SUPERREFLEXIVITY AND J–CONVEXITY OF BANACH SPACES

J. WENZEL

Abstract. A Banach space $X$ is superreflexive if each Banach space $Y$ that is finitely representable in $X$ is reflexive. Superreflexivity is known to be equivalent to $J$-convexity and to the non-existence of uniformly bounded factorizations of the summation operators $S_n$ through $X$.

We give a quantitative formulation of this equivalence.

This can in particular be used to find a factorization of $S_n$ through $X$, given a factorization of $S_N$ through $[L_2,X]$, where $N$ is ‘large’ compared to $n$.

1. Introduction

Much of the significance of the concept of superreflexivity of a Banach space $X$ is due to its many equivalent characterizations, see e.g. Beauzamy [1, Part 4].

Some of these characterizations allow a quantification, that makes also sense in non superreflexive spaces. Here are two examples.

Definition. Given $n$ and $0 < \varepsilon < 1$, we say that a Banach space $X$ is $J(n,\varepsilon)$-convex, if for all elements $z_1, \ldots, z_n \in U_X$ we have

$$\inf_{1 \leq k \leq n} \left\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h \right\| < n(1 - \varepsilon).$$

We let $J_n(X)$ denote the infimum of all $\varepsilon$, such that $X$ is not $J(n,\varepsilon)$-convex.

Definition. Given $n$ and $\sigma \geq 1$, we say that a Banach space $X$ factors the summation operator $S_n$ with norm $\sigma$, if there exists a factorization $S_n = B_n A_n$ with $A_n : l_1^n \to X$ and $B_n : X \to l_\infty^n$ such that $\|A_n\| \|B_n\| = \sigma$.

We let $S_n(X)$ denote the infimum of all $\sigma$, such that $X$ factors $S_n$ with norm $\sigma$.

Here, the summation operator $S_n : l_1^n \to l_\infty^n$ is given by

$$(\xi_k) \mapsto \left( \sum_{h=1}^{k} \xi_h \right)$$

and $U_X$ denotes the unit ball of the Banach space $X$.
It is known that a Banach space is superreflexive if and only if it is $J(n, \varepsilon)$-convex for some $n$ and $\varepsilon > 0$, or equivalently, if it does not factor the summation operators with uniformly bounded norm; see James [5, Th. 5, Lem. B], and Schäffer/Sundaresan [9, Th. 2.2].

Using the terminology introduced above, this can be reformulated as follows:

**Theorem 1.** For a Banach space $X$ the following properties are equivalent:

(i) $X$ is not superreflexive.

(ii) For all $n \in \mathbb{N}$ we have $J_n(X) = 0$.

(iii) There is a constant $\sigma \geq 1$ such that for all $n \in \mathbb{N}$ we have $S_n(X) \leq \sigma$.

(iv) For all $n \in \mathbb{N}$ we have $S_n(X) = 1$.

There are two conceptually different methods to prove that $X$ is superreflexive if and only if $[L_2, X]$ is. The one is to use Enflo’s renorming result [2, Cor. 3], which is not suited to be localized, the other is the use of $J$-convexity, see Pisier [7, Prop. 1.2]. It turns out that for fixed $n$

$$J_n([L_2, X]) \leq J_n(X) \leq 2n^2 J_n([L_2, X]);$$

see Section 2 for a proof. Similar results hold also in the case of $B$-convexity; see [8, p. 30].

**Theorem 2.** If for some $n$ and all $\varepsilon > 0$, $[L_2, X]$ contains $(1 + \varepsilon)$ isomorphic copies of $l_1^n$, then $X$ contains $(1 + \varepsilon)$ isomorphic copies of $l_1^n$.

**Theorem 3.** If for some $n$ and all $\varepsilon > 0$, $[L_2, X]$ contains $(1 + \varepsilon)$ isomorphic copies of $l_\infty^n$, then $X$ contains $(1 + \varepsilon)$ isomorphic copies of $l_\infty^n$.

On the other hand, no result of this kind for the factorization of $S_n$ is known, i.e. if for some $n$ and all $\varepsilon > 0$, $[L_2, X]$ factors $S_n$ with norm $(1 + \varepsilon)$, does it follow that $X$ factors $S_n$ with norm $(1 + \varepsilon)$?

