

## ON COMPLETE MEASURABILITY OF MULTIFUNCTIONS DEFINED ON PRODUCT SPACES

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### 1. INTRODUCTION

In the present note we occupy ourselves with the cases in which we can say that a multifunction of two variables is jointly measurable. In particular we generalize onto the case of multifunctions the Theorems 2 and 3 from paper [6, pp. 150–151]. We start with the concept of “measurable space with negligibles” which is intended as a common generalization of the two principal examples:  $(S, \mathcal{M}(S), \mathcal{N}(S))$ , where  $(S, \mathcal{M}(S), \mu)$ , is a measurable space and  $\mathcal{N}(S)$  is the  $\sigma$ -ideal of  $\mu$ -measure zero subsets of  $S$  and  $(S, \mathcal{B}(S), \mathcal{I}(S))$ , where  $(S, \mathcal{T}(S))$  is a topological space,  $\mathcal{B}(S)$  is the  $\sigma$ -algebra of subsets of  $S$  with the Baire property and  $\mathcal{I}(S)$  is the  $\sigma$ -ideal of meager subsets of  $S$ .

### 2. PRELIMINARIES

**Definition 1** ([1, Definition 1]). A measurable space with negligibles is a triple  $(S, \mathcal{M}(S), \mathcal{J}(S))$  where  $S$  is a set,  $\mathcal{M}(S)$  is a  $\sigma$ -algebra of subsets of  $S$  and  $\mathcal{J}(S) \subset \mathcal{P}(S)$  is a  $\sigma$ -ideal of the Boolean algebra  $\mathcal{P}(S)$  generated by  $\mathcal{J}(S) \cap \mathcal{M}(S)$ .

Such space  $(S, \mathcal{M}(S), \mathcal{J}(S))$  is said to be complete if  $\mathcal{J}(S) \subset \mathcal{M}(S)$ .

If  $(S, \mathcal{M}(S), \mathcal{J}(S))$  is an arbitrary measurable space with negligibles, we can determine its completion  $(S, \widehat{\mathcal{M}}(S), \mathcal{J}(S))$  by putting:

$$\widehat{\mathcal{M}}(S) = \{H \subset S : \text{there exist two } \mathcal{M}(S)\text{-measurable sets } A \text{ and } B \text{ such} \\ \text{that } A \subset H \subset B \text{ and } A \setminus B \in \mathcal{J}(S)\}.$$

Let  $(X, \mathcal{M}(X), \mathcal{J}(X))$  and  $(Y, \mathcal{M}(Y), \mathcal{J}(Y))$  be two measurable spaces with negligibles. Let  $\mathcal{M}(X) \otimes \mathcal{M}(Y)$  be the  $\sigma$ -algebra generated by  $\mathcal{M}(X) \times \mathcal{M}(Y)$  and let  $\mathcal{J}(X) \otimes \mathcal{J}(Y)$  denotes the  $\sigma$ -ideal generated by all the sets of the form

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$X \times J$  and  $I \times Y$  where  $I \in \mathcal{J}(X)$  and  $J \in \mathcal{J}(Y)$ . Let the product  $X \times Y$  be endowed with the  $\sigma$ -ideal  $\mathcal{J}(X \otimes Y)$ , including  $\mathcal{J}(X) \otimes \mathcal{J}(Y)$ , with the following ‘‘Fubini’s property’’:

For any set  $A \in \widehat{\mathcal{M}}(X \otimes Y)$  the following implications hold:

- (1)  $\{x \in X : A_x \notin \mathcal{J}(Y)\} \in \mathcal{J}(X) \implies A \in \mathcal{J}(X \otimes Y)$  and  
 (2)  $\{y \in Y : A^y \notin \mathcal{J}(X)\} \in \mathcal{J}(Y) \implies A \in \mathcal{J}(X \otimes Y)$ ,

where  $A_x = \{y \in Y : (x, y) \in A\}$  and  $A^y = \{x \in X : (x, y) \in A\}$  denote the  $x$ -section of  $A$  and  $y$ -section of  $A$  respectively and  $\widehat{M}(X \otimes Y)$  denotes  $\mathcal{J}(X \otimes Y)$ -completion of  $\mathcal{M}(X) \otimes \mathcal{M}(Y)$ .

### 3. THE ABSTRACT BAIRE CATEGORY CONCEPTS

Following [8] a pair  $(S, \mathcal{C}(S))$ , where  $\mathcal{C}(S) \subset \mathcal{P}(S)$  is a family of subsets of  $S$ , is called a category base, if the nonempty sets in  $\mathcal{C}(S)$ , called regions, satisfy the following axioms:

C.1. Every point of  $S$  belongs to some region, i.e.  $S = \bigcup\{A : A \in \mathcal{C}(S)\}$ .

C.2. Let  $A$  be a region and let  $\mathcal{D}(S)$  be any nonempty family of disjoint regions which has cardinality less than the cardinality of  $\mathcal{C}(S)$ .

- (a) If  $A \cap (\bigcup\{D : D \in \mathcal{D}(S)\})$  contains a region, then there is a region  $D_0 \in \mathcal{D}(S)$  such that  $A \cap D_0$  contains a region.  
 (b) If  $A \cap (\bigcup\{D : D \in \mathcal{D}(S)\})$  contains no region then there is a region  $B \subset A$  which is disjoint from every region in  $\mathcal{D}(S)$ .

