c\) then \(e_0(Q, A) = ad\).

**Proof.** See [4, Lemma 3.1].

**Proof of the Theorem.** On the ground of Lemma the only case to prove is \(b < a\) and \(d < c\). Let \(bc = \min\{ad, bc\}\), t.m. \(\frac{b}{d} < \frac{a}{c}\). Note, that the conditions of Theorem imply \(ad \neq bc\). Let \(\left\lfloor \frac{b}{d} \right\rfloor\) indicates the integer part of \(\frac{b}{d}\).

Received November 27, 1997.

We denote the integer part of \( q = 0 \). Let \( k \leq \frac{b}{d} < k + 1 < \cdots < k + \rho \leq \frac{a}{c} < k + \rho + 1, \ \rho \in N, \ \rho > 0 \).

For \( kc < a \) and \( kd \leq b \), we can write

\[
Q = (X^{kc}, X^{a-kc} - Y^{kd}, Y^{b-kd}, X^c - Y^d)
\]

\[
= (X^{kc}, X^{a-kc} - X^{kc} \cdot Y^{b-kd}, X^c - Y^d)
\]

because \( X^{kc} \equiv Y^{kd} \pmod{Q} \)

\[
= (X^{kc}, (X^{a-kc} - X^{b-kd}), X^c - Y^d)
\]

and therefore

\[
e_0(Q) = e_0(X^{kc}, X^c - Y^d) + e_0(X^c - Y^d, X^{a-kc} - Y^{b-kd})
\]

\[
= kcd + e_0(X^c - X^{a-kc} \cdot Y^{d-(b-kd)}, X^{a-kc} - Y^{b-kd})
\]

\[
= kcd + e_0(X^c, Y^{b-kd})
\]

since \((X^c, (1 - X^{a-kc-c}, Y^{d-(b-kd)}), X^{a-kc} - Y^{b-kd}) = (X^c, X^{a-ke} - Y^{b-kd}) = (X^c, Y^{b-kd}) in A.\) So we have \( e_0(Q) = kcd + c(b - kd) = bc \). This completes the proof if \( \left[ \frac{b}{a} \right] < \left[ \frac{a}{c} \right] \).

Let now \( k := \left[ \frac{a}{c} \right] = \left[ \frac{a}{c} \right] \). Then we have

\[
k \leq \frac{b}{d} < \frac{a}{c} < k + 1
\]

and from this follows

\[
a = kc + p, \quad 0 < p < c,
\]

\[
b = kd + q, \quad 0 \leq q < d.
\]

Then as above \( e_0(Q) = e_0(X^{kc}(X^{a-kc} - Y^{b-kd}), X^c - Y^d) = e_0(X^{kc}, Y^d) = bc \) if \( q = 0 \). Let \( q \neq 0 \). Then \( e_0(Q) = kcd + e_0(Q_1) \), where \( Q_1 = (X^p - Y^q, X^c - Y^d) \).

We denote the integer part of \( \frac{c}{p} \) as \( k_1 \). From \( k_1 = \left[ \frac{c}{p} \right] \) follows \( k_1 \leq \frac{c}{p} < \frac{d}{q} \) so there exist \( p_1, q_1 \) such that

\[
c = k_1p + p_1, \quad 0 \leq p_1 < p,
\]

\[
d = k_1q + q_1, \quad 0 < q_1.
\]

If \((k_1 + 1) \cdot q \leq d\), then

\[
Q_1 = (X^p - Y^q, X^c - Y^{(k_1+1)q}, Y^{d-(k_1+1)q})
\]

\[
= (X^p - Y^q, X^c(1 - X^{(k_1+1)p-c} \cdot Y^{d-(k_1+1)q}))
\]
while $X^{(k_1+1)p} \equiv Y^{(k_1+1)q} \pmod{Q_1}$

$$= (X^c, X^p - Y^q) \text{ in } A.$$ 

Then $e_0(Q) = kcd + cq = bc$.