Assuming $S_n([L_2, X]) \leq \sigma$ for some constant $\sigma$ and all $n \geq 1$, one can use Theorem 1 to obtain that $J_n([L_2, X]) = 0$ for all $n \geq 1$ and consequently $S_n(X) = 1$.

The intent of our paper is to keep $n$ fixed in this reasoning. Unfortunately, we don’t get a result as smooth as Theorems 2 and 3. Instead, we have to consider two different values $n$ and $N$. If $S_N([L_2, X]) = \sigma$ for some ‘large’ $N$, then $S_n(X) \leq (1 + \varepsilon)$ for some ‘small’ $n$. To make this more precise, let us introduce the iterated exponential (or TOWER) function $P_\delta(m)$. We let

$$P_0(m) := m \quad \text{and} \quad P_\delta+1(m) := 2P_\delta(m).$$

We will prove the following two theorems.

**Theorem 4.** For fixed $n \in \mathbb{N}$ and $\sigma > 1$ there is $\varepsilon > 0$ such that $J_n(X) \leq \varepsilon$ implies $S_n(X) < \sigma$. In particular $J_n(X) = 0$ implies $S_n(X) = 1$.  

Theorem 5. For fixed $n \in \mathbb{N}$, $\varepsilon > 0$ and $\sigma \geq 1$ there is a number $N(\varepsilon, n, \sigma)$, such that $S_N(X) \leq \sigma$ implies $J_n(X) < \varepsilon$. The number $N$ can be estimated by

$$N \leq P_m(cn),$$

where $m$ and $c$ depend on $\sigma$ and $\varepsilon$ only.

Using (1), we obtain the following consequence.

Corollary 6. For fixed $n \in \mathbb{N}$, $\sigma_1 > 1$, and $\sigma_2 \geq 1$ there is a number $N(\sigma_1, n, \sigma_2)$ such that $S_N([L_2, X]) \leq \sigma_2$ implies $J_n([L_2, X]) < \sigma_1$.

Proof. Determine $\varepsilon$ as in Theorem 4 such that $J_n(X) \leq \varepsilon$ implies $S_n(X) < \sigma_1$.

Choose $N = N(\sigma_1, n, \sigma_2)$ as in Theorem 5 such that $S_N([L_2, X]) \leq \sigma_2$ implies $J_n([L_2, X]) < \varepsilon$ and hence $S_n(X) < \sigma_1$. $\Box$

The estimate in Theorem 5 seems rather crude, and we have no idea, whether or not it is optimal.

2. Proofs

First of all, we list some elementary properties of the sequences $S_n(X)$ and $J_n(X)$.

Fact.

(i) The sequence $(S_n(X))$ is non-decreasing.

(ii) $1 \leq S_n(X) \leq (1 + \log n)$ for all infinite dimensional Banach spaces $X$.

(iii) The sequence $(nJ_n(X))$ is non-decreasing.

(iv) For all $n, m \in \mathbb{N}$ we have $J_n(X) \leq J_{nm}(X) \leq J_n(X) + 1/n$.

(v) If $J_n(X) \to 0$ then for all $n \in \mathbb{N}$ we have $J_n(X) = 0$.

(vi) $J_n(\mathbb{R}) \geq 1 - 1/n$ for all $n \in \mathbb{N}$.

(vii) If $q$ and $\varepsilon$ are related by $\varepsilon \geq (1 - \varepsilon)^q - 1$ then $J_n(l_q) \leq 4\varepsilon$ for all $n \in \mathbb{N}$.

Proof. The monotonicity properties (i) and (iii) are trivial.

The bound for $S_n(X)$ in (ii) follows from the fact that the summation operator $S_n$ factors through $l_2^n$ with norm $(1 + \log n)$ and from Dvoretzky’s Theorem.

To see (iv) assume that $X$ is $J(n, \varepsilon)$-convex. Given $z_1, \ldots, z_{nm} \in U_X$, let

$$x_h := \frac{1}{m} \sum_{k=1}^m z_{(h-1)m+k} \quad \text{for } h = 1, \ldots, n.$$

Then

$$\inf_{1 \leq k \leq nm} \left\| \sum_{h=1}^k z_h - \sum_{h=k+1}^{nm} z_h \right\| \leq m \inf_{1 \leq k \leq n} \left\| \sum_{h=1}^k x_h - \sum_{h=k+1}^n x_h \right\| < mn(1 - \varepsilon),$$

which proves that $X$ is $J(nm, \varepsilon)$-convex, and consequently $J_n(X) \leq J_{nm}(X)$. 