Notice (see [9]) that parts (a) and (b) of C.2 can be rewritten in the following form:

- (c) If  $A \cap B$  contains no region for each  $B \in \mathcal{C}(S)$  then there is a subregion of  $A$  which is disjoint from  $\bigcup \mathcal{D}(S)$ .

A set  $E$  is singular if every region contains a subregion which is disjoint from  $E$ . A countable union of singular sets is called a meager set. A set which is not meager set is called an abundant set. The family of all singular sets forms an ideal and the family  $\mathcal{J}_{\mathcal{C}}(S)$  of all meager sets forms a  $\sigma$ -ideal.

A set  $E$  is meager (resp. abundant) in a region  $A$  if  $E \cap A$  is a meager (resp. abundant) set.

A set  $E$  has the abstract Baire property if every region  $A \in \mathcal{C}(S)$  has a subregion  $B \subset A$  in which either  $E$  or  $X \setminus E$  is a meager set. The sets which have the abstract Baire property form a  $\sigma$ -algebra  $\mathcal{B}_{\mathcal{C}}(S)$ , which contains all regions and all meager sets. Thus the triple  $(S, \mathcal{B}_{\mathcal{C}}(S), \mathcal{J}_{\mathcal{C}}(S))$  creates a complete measurable space with negligibles.

A family  $\mathcal{A}$  of  $\mathcal{C}(S)$ -regions with the property that each abundant set is abundant everywhere in at least one region  $A$  in  $\mathcal{A}$  (i.e. it is abundant in every subregion

of  $A$ ) is called a quasi-base. A category base is called separable if it has a countable quasi-base.

Let  $(X, \mathcal{C}(X))$  and  $(Y, \mathcal{C}(Y))$  be category bases. It is known that  $(X \times Y, \mathcal{C}(X) \times \mathcal{C}(Y))$  is not necessarily a category base (see Example 2A, p. 110 in [7]).

If  $(X \times Y, \mathcal{C}(X) \times \mathcal{C}(Y))$  is a category base, then it is called a product base. A general theorem concerning the existence of product bases and many examples are given on pp. 112–114 in [7].

Assume that  $(X \times Y, \mathcal{C}(X) \times \mathcal{C}(Y))$  is a product base and  $(Y, \mathcal{C}(Y))$  is separable. Among the properties involving separability there is the following:

$$(3) \quad \text{If } A \in \mathcal{B}_{\mathcal{C}}(X \otimes Y) \wedge \{x : A_x \notin \mathcal{J}_{\mathcal{C}}(Y)\} \in \mathcal{J}_{\mathcal{C}}(X), \text{ then } A \in \mathcal{J}_{\mathcal{C}}(X \otimes Y),$$

where  $\mathcal{J}_{\mathcal{C}}(X \otimes Y)$  means the  $\sigma$ -ideal of meager sets with respect to  $\mathcal{C}(X) \times \mathcal{C}(Y)$  and  $\mathcal{B}_{\mathcal{C}}(X \otimes Y)$ , including  $\mathcal{B}_{\mathcal{C}}(X) \otimes \mathcal{B}_{\mathcal{C}}(Y)$ , the  $\sigma$ -algebra of sets with the abstract Baire property with respect to  $\mathcal{C}(X) \times \mathcal{C}(Y)$ .

#### 4. MAIN RESULTS

Let  $S$  and  $Z$  be some sets and let  $F: S \rightarrow Z$  be a multifunction (i.e.  $F(s) \subset Z$  for  $s \in S$ ). Then two counterimages of  $G \subset Z$  may be defined:

$$(4) \quad F^+(G) = \{s \in S : F(s) \subset G\} \quad \text{and} \quad F^-(G) = \{s \in S : F(s) \cap G \neq \emptyset\}.$$

It is clear that

$$(5) \quad F^-(G) = S \setminus F^+(Z \setminus G) \quad \text{and} \quad F^+(G) = S \setminus F^-(Z \setminus G).$$

**Definition 2.** Let  $(S, \mathcal{M}(S))$  be a measurable space and let  $(Z, \mathcal{T}(Z))$  be a topological space. We say that a multifunction  $F: S \rightarrow Z$  is lower (upper)  $\mathcal{M}(S)$ -measurable if the counterimage  $F^-(G)$  ( $F^+(G)$ ) is a  $\mathcal{M}(S)$ -measurable set for each  $G \in \mathcal{T}(Z)$ .

We describe the relationships between lower and upper  $\mathcal{M}(S)$ -measurability without any metrizability assumptions in contrast to the corresponding results from [2].