Let now $(k_1 + 1)q > d$. Then we have

\begin{align*}
c &= k_1p + p_1, & 0 \leq p_1 < p, \\
d &= k_1q + q_1, & 0 < q_1 < q.
\end{align*}

Then

\begin{align*}
Q_1 &= (X^p - Y^q, X^{k_1p + p_1} - Y^{k_1q + q_1}) \\
&= (X^p - Y^q, X^{k_1p}, X^{p_1} - Y^{k_1q}, Y^q)
\end{align*}

because $X^{k_1p} \equiv Y^{k_1q} \pmod{Q_1}$

$$= (X^p - Y^q, X^{k_1p}(X^{p_1} - Y^{q_1}))$$

and hence $e_0(Q) = kcd + e_0(X^p - Y^q, X^{k_1p}) + e_0(X^p - Y^q, X^{p_1} - Y^{q_1}) = kcd + k_1pq + e_0(Q_2)$ with $Q_2 = (X^p - Y^q, X^{p_1} - Y^{q_1})$.

We continue our algorithm.

Let $k_2$ denotes the integer part of $\frac{a}{q_1}$, i.e. $k_2q_1 \leq q$, but $(k_2 + 1)q_1 > q$. Then there are integers $p_2, q_2$ such that

\begin{align*}
p &= k_2p_1 + p_2, & 0 < p_2 \\
q &= k_2q_1 + q_2, & 0 \leq q_2 < q_1.
\end{align*}

If $(k_2 + 1)p_1 \leq p$, then

\begin{align*}
Q_2 &= (X^{(k_2+1)p_1}, X^{p-(k_2+1)p_1} - Y^q, X^{p_1} - Y^{q_1}) \\
&= (X^{(k_2+1)q_1}, X^{p-(k_2+1)p_1} - Y^q, X^{p_1} - Y^{q_1}) \\
&= (X^q(X^{(k_2+1)q_1} - q \cdot X^{p-(k_2+1)p_1} - 1), X^{p_1} - Y^{q_1}) \\
&= (X^q, X^{p_1} - Y^{q_1}) \text{ in } A.
\end{align*}

Then $e_0(Q) = kcd + k_1pq + qp_1 = bc$.

Let now $(k_2 + 1)p_1 > p$. Then we have $p_2 < p_1$,

\begin{align*}
Q_2 &= (X^{k_2p_1}, X^{p_2 - Y^{k_2q_1}}, Y^{q_2}, X^{p_1} - Y^{q_1}) \\
&= (X^{k_2p_1}, (X^{p_2} - Y^{q_2}), X^{p_1} - Y^{q_1})
\end{align*}
and within
\[ e_0(Q) = kcd + k_1pq + k_2p_1q_1 + e_0(Q_3), \quad 0 \leq p_2 < p_1. \]

with \( Q_3 = (X^{p_2} - Y^{q_2}, X^{p_1} - Y^{q_1}) \).

There are two descending chains of nonnegatives integers
\[ p > p_1 > p_2 > \ldots \]
\[ q > q_1 > q_2 > \ldots \]

which have to stop after \( n \) steps. Note that \( p_{2n} \neq 0 \) and \( q_{2n-1} \neq 0 \) for all \( n \). Let \( q_{2n} = 0 \) is the first zero and for all \( k < 2n \) \( q_k \neq 0 \). Then
\[ e_0(Q) = kcd + k_1pq + k_2p_1q_1 + k_3p_2q_2 + \cdots + k_{2n-1} \cdot p_{2n-1} \cdot q_{2n-1} \]
\[ = kcd + q(c - p_1) + p_1(q - q_2) + q_2(p_1 - p_3) + \cdots + p_{2n-1} \cdot (q_{2n-2} - q_{2n}) \]
\[ = kcd + qc = kcd + c(b - kd) = bc. \]

Let consequently \( p_{2n-1} = 0 \) (\( p_k \neq 0, q_k \neq 0 \) for all \( k < 2n - 1 \)).

Then it holds
\[ e_0(Q) = kcd + k_1pq + k_2p_1q_1 + k_3p_2q_2 + \cdots + k_{2n-1} \cdot p_{2n-2}q_{2n-2} \]
\[ = kcd + q(c - p_1) + p_1(q - q_2) + q_2(p_1 - p_3) + \cdots + q_{2n-2}(p_{2n-3} - p_{2n-1}) \]
\[ = kcd + c(b - kd) = bc, \]

which completes the proof for \( bc \) as a minimum of \( \{bc, ad\} \). The proof for the second case (\( ad = \min\{ad, bc\} \)) is the same as the first one. \( \square \)

References

1. Boďa E. and Solčan Š., On the multiplicity of \((X^{m_1}, X^{m_2}, X^{k_1}Y^{l_1})\), Acta Math. Univ. Comenianae LII-LIII (1987), 297–299.

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