Assume now that $X$ is $J(nm, \varepsilon)$-convex. Given $z_1, \ldots, z_n \in U_X$, let

$$x_1 = \ldots = x_m := z_1$$
$$\vdots$$
$$x_{(n-1)m+1} = \ldots = x_{nm} := z_n.$$  

If

$$\inf_{1 \leq k \leq nm} \left\| \sum_{h=1}^{k} x_h - \sum_{h=k+1}^{nm} x_h \right\|$$

is attained for $k_0$,

there is $l \in \{0, \ldots, n\}$ such that

$$m/2 + (l-1)m < k_0 \leq m/2 + lm,$$

hence

$$\left\| \sum_{h=1}^{k_0} x_h - \sum_{h=k_0+1}^{nm} x_h \right\| \geq \left\| \sum_{h=1}^{lm} x_h - \sum_{h=lm+1}^{nm} x_h \right\| - 2 \sum_{h \in I} \|x_h\|,$$

where $I = \{k_0 + 1, \ldots, lm\}$ or $I = \{lm + 1, \ldots, k_0\}$ according to whether $k_0 \leq lm$ or $k_0 > lm$. It follows that

$$nm(1 - \varepsilon) > m \inf_{1 \leq k \leq n} \left\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h \right\| - m,$$

and hence $J_n(X) \geq \varepsilon - 1/n$. This proves (iv).

(v) is a consequence of (iv).

For (vi) and (vii) see Section 3.

For the convenience of the reader, let us repeat the argument for the proof of (1) from [1]. The left-hand part of (1) is obvious, since $X$ can be isometrically embedded into $[L_2, X]$. To see the right-hand inequality, assume that for all $z_1, \ldots, z_n \in U_X$

$$\inf_{1 \leq k \leq n} \left\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h \right\| < n(1 - \varepsilon).$$

Obviously, if $\|z_1\| = \ldots = \|z_n\|$ it follows by homogeneity that

$$\inf_{1 \leq k \leq n} \left\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h \right\| < (1 - \varepsilon) \sum_{k=1}^{n} \|z_k\|,$$

If $z_1, \ldots, z_n$ are arbitrary, let $m := \min_{1 \leq k \leq n} \|z_k\|$, $\lambda_k := m/\|z_k\|$, and $\tilde{z}_k := (1 - \lambda_k)z_k$. It turns out that $\|z_k - \tilde{z}_k\| = m$ and therefore by (2)

$$\inf_{1 \leq k \leq n} \left\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h \right\| < \sum_{k=1}^{n} \|\tilde{z}_k\| + (1 - \varepsilon) \sum_{k=1}^{n} \|z_k - \tilde{z}_k\|$$

$$\leq \sum_{k=1}^{n} ((1 - \lambda_k) + (1 - \varepsilon)\lambda_k) \|z_k\| \leq \left( \sum_{k=1}^{n} (1 - \varepsilon\lambda_k)^2 \right)^{1/2} \left( \sum_{k=1}^{n} \|z_k\|^2 \right)^{1/2}.$$
Now, at least one of the $\lambda_k$’s equals one, while the others are greater than or equal to zero. This yields

$$\inf_{1 \leq k \leq n} \left\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h \right\| < \left( (1 - \varepsilon)^2 + n - 1 \right)^{1/2} \left( \sum_{k=1}^{n} \| z_k \|^2 \right)^{1/2}$$

$$\leq (n - 2\varepsilon + \varepsilon^2)^{1/2} \left( \sum_{k=1}^{n} \| z_k \|^2 \right)^{1/2}.$$ 

On the other hand, we trivially get that for all $1 \leq k \leq n$

$$\left\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h \right\| \leq n^{1/2} \left( \sum_{k=1}^{n} \| z_k \|^2 \right)^{1/2}.$$ 

Therefore

$$\sum_{k=1}^{n} \left\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h \right\|^2 < (n - 2\varepsilon + \varepsilon^2) + (n - 1)n \sum_{k=1}^{n} \| z_k \|^2$$

for all $z_1, \ldots, z_n \in X$. If in particular $f_1, \ldots, f_n \in U_{[L_2, X]}$, then

$$\sum_{k=1}^{n} \left\| \sum_{h=1}^{k} f_h(t) - \sum_{h=k+1}^{n} f_h(t) \right\|^2 < (n^2 - 2\varepsilon + \varepsilon^2) \sum_{k=1}^{n} \| f_k(t) \|^2.$$ 