**Proposition 1** (cf. [2, Theorem 3.1, p. 55] in the metric case). *Let  $(S, \mathcal{M}(S))$  be a measurable space,  $(Z, \mathcal{T}(Z))$  a topological space and let  $F: S \rightarrow Z$  be a multifunction. Then*

- (i) *If  $(Z, \mathcal{T}(Z))$  is a perfect space and  $F$  is upper  $\mathcal{M}(S)$ -measurable, then it is lower  $\mathcal{M}(S)$ -measurable.*
- (ii) *If  $(Z, \mathcal{T}(Z))$  is perfectly normal and  $F$  is a compact-valued lower  $\mathcal{M}(S)$ -measurable multifunction, then it is upper  $\mathcal{M}(S)$ -measurable.*

*Proof.* Part (i) is obvious because we have

$$(6) \quad F^-(G) = \bigcup_{n \in N} F^-(B_n) \in \mathcal{M}(S),$$

where  $Z \setminus B_n \in \mathcal{T}(Z)$ , whenever  $G \in \mathcal{T}(Z)$ .

Let  $B$  be a closed subset of  $Z$ . By virtue of perfect normality of  $(Z, \mathcal{T}(Z))$  there is a sequence  $(G_n)_{n \in N}$  of  $\mathcal{T}(Z)$ -open sets such that

$$(7) \quad B = \bigcap_{n \in N} G_n = \bigcap_{n \in N} Cl(G_n) \quad \text{and}$$

$$(8) \quad G_{n+1} \subset Cl(G_{n+1}) \subset G_n \quad \text{for } n = 1, 2, \dots,$$

By (7) and (5) we have

$$(9) \quad F^-(B) = S \setminus F^+\left(\bigcup_{n \in N} (Z \setminus G_n)\right) = S \setminus F^+\left(\bigcup_{n \in N} (Z \setminus Cl(G_n))\right).$$

The family  $\{Z \setminus Cl(G_n) : n \in N\}$  forms an open covering of compact subset  $F(s)$  for each fixed  $s \in F^+(\bigcup_{n \in N} (Z \setminus G_n))$ . By (8) this covering is increasing. Consequently we have

$$(10) \quad F(s) \subset \bigcup_{n \in N} (Z \setminus G_n) \text{ if and only if there exists } n(s) \in N \text{ such that}$$

$$F(s) \subset Z \setminus Cl(G_{n(s)}) \subset Z \setminus G_{n(s)+1}.$$

Applying (10) we infer that

$$(11) \quad F^+\left(\bigcup_{n \in N} (Z \setminus G_n)\right) = \bigcup_{n \in N} F^+(Z \setminus G_n).$$

So (9) completes the argument and proof is finished.  $\square$

**Proposition 2.** *Let  $(S, \mathcal{M}(S))$  be a measurable space and let  $(Z, \mathcal{T}(Z))$  be a second countable Hausdorff space. Let  $F_1, F_2 : S \rightarrow Z$  be two compact-valued lower  $\mathcal{M}(S)$ -measurable multifunctions. Then*

$$(12) \quad \{s \in S : F_1(s) \neq F_2(s)\} \in \mathcal{M}(S).$$

*Proof.* Observe that

$$(13) \quad F_1(s) \neq F_2(s) \text{ if and only if there exists } z \in Z \text{ such that}$$

$$z \in F_1(s) \div F_2(s), \text{ where } \div \text{ denotes the symmetric difference.}$$

Let  $z_0 \in F_1(s)$  and  $z_0 \notin F_2(s)$ . For each  $z \in F_2(s)$  let  $U(z)$  and  $V(z)$  denote open sets with the property:

$$(14) \quad z \in U(z) \text{ and } z_0 \in V(z) \text{ and } U(z) \cap V(z) = \emptyset.$$

The family  $\{U(z) : z \in F_2(s)\}$  forms an open covering of the compact set  $F_2(s)$ . Thus

$$(15) \quad \text{there exists } n \in \mathbb{N} \text{ and } \{z_1, z_2, \dots, z_n\} \subset Z \text{ such that } F_2(s) \subset \bigcup_{i=1}^n U(z_i).$$

Moreover

$$(16) \quad \bigcap_{i=1}^n V(z_i) \cap \bigcup_{i=1}^n U(z_i) = \emptyset.$$

There is a basic open set  $V$  in  $Y$  such that:

$$(17) \quad z_0 \in V \subset V(z_1) \cap V(z_2) \cap \dots \cap V(z_n).$$

From the fact that  $z_0 \in F_1(s) \cap V$  we infer that:

$$(18) \quad s \in F_1^-(V) \quad \text{and} \quad s \in S \setminus F_2^-(V) = F_2^+(Z \setminus V).$$

Consequently,

$$(19) \quad s \in F_1^-(V) \cap F_2^+(Z \setminus V) \in \mathcal{M}(S).$$

If on the contrary,  $z_0 \in F_2(s) \setminus F_1(s)$ , then symmetrically

$$s \in F_2^-(V) \cap F_1^+(Z \setminus V) \in \mathcal{M}(S)$$

for chosen in a suitable manner basic open set  $V \subset Z$ .

