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A NOTE ON THE CIRCUMFERENCE OF GRAPHS

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Abstract. The well-known Bondy's Theorem [**1**] guarantees (in terms of vertex degrees) a sufficiently "large" cycle in a block. We show that adding a condition on connectivity of these blocks yields an improvement of the lower bound in Bondy's Theorem.

Introduction

Throughout, the graphs considered are finite, simple, undirected and of order $n \geq 3$. The **degree** $d_G(v)$ (or simply $d(v)$) of a vertex v in a graph G is the number of edges in G incident with v. A graph G is called **k-solid** if for each i-cut $\{u_1, u_2, \dots, u_i\}$ of G, $i \leq k$, it holds that $G - \{u_1, u_2, \dots, u_i\}$ has at most two components. The maximum cycle length in G is the **circumference** $c(G)$. If $c(G) = n$, G is said to be **Hamiltonian**. The characterization of Hamiltonian graphs is apparently a very hard problem, though various sufficient conditions are known (cf. [**3**] for a survey). Many of these conditions are based on vertex degrees; such as the following result of Bondy.

Theorem 1 [**1**]. *Let G be a block with vertex degrees $d_1 \leq d_2 \leq \dots \leq d_n$. If*

$$d_j \leq j,\ d_k \leq k,\ (j \neq k) \Longrightarrow d_j + d_k \geq c,$$

then G has a cycle of length at least $\min(c, n)$.

A special case (if we set $c = n$) of this result has been generalized in [**2**], [**4**] and [**5**]. The aim of this paper is to strengthen the general Bondy's result by adding a condition on connectivity of blocks. Our main result is

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Theorem 2. *Let G be a 4-solid block with vertex degrees $d_1 \leq d_2 \leq \cdots \leq d_n$. If*

$$(1) \qquad d_j \leq j,\ d_k \leq k,\ (j \neq k) \Longrightarrow \begin{array}{l} d_j + d_k \geq c, \quad \text{and} \\ \left(d_{n-\lfloor \frac{c}{2} \rfloor} \geq c\right) \vee \left(d_{n-c} > \dfrac{c}{2}\right), \end{array}$$

then G has a cycle of length at least $\min(c+1, n)$.

For the reader's convenience we start with some definitions and notations. Let P_i be a path. For simplicity, we will refer to the first vertex of P_i as f_i and to the last vertex of P_i as l_i. Following this, if $P = (f, x_1, x_2, \ldots, x_k, l)$, then the reverse path to P is the path $\overline{P} = (\overline{f}, x_k, x_{k-1}, \ldots, x_1, \overline{l})$, where $\overline{f} = l$ and $\overline{l} = f$. When $u, v \in V(P)$ and u precedes v on P we write $u \prec_P v$. The subpath of P starting at u and ending at v will be denoted by $[u, v]$; similarly, $[u, v]_i$ will denote the section of P_i. We write $p(v)$ and $s(v)$ for the predecessor and successor of v on P, respectively. If P_i and P_j are two paths for which $l_i = f_j$, then the **composition** $P_i \cdot P_j$ is the path $[f_i, p(l_i)]_i$ followed by P_j. A path P has **length** $\ell(P) = |V(P)| - 1$; a cycle C has length $\ell(C) = |V(C)|$. Let P, P_i and P_j be paths such that $V(P) \cap V(P_i) = \{f_i, l_i\}$, $V(P) \cap V(P_j) = \{f_j, l_j\}$ and $V(P_i) \cap V(P_j) = \emptyset$. Then P_i **overlaps with** P_j on P if $f_i \prec_P f_j \prec_P l_i \prec_P l_j$.

We will need the following Lemma which has been proved in [**1**].

Lemma 1 [**1**]. *Let G be a block and let P be any path in G. Then for some $m \geq 1$, there is a sequence of m pairwise edge-disjoint paths $P_1, \ldots, P_m$, satisfying*

$$f_1 = f,\ l_m = l,\ V(P) \cap V(P_i) = \{f_i, l_i\}, \quad 1 \leq i \leq m,$$

and such that, for $1 \leq i < m - 1$, P_i overlaps with P_{i+1} on P.

Proof of Theorem 2.

By Theorem 1, G has a cycle of length at least $\min(c, n)$. Suppose by way of contradiction that $c(G) = c < n$; we will refer to the cycle of length c as C. Then there are $n - c$ vertices which do not lie on C.

Let P be a path of maximum length in G, chosen so that the sum of degrees $d(f) + d(l)$ is as large as possible. Let $d(f) = j$, $d(l) = k$, with $j \leq k$, and let J and K be the sets of vertices adjacent to f and l, respectively. Let $p(J) = \{p(v) \mid v \in J\}$. Then $d(x) \leq d(f) = j$ for each $x \in p(J)$, since otherwise we can find a longest path with larger sum of degrees of its endvertices. Therefore the j vertices of $p(J)$ have degrees at most j so that $d_j \leq j$. Analogously we have $d_k \leq k$, and so by (1) $d(f) + d(l) = j + k \geq d_j + d_k \geq c$.

Let the path P have length $\ell(P) = p$. We claim that $p \geq c + 1$. This is true if there is at least one edge between vertices which do not lie on C (which follows from the connectivity of G). So, let us assume that there is no edge between these vertices. Now, the degree of each of these vertices is at most $\frac{c}{2}$.

It follows from (1) that either $d_{n-\lfloor \frac{c}{2} \rfloor} \geq c$ or $d_{n-c} > \frac{c}{2}$. In the first case each of the $\lfloor \frac{c}{2} \rfloor + 1$ vertices of degree at least c must lie on C and at least two of them are consecutive on C. Both these vertices must be adjacent to at least one vertex not on C. The claim follows immediately. In the second case there are at least $c+1$ vertices of degree greater than $\frac{c}{2}$. Thus, at least one of these vertices does not lie on C, a contradiction. This proves our claim.