Integration with respect to $t$ yields

$$\sum_{k=1}^{n} \left\| \sum_{h=1}^{k} f_h - \sum_{h=k+1}^{n} f_h \right\|^2_{L_2} < (n^2 - 2\varepsilon + \varepsilon^2) \sum_{k=1}^{n} \| f_k \|^2_{L_2} \leq n(n^2 - 2\varepsilon + \varepsilon^2).$$ 

This implies that

$$\inf_{1 \leq k \leq n} \left\| \sum_{h=1}^{k} f_h - \sum_{h=k+1}^{n} f_h \right\|_{L_2} < (n^2 - 2\varepsilon + \varepsilon^2)^{1/2} \leq n(1 - \delta)$$

for $\delta = \varepsilon/2n^2$. Therefore $J_n([L_2, X]) \geq J_n(X)/2n^2$. \qed

Let us now prove Theorem 4.

**Proof of Theorem 4.** Choose $\varepsilon < \frac{1}{2(n+2)!}$ such that $1 + 2(n+2)!\varepsilon < \sigma$. If $J_n(X) \leq \varepsilon$, we find $z_1, \ldots, z_n \in U_X$ be such that

$$\inf_{1 \leq k \leq n} \left\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h \right\| \geq n(1 - \varepsilon).$$
By the Hahn-Banach theorem, we find $y_k \in U_{X^*}$ such that

$$n(1 - \varepsilon) \leq \sum_{h=1}^{k} \langle z_h, y_k \rangle - \sum_{h=k+1}^{n} \langle z_h, y_k \rangle.$$ 

Obviously $|\langle z_h, y_k \rangle| \leq 1$. If for some $h \leq k$ we even have

$$\langle z_h, y_k \rangle < 1 - n\varepsilon,$$

then

$$n(1 - \varepsilon) \leq \sum_{i=1}^{k} \langle z_i, y_k \rangle - \sum_{i=k+1}^{n} \langle z_i, y_k \rangle < (n - 1) + (1 - n\varepsilon) = n(1 - \varepsilon),$$

which is a contradiction. Hence

(3) $1 - n\varepsilon \leq \langle z_h, y_k \rangle \leq 1$ for all $h \leq k$.

Similarly

(4) $1 - n\varepsilon \leq -\langle z_h, y_k \rangle \leq 1$ for all $h > k$.

Let $x_h := (z_1 + z_h)/2$. Then it follows from (3) and (4) that there are $x_1, \ldots, x_n \in U_X$ and $y_1, \ldots, y_n \in U_{X^*}$ so that

$$\langle x_h, y_k \rangle \in \begin{cases} (1 - n\varepsilon, 1) & \text{if } h \leq k, \\ (-n\varepsilon, +n\varepsilon) & \text{if } h > k. \end{cases}$$

The assertion now follows from the following distortion lemma. $\square$

**Lemma 7.** Suppose that $\varepsilon < \frac{1}{2(n+1)!}$ and that there are $x_1, \ldots, x_n \in U_X$ and $y_1, \ldots, y_n \in U_{X^*}$ such that

$$\langle x_h, y_k \rangle \in \begin{cases} (1 - \varepsilon, 1) & \text{if } h \leq k, \\ (-\varepsilon, +\varepsilon) & \text{if } h > k. \end{cases}$$

Then $S_n(X) \leq 1 + 2(n + 1)!\varepsilon$.

**Proof.** Fix $h \in \{1, \ldots, n\}$. Let $\alpha_{lk} := \langle x_l, y_k \rangle$. Consider the system of linear equations

$$\sum_{l=1}^{n} \alpha_{lk} \xi_l = \begin{cases} 1 - \alpha_{hk} & \text{if } h \leq k, \\ -\alpha_{hk} & \text{if } h > k, \end{cases} \quad k = 1, \ldots, n.$$
in the $n$ variables $\xi_1, \ldots, \xi_n$. Its solution is given by

$$\xi_m^{(h)} = \frac{\det(\beta_l^{(m)})}{\det(\alpha_l)}$$

where $(\beta_l^{(m)})$ is the matrix $(\alpha_l)$ but with its $m$-th column replaced by the right-hand side of our system of equations. It follows that

$$|\det(\beta_l^{(m)})| = \left| \sum_\pi \sgn(\pi) \prod_{k=1}^n \beta_{k\pi(k)}^{(m)} \right| \leq n!|\beta_{m\pi(m)}^{(m)}| \leq n!\varepsilon.$$