Thus, if  $\{V_1, V_2, \dots\}$  create a countable basis of  $Z$ , we have

$$(20) \quad \{s \in S : F_1(s) \neq F_2(s)\} = \bigcup_{i \in \mathbb{N}} [(F_1^-(V_i) \cap F_2^+(Z \setminus V_i)) \cup (F_2^-(V_i) \cap F_1^-(Z \setminus V_i))] \in \mathcal{M}(S)$$

completing the proof. □

Let as remark that:

$$(21) \quad \text{the equality (20) holds also in the case when } F_1 \text{ and } F_2 \text{ are closed-valued and } Z \text{ is regular and second countable.}$$

Let  $(S, \mathcal{M}(S), \mathcal{J}(S))$  be a measurable space with negligibles and let  $\widehat{\mathcal{M}}(S)$  be the  $\mathcal{J}(S)$ -completion of  $\mathcal{M}(S)$ . Let  $Z$  be a topological space and let  $\mathcal{B}_0(Z)$  denotes the Borel  $\sigma$ -algebra of the space  $Z$ .

We define the following “projection property”:

$$(22) \quad \begin{aligned} &\text{If } A \in \mathcal{M}(S) \otimes \mathcal{B}_0(Z), \text{ then} \\ &\Pi_S(A) = \{s \in S : \text{there exists } z \in Z \text{ such that } (s, z) \in A\} \in \widehat{\mathcal{M}}(S). \end{aligned}$$

Among examples of spaces fulfilling “projection property” (22) are complete measure spaces  $(S, \mathcal{M}(S), \mu)$  ( see [1, 6B(f)]) and the  $(S, \mathcal{B}_0(S), \mathcal{J}(S))$ , where  $S$  is an arbitrary topological space (see [1, 7C, p. 67]). Such spaces are proto-decomposable in the meaning of Definition 1B(h), in [1], and thus, by [1, 1D(b)(iii) and 1H, p. 10], their completions have  $\sigma$ -algebras closed under Souslin’s operation, which in turn insures (22). Note that the notion of proto-decomposability of measurable spaces with negligibles were offered by D. H. Fremlin as a generalization of the localization principle of Banach, which also applies to the most important measure spaces. An abstract measurable space with negligibles  $(S, \mathcal{M}(S), \mathcal{J}(S))$  which is also  $\omega_1$ -saturated, that means:

$$(23) \quad \begin{aligned} &\text{If for every } \mathcal{A} \subset \mathcal{M}(S) \text{ card } \mathcal{A} = \omega_1, \text{ then there exist two sets} \\ &A \in \mathcal{A} \text{ and } B \in \mathcal{A} \text{ such that } A \neq B \text{ and } A \cap B \notin \mathcal{J}(S), \end{aligned}$$

is also known to be proto-decomposable and thus its completion  $(S, \widehat{\mathcal{M}}(S), \mathcal{J}(S))$  has the required property (22).

**Proposition 3.** *Let  $(S, \mathcal{M}(S), \mathcal{J}(S))$  be a measurable space with negligibles and  $Z$  let be a separable metrizable space. Assume that property (22) holds. Let for  $n \in N$   $F_n: S \rightarrow Z$  be a sequence of closed-valued lower  $\mathcal{M}(S)$ -measurable multifunctions. Then multifunction  $F: S \rightarrow Z$  given by formula:*

$$(24) \quad F(s) = \left( \bigcap_{n \in N} F_n \right)(s) = \bigcap_{n \in N} F_n(s)$$

is upper  $\widehat{\mathcal{M}}(S)$ -measurable.

*Proof.* Define functions  $f_n: S \times Z \rightarrow R$  as follows:

$$(25) \quad f_n(s, z) = \text{dist}(z, F_n(s)) \text{ for } (s, z) \in S \times Z,$$

where (denoting by  $d$  a metric on  $Z$ )  $\text{dist}(z, B) = \inf\{d(z, b) : b \in B\}$ .

Then observe, that the graph of  $F_n$  is the kernel of  $f_n$ :

$$\text{Gr } F_n = \{(s, z) \in S \times Z : z \in F_n(s)\} = f_n^{-1}(0).$$

All the sections  $(f_n)^z, z \in Z$ , are  $\mathcal{M}(S)$ -measurable and all the sections  $(f_n)_s, s \in S$ , are continuous on  $Z$ . Thus, by virtue of the known theorem (see e.g. Theorem 2, p. 65 in [5])  $f_n$  is  $\mathcal{M}(S) \otimes \mathcal{B}_0(Z)$ -measurable, so that, by (26) we have:

$$(27) \quad \text{Gr } F_n \in \mathcal{M}(S) \otimes \mathcal{B}_0(Z).$$

Hence

$$(28) \quad \text{Gr } F = \text{Gr} \left( \bigcap_{n \in N} F_n \right) = \bigcap_{n \in N} (\text{Gr } F_n) \in \mathcal{M}(S) \otimes \mathcal{B}_0(Z).$$

Let  $B \in \mathcal{B}_0(Z)$  be a closed subset of  $Z$ . By virtue of (22) we obtain

$$(29) \quad F^-(B) = \Pi_S(\text{Gr } F \cap (S \times B)) \in \widehat{\mathcal{M}}(S)$$

as a projection of the intersection of two  $\mathcal{M}(S) \otimes \mathcal{B}_0(Z)$ -measurable subsets of  $S \times Z$ , which finishes the proof.  $\square$

In context of Proposition 3 let us remark that there is an example (see Example 2, p. 166 in [3]) showing, that the intersection of two lower  $\mathcal{M}(S)$ -measurable multifunctions  $F_1$  and  $F_2$  with closed values may fail to be lower  $\mathcal{M}(S)$ -measurable, even if  $Z$  is Polish,  $S$  is the unit interval endowed with the Borel  $\sigma$ -algebra and  $\text{dom}(F_1 \cap F_2) = \{s \in S : F_1 \cap F_2 \neq \emptyset\} = S$ .

