RNP AND KMP ARE INCOMPARABLE PROPERTIES IN NONCOMPLETE SPACES

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Abstract. We exhibit an example in a noncomplete space of a closed, bounded and convex subset verifying KMP and failing RNP and, another such example verifying RNP and failing KMP.

We begin this note by recalling some definitions: (See [2] and [3]).

Let $X$ be a normed linear space and let $C$ be a closed, bounded and convex subset of $X$.

$C$ is said to be dentable if for each $\varepsilon > 0$ there is $x \in C$ such that $x \notin \overline{co}(C \setminus B(x,\varepsilon))$, where $\overline{co}$ denotes the closed convex hull and $B(x,\varepsilon)$ is the closed ball with centrum $x$ and radius $\varepsilon$.

$C$ is said to have the Radon-Nikodym property (RNP) if every nonempty subset of $C$ is dentable.

$C$ is said to have the Krein-Milman property if every closed and convex subset, $F$, of $C$ verifies $F = \overline{co}(\text{Ext } F)$, where $\text{Ext } F$ denotes the set of extreme points of $F$.

It is known that $C$ has KMP if every closed and convex subset of $C$ has some extreme point. (Even in noncomplete spaces.)

The above definition of RNP working in noncomplete spaces and, today, the most authors define RNP in Banach spaces as here.

For a Banach space $X$ it is known that RNP implies KMP and the converse is an well known open problem.

We prove that KMP does not imply RNP in noncomplete spaces. For this we consider a closed, bounded and convex subset, $STS$, which appears in [1], of $c_0(\Gamma)$.

In [1] it is shown that $\overline{STS_0} = STS$ in $c_0(\Gamma)$.

Our goal is to prove that $STS_0$ is a closed, bounded and convex subset of $c_{00}(\Gamma)$ verifying KMP and failing RNP.

Now we descript briefly the set $STS_0$ of $c_{00}(\Gamma)$.

$\Gamma$ denotes the set of finite sequences of natural numbers and 0 denotes the empty sequence in $\Gamma$.

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For \( \alpha, \beta \in \Gamma \) we define \( \alpha \leq \beta \) if \( |\alpha| \leq |\beta| \) and \( \alpha_i = \beta_i \) for \( 1 \leq i \leq |\alpha| \), where \( |\alpha| \) is the length of \( \alpha \). Of course \( |0| = 0 \) and \( 0 \leq \alpha \ \forall \ \alpha \in \Gamma \).

\[
\text{co}(\Gamma) = \{ x \in \mathbb{R}^\Gamma : \{ \alpha \in \Gamma : x(\alpha) \neq 0 \} \text{ is finite} \}
\]

For each \( \alpha \in \Gamma \) we define \( b_\alpha \in \text{co}(\Gamma) \) by \( b_\alpha(\gamma) = 1 \) if \( \gamma \leq \alpha \) and \( b_\alpha(\gamma) = 1 \) in other case.

\[
\text{STS}_0 = \text{co}\{ b_\alpha : \alpha \in \Gamma \} \subset \text{co}(\Gamma).
\]

So, \( \text{STS}_0 \) is a nonempty closed, bounded and convex subset of \( \text{co}(\Gamma) \).

**Theorem.** \( \text{STS}_0 \) has KMP and fails RNP.

**Proof.** It is easy to see that

\[
b_\beta \in \text{co}(A \setminus B(b_\beta, 1)) \quad \forall \beta \in \Gamma,
\]

where \( A = \{ b_\alpha : \alpha \in \Gamma \} \), because

\[
\lim_{n \to +\infty} \frac{b_{\alpha,1} + \ldots + b_{\alpha,n}}{n} = b_\alpha \quad \forall \alpha \in \Gamma.
\]

Then \( A \) is not dentable and so \( \text{STS}_0 \) fails RNP.

Now let \( C \) be a nonempty closed and convex subset of \( \text{STS}_0 \). We will see that \( \text{Ext}(C) \neq \emptyset \).

Let \( z \in C \), and \( K = \{ x \in C : \text{supp}(x) \subseteq \text{supp}(z) \} \), where for each \( x \in C \), \( \text{supp}(x) = \{ \alpha \in \Gamma : x(\alpha) \neq 0 \} \).

Now \( K \) is a nonempty, convex and compact face of \( C \). The Krein-Milman theorem says us that \( \text{Ext}(K) \neq \emptyset \) and so, \( \text{Ext}(C) \neq \emptyset \) because \( K \) is a face of \( C \).

**Remark.** As in [1] it is easy to see that \( \text{STS}_0 \) fails PCP (the point of continuity property) because \( \{ b_{\alpha,i} \} \) converges weakly to \( b_\alpha \) when \( i \to +\infty \), \( \forall \alpha \in \Gamma \) and \( \| b_{\alpha,i} - b_\alpha \| = 1 \ \forall \alpha \in \Gamma \). (This is not immediate because our environment space is not complete.)

Now, we give an example of a closed, bounded and convex set in a noncomplete space verifying RNP and failing KMP.

For this, we consider \( c_0 \) the Banach space of real null sequences with the maximum norm and, \( c_{00} \) the nonclosed subspace of \( c_0 \) of real sequences with a finite numbers of terms nonzero. So, \( c_{00} \) is a noncomplete normed linear space. We define:

\[
F_0 = \left\{ x \in c_{00} : |x_n| \leq \frac{1}{n} \ \forall \ n \in \mathbb{N} \right\}
\]

Then \( F_0 \) is a closed, bounded and convex subset of \( c_{00} \).

It is clear that \( F_0 \) has not extreme points because if \( x \in F_0 \) and \( k \in \mathbb{N} \) such that \( x(n) = 0 \ \forall \ n \geq k \), then \( y = x + \frac{1}{k} e_k \) and \( z = x - \frac{1}{k} e_k \) are elements of \( F_0 \) such that \( x = \frac{y + z}{2} \). (\( e_k \) is the sequence with value 1 in \( k \) and value 0 in \( n \neq k \).)
Therefore, $F_0$ fails KMP.
Let us see, now, that $F_0$ has RNP. If $C$ is a subset of $F_0$, then $\overline{C}$ is a weakly compact of $c_0$, since the closure of $F_0$ in $c_0$, $F$ is it. So $C$ is dentable. (See [2, Th. 2.3.6].)
Then $F_0$ has RNP and fails KMP.

References


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