Since for all permutations $\pi$ that are not the identity, there exists at least one $k$ such that $\pi(k) > k$, we have $|\alpha_{\pi(k)k}| < \varepsilon$ and hence

$$|\det(\alpha_l)| = \left| \sum_\pi \sgn(\pi) \prod_{k=1}^n \alpha_{\pi(k)k} \right| \geq \left| \prod_{k=1}^n \alpha_{kk} \right| - \sum_{\pi \neq \text{id}} \varepsilon$$

$$\geq (1 - \varepsilon)^n - n!\varepsilon \geq 1 - n\varepsilon - n!\varepsilon \geq 1 - (n + 1)!\varepsilon.$$

Hence if $\varepsilon < \frac{1}{2(n+1)!}$ the solutions $\xi_m^{(h)}$ satisfy

$$|\xi_m^{(h)}| \leq 2n!\varepsilon.$$

Defining $A_n : l_1^n \to X$ by

$$A_ne_h := \sum_{m=1}^n x_m s_m^{(h)} + x_h,$$

we get that $\|A_n\| \leq 1 + \sup_h \sum_{m=1}^n |s_m^{(h)}| \leq 1 + 2(n+1)!\varepsilon$. Defining $B_n : X \to l_\infty^n$ by

$$B_n x := ((x,y_k))_{k=1}^n,$$

we get that $\|B_n\| \leq 1$ and $S_n = B_n A_n$. This completes the proof, since $S_n(X) \leq \|A_n\| \|B_n\| \leq 1 + 2(n+1)!\varepsilon$.

**Interlude on Ramsey theory**

Our proof of Theorem 5 makes massive use of the general form of Ramsey’s Theorem. Therefore, for the convenience of the reader, let us recall, what it says; see [3] and [6].

For a set $M$ and a positive integer $k$, let $M[k]$ be the set of all subsets of $M$ of cardinality $k$. 
Theorem 8. Given \( r, k \) and \( n \), there is a number \( R_k(n, r) \) such that for all \( N \geq R_k(n, r) \) the following holds:

For each function \( f : \{1, \ldots, N\}^k \to \{1, \ldots, r\} \) there exists a subset \( M \subseteq \{1, \ldots, N\} \) of cardinality at least \( n \) such that \( f(M^k) \) is a singleton.

The following estimate for the Ramsey number \( R_k(l, r) \) can be found in [3, p. 106].

Lemma. There is a number \( c(r, k) \) depending on \( r \) and \( k \), such that

\[ R_k(l, r) \leq P_k(c(r, k) \leq l) \]

We can now turn to the proof of Theorem 5.

Proof of Theorem 5. The proof follows the line of James’s proof in [4, Th. 1.1].

The main new ingredient is the use of Ramsey’s Theorem to estimate the number \( N \).

Let \( n, \varepsilon > 0 \), and \( \sigma \) be given. Define \( m \) by

\[ 2m\sigma < \left( \frac{1}{1 - \varepsilon} \right)^{m-1} \]

and let

\[ N := R_{2m}(R_{2m}(2nm + 1, m), m) \]

where \( R \) denotes the Ramsey number introduced in the previous paragraph.

The required estimate for \( N \) then follows from Lemma 9 as follows

\[ N \leq P_{2m}(c_1 P_{2m}(c_2 2nm)) \leq P_{4m}(c_3 n) \]

where \( c_1, c_2, \) and \( c_3 \) are constants depending on \( m \), which in turn depends on \( \sigma \) and \( \varepsilon \).

Replacing, e.g. \( \sigma \) by \( 2\sigma \), we may assume that in fact \( S_N(X) < \sigma \) in order to avoid using an additional \( \delta \) in the notation.