**Definition 3.** Let  $(S, \mathcal{T}(S))$  and  $(Z, \mathcal{T}(Z))$  be two topological spaces and  $F: S \rightarrow Z$  let be a multifunction.  $F$  is called lower semicontinuous at a point  $s_0 \in S$  when for each  $U \in \mathcal{T}(Z)$  we have:

$$(30) \quad \text{If } F(s_0) \cap U \neq \emptyset, \text{ then there exists a set } G \in \mathcal{T}(S) \text{ such that } s_0 \in G \text{ and } F(s) \cap U \neq \emptyset \text{ for each } s \in G.$$

Dually,  $F$  is called upper semicontinuous at a point  $s_0 \in S$  when for each  $U \in \mathcal{T}(Z)$  we have:

$$(31) \quad \text{If } F(s_0) \subset U, \text{ then there exists a set } G \in \mathcal{T}(S) \text{ such that } s_0 \in G \text{ and } F(s) \subset U \text{ for each } s \in G.$$

$F$  is lower (resp. upper) semicontinuous if it is lower (resp. upper) semicontinuous at each point  $s_0 \in S$ .

Let  $\mathcal{U}(s_0)$  denotes a filterbase of open neighborhoods of the point  $s_0 \in S$ . The grill of  $\mathcal{U}(s_0)$ , denoted here by  $\mathcal{U}''(s_0)$ , is defined as follows:

$$(32) \quad \mathcal{U}''(s_0) = \{A(s_0) \subset S : A(s_0) \cap U \neq \emptyset \text{ for each } U \in \mathcal{U}(s_0)\}.$$

Observe that:

$$(33) \quad \text{If } A \in \mathcal{U}''(s_0), \text{ then } s_0 \in Cl(A).$$

Following [4] we define the upper and lower limit of a multifunction  $F: S \rightarrow Z$  as follows:

$$(34) \quad \text{p-Lim sup}_{s \rightarrow s_0} F(s) = \bigcap_{U \in \mathcal{U}(s_0)} Cl\left(\bigcup_{s \in U} F(s)\right),$$

$$(35) \quad \text{p-Lim inf}_{s \rightarrow s_0} F(s) = \bigcap_{A \in \mathcal{U}''(s_0)} Cl\left(\bigcup_{s \in A} F(s)\right).$$

Our multifunction  $F$  is lower semicontinuous at  $s_0 \in S$  if and only if  $F(s_0) \subset \text{p-Lim inf}_{s \rightarrow s_0} F(s)$ . If the space  $Z$  is regular,  $F$  has closed values and it is continuous at  $s_0 \in S$  (that means it is simultaneously lower and upper semicontinuous), then

$$(36) \quad \text{p-Lim inf}_{s \rightarrow s_0} F(s) = F(s_0) = \text{p-Lim sup}_{s \rightarrow s_0} F(s).$$

Let  $\mathcal{B}$  be a basis for  $S$ . Let us replace  $\mathcal{U}''(s_0)$  in (35) by equality:

$$(37) \quad \mathcal{U}''(s_0) \cap \mathcal{B} = \{U \in \mathcal{B} : s_0 \in Cl(U)\}$$

and denote the resulting operation by  $\text{q-Lim inf}$ . We have:

$$(38) \quad \text{p-Lim inf} \subset \text{q-Lim inf} \subset \text{p-Lim sup}.$$

At each continuity point  $s_0$  of  $F$  we have also

$$(39) \quad \text{q-Lim inf}_{s \rightarrow s_0} F(s) = F(s_0) = \text{p-Lim sup}_{s \rightarrow s_0} F(s).$$

Thus the set  $\{s_0 \in S : \text{q-Lim inf}_{s \rightarrow s_0} F(s) \neq \text{p-Lim sup}_{s \rightarrow s_0} F(s)\}$  is contained in the set  $D(F)$  of all discontinuity points of  $F$ .

**Proposition 4.** *Let  $(X, \mathcal{M}(X))$  be a measurable space,  $(Y, \mathcal{T}(Y))$  a second countable topological space and let  $(Z, \mathcal{T}(Z))$  be a second countable perfectly normal topological space. Let  $F: X \times Y \rightarrow Z$  be a closed-valued multifunction with lower  $\mathcal{M}(X)$ -measurable all sections  $F^y$ ,  $y \in Y$ . Denote by  $\mathcal{J}(X \otimes Y)$  a  $\sigma$ -ideal in  $X \times Y$  including  $\mathcal{J}(X) \otimes \mathcal{J}(Y)$  such that (22) holds. Let  $P$  be a countable dense subset of  $Y$ . Then multifunction  $G_*: X \times Y \rightarrow Z$  defined by formula:*

$$(40) \quad G_*(x, y) = \text{q-Lim inf}_{t \rightarrow y \wedge t \in P} (F_x)(t)$$

is upper  $\widehat{\mathcal{M}}(X \otimes Y)$ -measurable, where  $\widehat{\mathcal{M}}(X \otimes Y)$  denotes  $\mathcal{J}(X \otimes Y)$ -completion of  $\mathcal{M}(X) \otimes \mathcal{B}_0(Y)$ .

*Proof.* Let  $\mathcal{B}$  denotes a countable base of  $Y$ . We have

$$(41) \quad G_*(x, y) = \bigcap_{U \in \mathcal{B} \wedge y \in Cl(U)} Cl\left(\bigcup_{t \in U \cap P} F(x, t)\right).$$

Define for each  $U \in \mathcal{B}$  a multifunction  $H_U$  by formula:

$$(42) \quad H_U(x, y) = \bigcup_{t \in U \cap P} F(x, t) \subset Z,$$

and observe that for  $V \in \mathcal{T}(Z)$  we have:

$$(43) \quad \begin{aligned} H_U^-(V) &= \{(x, y) : \text{there is } t \in U \cap P \text{ such that } F(x, t) \cap V \neq \emptyset\} \\ &= \bigcup_{t \in U \cap P} (\{x \in X : F(x, t) \cap V \neq \emptyset\} \times Y) \\ &= \bigcup_{t \in U \cap P} ((F^t)^-(V) \times Y) \in \mathcal{M}(X) \otimes \mathcal{B}_0(Y) \end{aligned}$$

since  $U \cap P$  is countable and  $F^t$  are lower  $\mathcal{M}(X)$ -measurable. So by the well known fact (see [2, Prop. 2.6, p. 55]) multifunction  $\overline{H}_U: X \times Y \rightarrow Z$  defined by equality:

$$(44) \quad \overline{H}_U(x, y) = Cl(H_U(x, y))$$

is also  $\mathcal{M}(X) \otimes \mathcal{B}_0(Y)$ -measurable.

Observe that

$$(45) \quad G_*(x, y) = \bigcap \{\overline{H}_U(x, y) : U \in \mathcal{B} \wedge y \in Cl(U)\}.$$

Define multifunction  $G_U: X \times Y \rightarrow Z$  by formula:

$$G_U(x, y) = \begin{cases} \overline{H}_U(x, y) & \text{if } y \in Cl(U), \\ Z & \text{if } y \notin Cl(U) \end{cases}$$

and observe that

$$(46) \quad G_U^-(V) = H_U^-(V) \cap (X \times Cl(U)) \cup (X \times (Y \setminus Cl(U))) \in \mathcal{M}(X) \otimes \mathcal{B}_0(Y).$$

We have

$$(47) \quad G_*(x, y) = \bigcap_{U \in \mathcal{B}} G_U(x, y).$$

By virtue of Proposition 3 multifunction  $G_*$  is upper  $\widehat{\mathcal{M}}(X \otimes Y)$ -measurable (according to Proposition 1 it is also lower  $\widehat{\mathcal{M}}(X \otimes Y)$ -measurable) and the proof is finished.  $\square$

**Proposition 5.** *Let  $X, Y, Z, P$  and  $F$  be the same as in Proposition 4. Define multifunction  $G^* : X \times Y \rightarrow Z$  as follows:*

$$(48) \quad G^*(x, y) = \text{p-Lim sup}_{t \rightarrow y \wedge t \in P} (F_x)(t).$$

*Then multifunction  $G^*$  is upper  $\widehat{\mathcal{M}}(X \otimes Y)$ -measurable.*

*Proof.* The proof of this proposition is very similar to the preceding one.  $\square$

We are now in position to state and prove our main theorem, serving as a unification and generalization in several aspects of Theorems 2 on p. 150 and 3 on p. 151 from famous paper [6].

**Theorem 1.** *Let  $(X, \mathcal{M}(X), \mathcal{J}(X))$  be a measurable space with negligibles,  $(Y, \mathcal{T}(Y))$  a second countable topological space and  $(Z, \mathcal{T}(Z))$  a second countable perfectly normal topological space. Let  $\mathcal{J}(Y) \subset \mathcal{B}_0(Y)$  be a Borel  $\sigma$ -ideal in  $Y$  such that there exists a  $\sigma$ -ideal  $\mathcal{J}(X \otimes Y)$  including  $\mathcal{J}(X) \otimes \mathcal{J}(Y)$  fulfilling (1) and (22). Assume that  $F : X \times Y \rightarrow Z$  is a closed valued multifunction with the following three properties:*

- (i) *All the sections  $F^y, y \in Y$ , are lower  $\widehat{\mathcal{M}}(X)$ -measurable.*
- (ii) *For all  $x \in X$  the set  $D(F_x)$  of discontinuity points of the section  $F_x$  is  $\mathcal{J}(Y)$ -negligible.*
- (iii) *For all  $(x, y) \in X \times Y$  the inclusions*

$$(49) \quad G_*(x, y) \subset F(x, y) \subset G^*(x, y)$$

*hold, where  $G_*$  and  $G^*$  are multifunctions constructed from  $F$  according to (40) and (48) by using some fixed countable dense subset  $P \subset Y$ , the existence of which we assume. Then  $F$  is lower measurable with respect to the  $\mathcal{J}(X \otimes Y)$ -completion of  $\mathcal{M}(X) \otimes \mathcal{B}_0(Y)$ .*