If \( S_N(X) < \sigma \) then there are \( A_N : l_1^N \to X \) and \( B_N : X \to l_\infty^N \) such that \( S_N = B_N A_N \) and \( \| A_N \| = 1 \), \( \| B_N \| \leq \sigma \). Let \( x_h := A_N e_h \) and \( y_k := B_N e_k \). Note that

\[ \| x_h \| \leq 1, \quad \| y_k \| \leq \sigma, \quad \text{and} \quad (x_h, y_k) = \begin{cases} 1 & \text{if } h \leq k, \\ 0 & \text{if } h > k. \end{cases} \]

For each subset \( M \subseteq \{1, \ldots, N\} \), we let \( \mathcal{F}_m(M) \) denote the collection of all sequences \( F = (F_1, \ldots, F_m) \) of consecutive intervals of numbers, whose endpoints are in \( M \), i.e.

\[ F_j = \{ l_j, l_j + 1, \ldots, r_j \}, \quad l_j, r_j \in M, \quad l_j < r_j < l_{j+1}, \]

for \( j = 1, \ldots, m \). Note that \( \mathcal{F}_m(M) \) can be identified with \( M^{[2m]} \).
The outline of the proof of Theorem 5 is as follows. To each \( F = (F_1, \ldots, F_m) \), we assign an element \( x(F) \) which in fact is a linear combination of the elements \( x_1, \ldots, x_N \). Next, we extract a ‘large enough’ subset \( M \) of \( \{1, \ldots, N\} \), such that all \( x(F) \) with \( F \in \mathcal{F}_m(M) \) have about equal norm. Finally, we look at special sequences \( F(1), \ldots, F(n) \) and \( E(1), \ldots, E(n) \) in \( \mathcal{F}_m(M) \) such that

\[
\left\| \sum_{h=1}^{k} x(F^{(h)}) - \sum_{h=k+1}^{n} x(E^{(h)}) \right\| \geq n \|x(E^{(k)})\|.
\]

Since \( \|x(E^{(k)})\| \asymp \|x(F^{(h)})\| \), normalizing the elements \( x(F^{(h)}) \) yields the required elements \( z_1, \ldots, z_n \) to prove that \( J_n(X) < \varepsilon \).

Let us start by choosing the elements \( x(F) \). For a sequence \( F \in \mathcal{F}_m(M) \), we define

\[
S(F) := \left\{ x = \sum_{h=1}^{N} \xi_h x_h : \sup_{h} |\xi_h| \leq 2, \ (x, y_l) = (-1)^j \text{ for all } l \in F_j \text{ and } j = 1, \ldots, m \right\}.
\]

By compactness, there is \( x(F) \in S(F) \) such that

\[
\|x(F)\| = \inf_{x \in S(F)} \|x\|.
\]

**Lemma 10.** We have \( 1/\sigma \leq \|x(F)\| \leq 2m \) for all \( F \in \mathcal{F}_m(\{1, \ldots, N\}) \).

**Proof.** Write \( F_j = \{l_j, \ldots, r_j\} \) and let

\[
x := -x_{l_1} + 2 \sum_{i=2}^{m} (-1)^i x_{l_i}.
\]

Then for \( l \in F_j \), we have

\[
(x, y_l) = -1 + 2 \sum_{i=2}^{j} (-1)^i \cdot 1 + 2 \sum_{i=j+1}^{m} (-1)^i \cdot 0 = (-1)^j,
\]

hence \( x \in S(F) \) and \( \|x(F)\| \leq \|x\| \leq 2m - 1 \).

On the other hand,

\[
1 = |(x(F), y_{l_1})| \leq \sigma \|x(F)\|.
\]

Hence \( 1/\sigma \leq \|x(F)\| \).

By (5), we can write the interval \([1/\sigma, 2m]\) as a disjoint union as follows

\[
\left[ \frac{1}{\sigma}, 2m \right] \subseteq \bigcup_{i=1}^{m-1} A_i, \text{ where } A_i := \frac{1}{\sigma} \left[ \left( \frac{1}{1-\varepsilon} \right)^{i-1}, \left( \frac{1}{1-\varepsilon} \right)^i \right].
\]
For $F = (F_1, \ldots, F_m) \in \mathcal{F}_m(\{1, \ldots, N\})$ and $1 \leq j \leq m$, let
\[ P_j(F) := (F_1, \ldots, F_j) \in \mathcal{F}_j(\{1, \ldots, N\}). \]
Obviously
\[ \|x(P_{j-1}(F))\| \leq \|x(P_j(F))\| \leq 2m \quad \text{for } j = 2, \ldots, m. \]
It follows that for each $F \in \mathcal{F}_m(\{1, \ldots, N\})$ there is at least one index $j$ for which
the two values $\|x(P_{j-1}(F))\|$ and $\|x(P_j(F))\|$ belong to the same interval $A_i$.
Applying Ramsey’s Theorem to that function, yields the existence of a number $j_0$
and a subset $L$ of $\{1, \ldots, N\}$ of cardinality $|L| \geq R_{2m}(2nm + 1, m)$ such that for
all $F \in \mathcal{F}_m(L)$ the two values $\|x(P_{j_0-1}(F))\|$ and $\|x(P_{j_0}(F))\|$ belong to the same
of the intervals $A_i$.
Next, for each $F \in \mathcal{F}_m(L)$ there is a unique number $i$ for which the value
$\|x(P_{j_0}(F))\|$ belongs to the interval $A_i$. Letting $g(F)$ be that number $i$, defines a
function $g: L[2^m] \to \{1, \ldots, m\}$.
We now define sequences
\[ F(h) := (F_1^{(h)}, \ldots, F_m^{(h)}) \quad \text{and} \quad E(k) := (E_1^{(k)}, \ldots, E_{m-1}^{(k)}) \]
of nicely overlapping intervals.
Write $M = \{p_1, \ldots, p_{2nm+1}\}$, where $p_1 < p_2 < \cdots < p_{2nm+1}$ and define
\[ F^{(h)} := (F_1^{(h)}, \ldots, F_m^{(h)}) \in \mathcal{F}_m(M) \quad \text{for } h = 1, \ldots, n \]
as follows
\[ F_j^{(h)} := \begin{cases} \{p_{2h-1}, \ldots, p_{n+2h-1}\} & \text{if } j = 1, \\ \{p_{n+2(j-1)+2h}, \ldots, p_{n+2j-1}+2h-1\} & \text{if } j = 2, \ldots, m-1, \\ \{p_{n(2m-3)+2h}, \ldots, p_{n(2m-1)+h}\} & \text{if } j = m. \end{cases} \]
It turns out that

(9) \[ E_j^{(k)} := \bigcap_{h=1}^{k} F_j^{(h)} \cap \bigcap_{h=k+1}^{n} F_j^{(h)} \quad k = 1, \ldots, n \]

is given by

\[ E_j^{(k)} := \{ p_{n(2j-1)+2k}, \ldots, p_{n(2j-1)+2k+1} \} \quad \text{if} \; j = 1, \ldots, m - 1. \]

Hence \((E_1^{(k)}, \ldots, E_m^{(k)}) \in \mathcal{F}_m(M)\). In order to obtain an element of \(\mathcal{F}_m(M)\) we add the auxiliary set \(E_m^{(k)} := \{ p_{2nm}, \ldots, p_{2nm+1} \}\), this can be done for \(n \geq 2\), which is the only interesting case anyway since \(J_1(X) = 0\) for any Banach space \(X\). We have \((E_1^{(k)}, \ldots, E_m^{(k)}) \in \mathcal{F}_m(M)\).

The following picture shows the sets \(E_j^{(k)}\) and \(F_j^{(k)}\) in the case \(n = 3\) and \(m = 4\):

![Diagram showing sets E_j^{(k)} and F_j^{(k)}](image)

It follows from (9) that for \(1 \leq k \leq n\)

\[ \frac{1}{n} \left( - \sum_{h=1}^{k} x(P_{j_0}([E^{(h)}])) + \sum_{h=k+1}^{n} x(P_{j_0}([E^{(h)}])) \right) \in S(P_{j_0-1}([E^{(k)}])) \]

hence

\[ \left\| \sum_{h=1}^{k} x(P_{j_0}([E^{(h)}])) - \sum_{h=k+1}^{n} x(P_{j_0}([E^{(h)}])) \right\| \geq n \| x(P_{j_0-1}([E^{(k)}])) \|. \]

Let \(z_h := \sigma (1 - \varepsilon)^{i_h} x(P_{j_0}([E^{(h)}]))\). Then

\[ \left\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h \right\| \geq n \sigma (1 - \varepsilon)^i \| x(P_{j_0-1}([E^{(k)}])) \|. \]