*Proof.* Let us consider the set:

$$(50) \quad A = \{(x, y) : G_*(x, y) \neq G^*(x, y)\}.$$

Both multifunction  $G_*$  and  $G^*$  are upper  $\widehat{\mathcal{M}}(X \otimes Y)$ -measurable by virtue of Propositions 4 and 5 respectively. Therefore by the remark (21) ( $Z$  being perfectly normal is also regular) we infer that  $A \in \widehat{\mathcal{M}}(X \otimes Y)$ . Observe that by the assumption (ii) all  $x$ -sections of the set  $A$  are  $\mathcal{J}(Y)$ -negligibles:

$$(51) \quad A_x = \{y \in Y : G_*(x, y) \neq G^*(x, y)\} \subset D(F_x) \in \mathcal{J}(Y).$$

Consequently we have

$$(52) \quad \{x \in X : A_x \notin \mathcal{J}(Y)\} = \emptyset \in \mathcal{J}(X),$$

which, by using (1), insures the appartenance of  $A$  to the  $\mathcal{J}(X \otimes Y)$ . The double inclusion (49) entrains, by transitivity, implication:

$$(53) \quad \text{If } G_*(x, y) = G^*(x, y), \text{ then } G_*(x, y) = F(x, y),$$

which, in tour, guarantees the  $\mathcal{J}(X \otimes Y)$ -negligibility of  $A_1$ :

$$(54) \quad A_1 = \{(x, y) : G_*(x, y) \neq F(x, y)\} \subset A \in \mathcal{J}(X \otimes Y).$$

Next, let  $U$  be an arbitrary open subset of  $Z$ .  $G_*$  is upper  $\widehat{\mathcal{M}}(X \otimes Y)$ -measurable and thus, by Proposition 1 it is also lower  $\widehat{\mathcal{M}}(X \otimes Y)$ -measurable. So that we have:

$$(55) \quad G_*^-(U) = (B \setminus A_2) \cup A_3 \in \widehat{\mathcal{M}}(X \otimes Y)$$

for some  $B \in \mathcal{M}(X) \otimes \mathcal{B}_0(Y)$  and  $A_2, A_3 \in \mathcal{J}(X \otimes Y)$ .

Next let us remark that by (53) and (54):

$$\begin{aligned} F^-(U) &= (F^-(U) \cap (X \times Y \setminus A_1)) \cup (F^-(U) \cap A_1) \\ &= (G_*^-(U) \cap (X \times Y \setminus A_1)) \cup A_4 = (B \setminus A_5) \cup A_4, \end{aligned}$$

where  $A_4 = F^-(U) \cap A_1$ ,  $A_5 = A_1 \cup [A_2 \cap (X \times Y \setminus A_1)]$ .

All the sets  $A_i$ ,  $i = 1, 2, \dots, 5$ , are  $\mathcal{J}(X \otimes Y)$ -negligibles members of  $\widehat{\mathcal{M}}(X \otimes Y)$ . Therefore  $F$  is lower measurable with respect to the  $\mathcal{J}(X \otimes Y)$ -completion of  $\mathcal{M}(X) \otimes \mathcal{B}_0(Y)$  and if it is moreover compact-valued also upper  $\widehat{\mathcal{M}}(X \otimes Y)$ -measurable. The proof of theorem is finished.  $\square$

The same proof works in the case of multifunctions defined on product category base  $(X \times Y, \mathcal{C}(X) \times \mathcal{C}(Y))$  (cf. [7]), where  $(Y, \mathcal{C}(Y))$  is a separable category base. We use (3) instead of (1), we take as  $P$  a subset obtained by selecting a point from each member of countable quasi-base of  $Y$ , and we generalize the notion of continuity by taking in all Definitions 3 the set all members of this quasi-base containing  $y_0 \in Y$  instead of the filter base  $\mathcal{U}(y_0)$ . The thesis of Proposition 3 holds in the presence of proto-decomposability of product space  $X \times Y$  endowed with the  $\sigma$ -algebra of sets with the abstract Baire property. This version seems to be new even in the single-valued case of real functions defined on the product of category bases.

**Question 1.** *Let  $(Y, \mathcal{T}(Y), \mathcal{S}(Y), \mathcal{M}(Y), \mathcal{J}(Y))$  be a bitopological space which is simultaneously a measurable space with negligibles  $\mathcal{J}(Y)$ .*