By (7) we have \(\| x(P_{j_0}([E^{(h)}])) \| \in A_{i_0}\), which implies \(\| z_h \| \leq 1\). On the other hand, by (8) we have \(\| x(P_{j_0-1}([E^{(k)}])) \| \in A_{i_0}\), which implies

\[ \left\| \sum_{h=1}^{k} z_h - \sum_{h=k+1}^{n} z_h \right\| \geq n \sigma (1 - \varepsilon)^{i_0} \frac{1}{\sigma} \left( \frac{1}{1 - \varepsilon} \right)^{i_0-1} = n (1 - \varepsilon). \]

Consequently \(J_n(X) \leq \varepsilon. \) \(\square\)
3. Problems and Examples

**Example 1.** $J_n(\mathbb{R}) \geq 1 - 1/n$.

*Proof.* Let $|\xi_h| \leq 1$ for $h = 1, \ldots, n$. For $k = 1, \ldots, n$ define

$$\eta_k := \sum_{h=1}^{k} \xi_h - \sum_{h=k+1}^{n} \xi_h$$

and let $\eta_0 := -\eta_n$. Obviously $|\eta_k - \eta_{k+1}| \leq 2$ for $k = 0, \ldots, n-1$. Since $\eta_0 = -\eta_n$ there exists at least one $k_0$ such that $\text{sgn} \eta_{k_0} \neq \text{sgn} \eta_{k_0+1}$. Assume that $|\eta_{k_0}| > 1$ and $|\eta_{k_0+1}| > 1$, then $|\eta_{k_0} - \eta_{k_0+1}| > 2$, a contradiction. Hence there is $k$ such that $|\eta_k| \leq 1$. This proves that

$$\inf_{1 \leq k \leq n} \left| \sum_{h=1}^{k} \xi_h - \sum_{h=k+1}^{n} \xi_h \right| \leq 1 = n \frac{1}{n},$$

and hence $J_n(\mathbb{R}) \geq 1 - \frac{1}{n}$. □

**Example 2.** If $q$ and $\varepsilon$ are related by

$$\varepsilon \geq (1 - \varepsilon)^{q-1}$$

then $J_n(l_q) \leq 4\varepsilon$ for all $n \in \mathbb{N}$.

*Proof.* Given $\varepsilon > 0$ find $n_0$ such that

$$\frac{1}{n_0} < \varepsilon \leq \frac{1}{n_0 - 1},$$

then

$$\left( \frac{1}{n_0} \right)^{1/q} \geq \left( 1 - \frac{1}{n_0} \right)^{1/q} \varepsilon^{1/q} \geq 1 - \varepsilon.$$

If $n \leq n_0$, choosing

$$x_h := (-1, \ldots, -1, \overbrace{+1, \ldots, +1}^{n-h}, 0, \ldots),$$

we obtain

$$\left\| \sum_{h=1}^{k} x_h - \sum_{h=k+1}^{n} x_h \right\|_q \geq \left\| \sum_{h=1}^{k} x_h - \sum_{h=k+1}^{n} x_h \right\|_\infty = n.$$

And since

$$\left\| x_h \right\|_q = n^{1/q} \leq n_0^{1/q} \leq 1/(1 - \varepsilon)$$

it follows that $J_n(l_q) \leq \varepsilon$.

If $n > n_0$, there is $m \geq 2$ such that $(m-1)n_0 < n \leq mn_0$. Hence, by Properties (iii) and (iv) in the fact in Section 2 it follows that

$$J_n(X) \leq \frac{mn_0}{n} J_{mn_0}(X) \leq \frac{mn_0}{n} (J_{n_0} + \frac{1}{n_0}) \leq \frac{mn_0}{n} 2\varepsilon \leq 4\varepsilon.$$ □

The main open problem of this article is the optimality of the estimate for $N$ in Theorem 5.
**Problem.** Are there $\sigma \geq 1$ and $\varepsilon > 0$ and a sequence of Banach spaces $(X_n)$ such that

$$S_{f(n)}(X_n) \leq \sigma \quad \text{and} \quad J_n(X_n) \geq \varepsilon,$$

where $f(n)$ is any function such that $f(n) > n$?

In particular $f(n) > P_m(n)$, where $m$ is given by (5) would show that the estimate in Theorem 5 for $N$ is sharp in an asymptotic sense.

**References**


J. Wenzel, Texas A&M University, Department of Mathematics, College Station, Texas 77843-3368, U.S.A.

*current address:* Mathematisches Institut, FSU Jena, 07740 Jena, Germany