*Two topologies  $\mathcal{T}(Y)$  and  $\mathcal{S}(Y)$  are assumed to be related modulo  $\sigma$ -ideal  $\mathcal{J}(Y)$  namely the symmetric difference  $Cl_{\mathcal{T}}(A) \div Cl_{\mathcal{S}}(A)$  is  $\mathcal{J}(Y)$ -negligible for each subset  $A \subset Y$ . From Th. 2 in [10] it follows that for each multifunction  $H : Y \rightarrow Z$  (where  $Z$  is a second countable Hausdorff space) which is at every point  $y \in Y$*

either  $\mathcal{T}(Y)$ -continuous or  $\mathcal{S}(Y)$ -continuous the set  $\mathcal{T} \cap \mathcal{S}\text{-}D(H) \in \mathcal{J}(Y)$ . Under what conditions imposed on  $\mathcal{T}(Y)$  and  $\mathcal{S}(Y)$  we have

$$(56) \quad \mathcal{T} \cap \mathcal{S}\text{-}q\text{-}\text{Lim inf}_{t \rightarrow y \wedge t \in P} H(t) \subset H(y) \subset \mathcal{T} \cap \mathcal{S}\text{-}p\text{-}\text{Lim sup}_{t \rightarrow y \wedge t \in P} H(t)?$$

Let  $(X, \mathcal{M}(X), \mathcal{J}(X))$  be a (complete) measurable space with negligibles. Under what conditions a multifunction  $F: X \times Y \rightarrow Z$ , whose all  $x$ -sections are at each  $y \in Y$  either  $\mathcal{T}(Y)$ -continuous or  $\mathcal{S}(Y)$ -continuous and all  $y$ -section are  $\widehat{\mathcal{M}}(X)$ -measurable, is  $\widehat{\mathcal{M}}(X \otimes Y)$ -measurable. Evidently  $(Y, \mathcal{T}(Y) \cap \mathcal{S}(Y))$  is assumed to be second countable Baire space and  $\mathcal{M}(Y)$  is related with the  $\mathcal{T}(Y) \cap \mathcal{S}(Y)$ -Borel  $\sigma$ -algebra  $\mathcal{B}_0(Y, \mathcal{T}(Y) \cap \mathcal{S}(Y))$ . Is the condition  $\mathcal{T}\text{-}\text{Lim } H(y) \subset H(y) \subset \mathcal{S}\text{-}\text{Lim } H(y)$  sufficient for the double inclusion (56)?

In that manner we have the possibility to obtain many applications of Theorem 1, e.g. for multifunctions whose  $x$ -sections are monotone in certain generalized meaning.

**Question 2.** Let  $Y$  be a (finite dimensional) Euclidean space with the scalar product  $\langle \cdot | \cdot \rangle$ . Let us consider the unit sphere

$$(57) \quad S^1 = \{y \in Y : \|y\| = \sqrt{\langle y | y \rangle} = 1\}$$

endowed with the metric  $\varrho(y_1, y_2) = \arccos \langle y_1 | y_2 \rangle$ . By an angular region in  $Y$  is called the subset of the form

$$(58) \quad \widehat{\Omega}(y_0, V) = y_0 + \left\{ y \in Y : \frac{y}{\|y\|} \in V \right\},$$

where  $V \subset S^1$  is a  $\varrho$ -open subset of the unit sphere  $S^1$  (cf. [11, p. 318]).

A multifunction  $H: Y \rightarrow Z$ , where  $Z$  is an arbitrary topological space, is called lower semicontinuous at  $y_0 \in Y$  from the angular region  $\widehat{\Omega}(y_0, V)$  if for each open subset  $G \subset Z$  such that  $G \cap H(y_0) \neq \emptyset$  the big inverse image  $H^-(G)$  contains  $\widehat{\Omega}(y_0, V) \cap \{y \in Y : \|y - y_0\| < r\}$  for some  $r > 0$ .

Analogously,  $H$  is called upper semicontinuous at  $y_0 \in Y$  from  $\widehat{\Omega}(y_0, V)$  if for each subset  $G \subset Z$  such that  $H(y_0) \subset G$  the small inverse image  $H^+(G)$  contains  $\widehat{\Omega}(y_0, V) \cap \{y \in Y : \|y - y_0\| < r\}$  for some  $r > 0$ .

A multifunction  $H$  is said to be  $\Omega$ -lower (resp. upper) semicontinuous on  $Y$ , if for each  $y \in Y$  there is an angular region  $\widehat{\Omega}(y, V(y))$  such that  $H$  is lower (resp. upper) semicontinuous at  $y$  from  $\widehat{\Omega}(y, V(y))$ .

A multifunction simultaneously  $\Omega$ -lower and  $\Omega$ -upper semicontinuous is called  $\Omega$ -continuous.

Under what conditions a multifunction  $F: X \times Y \rightarrow Z$ , where  $Y$  is as above,  $X$  is a complete measurable space with negligibles and  $Z$  is a second countable perfectly normal topological space, with  $\Omega$ -continuous  $x$ -sections and  $\mathcal{M}(X)$ -measurable  $y$ -sections is  $\widehat{\mathcal{M}}(X \otimes Y)$ -measurable with respect to the  $\mathcal{J}(X \otimes Y)$ -completion

of  $\mathcal{M}(X) \otimes \mathcal{B}_0(Y)$ . The  $\sigma$ -ideal  $\mathcal{J}(X \otimes Y)$  including  $\mathcal{J}(X) \otimes \mathcal{J}(Y)$  means here an  $\sigma$ -ideal fulfilling the conditions of Theorem 1, where  $\mathcal{J}(Y)$  is a Borel  $\sigma$ -ideal in  $Y$ , e.g. of subsets of Lebesgue measure zero or of the first category. Is the finite-dimensionality of  $Y$  essential?

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