Choose the minimum possible system of paths $P_1, \ldots, P_m$ satisfying Lemma 1. From the maximality of P, the paths P_1 and P_m both have length 1.

(i) $m = 1$. Then the edge $(l, f) \in E(G)$ and the cycle $P \cdot (l, f)$ has length $p + 1 \geq c + 2$.

(ii) $m = 2$. Choose the paths P_1 and P_2 so that the length of the path $[f_2, l_1]$ is as small as possible. Suppose that $\ell([f_2, l_1]) \geq p - c + 3$. Let H' be the graph induced by the set of vertices $V([f_1, f_2]) \cup V([l_1, l_2])$ and $H = H' + (f_2, l_1) - (f_1, l_1)$. The order of H is at most $|V(P)| - |V([s(f_2), p(l_1)])| \leq p + 1 - p + c - 2 = c - 1$. From the maximality of P, $d_H(f_1) = d_G(f_1) - 1$ and $d_H(l_2) = d_G(l_2)$, and hence $d_H(f_1) + d_H(l_2) \geq c - 1$. Let J', K' be sets of vertices adjacent to f_1, l_2, in H, respectively; and let $p(J') = \{p(v) \mid v \in J'\}$. For $x \in p(J') \cap K'$, the paths $P'_1 = (f_1, s(x))$ and $P'_2 = (x, l_2)$ satisfy conditions of Lemma 1 with $\ell([f'_2, l'_1]) = 1$, contradicting the choice of P_1, P_2. Obviously, $|p(J')| = d_H(f_1)$. It follows that $d_H(l_2) \leq c - 2 - d_H(f_1)$, a contradiction. Therefore $\ell([f_2, l_1]) \leq p - c + 2$. If $\ell([f_2, l_1]) \leq p - c + 1$, then the cycle $P_1 \cdot [l_1, l_2] \cdot \overline{P_2} \cdot \overline{[f_1, f_2]}$ has length at least $|V(P)| - |V([s(f_2), p(l_1)])| \geq p + 1 - p + c = c + 1$.

Now suppose that $\ell([f_2, l_1]) = p - c + 2 \geq 3$. First consider the following three cases. Let $N(x)$ denote the **neighbourhood** of the vertex x in G.

(a) there is $u \in N(f_1), v \in N(l_2)$ such that $p(l_1) \prec_P v$ and $v \prec_P u$;

Assume that vertices u, v are chosen in such way that $\ell([v, u])$ is as small as possible. Consider the graph H', induced by the set of vertices $V([f_1, f_2]) \cup V([l_1, s(v)]) \cup V([u, l_2])$. Let $H = H' + (f_2, l_1) + (s(v), u) - (f_1, l_1)$. Since $\ell([v, u]) \geq p - c + 2 \geq 3$, the order of H is at most $|V(P)| - |V([s(f_2), p(l_1)])| - |V([s(s(v)), p(u)])| \leq p + 1 - p + c - 1 - p + c = 2c - p \leq c - 1$. One can show by a method similar to the above that there are two consecutive vertices $p(x)$ and x such that $p(x) \in N(l_2)$ and $x \in N(f_1)$, a contradiction.

(b) there is $u \in N(f_1), v \in N(l_2)$ such that $u \prec_P s(f_2)$ and $v \prec_P u$;

This case can be handled similarly to the case (a).

(c) there are vertices $u, v \in N(f_1)$, such that $v \prec_P u$, $u \prec_P s(f_2)$ or $p(l_1) \prec_P v$, $\ell([v, u]) \geq 2$ and no vertex from $V([v, u]) - \{u, v\}$ is adjacent to f_1;

Without loss of generality assume that $u \prec_P s(f_2)$ (the case $p(l_1) \prec_P v$ is analogous). Assume that vertices u, v are chosen in such way that $\ell([v, u])$ is as small as possible. By (a) and (b) no vertex from $V([v,u]) - \{u\}$ is adjacent to l_2. Let H' be the graph induced by the set $V([f_1,v]) \cup V([u,f_2]) \cup V([l_1,l_2])$ and let $H = H' + (v,u) + (f_2,l_1) - (f_1,l_1)$. Since $\ell([v,u]) \geq 2$, the order of H is at most $|V(P)| - |V([s(f_2),p(l_1)])| - |V([s(v),p(u)])| \leq p+1-p+c-1-1 = c-1$. Again it can be shown that there are two consecutive vertices $p(x)$ and x such that $p(x) \in N(l_2)$ and $x \in N(f_1)$, a contradiction.

Now, there is only one case left for the neighbours of the vertex f_1, namely, when its neighbours are vertices $s(f_1)$, $s(s(f_1))$, $\dots$, $s(s(\dots s(f_1))) = x$, l_1, $s(l_1)$, $\dots$, $s(s(\dots s(l_1))) = y$. It follows from $d(f_1) + d(l_2) \geq c$ and from $|V(P)| - |V([s(f_2),p(l_1)])| = p+1-p+c-1 = c$ that the neighbours of l_2 are $p(l_2)$, $p(p(l_2))$, $\dots$, $p(p(\dots p(l_2)))$, y, f_2, $p(f_2)$, $\dots$, x. In what follows we show that $G' = G - \{f_2, l_1, x, y\}$ has at least three components, a contradiction. The first of them, say A, will be induced by vertices f_1, $s(f_1)$, $\dots$, $p(x)$, $s(l_1)$, $s(s(l_1))$, $\dots$, $p(y)$. The second one, say B, will be formed by vertices $s(x)$, $s(s(x))$, $\dots$, $p(f_2)$, $s(y)$, $s(s(y))$, $\dots$, l_2. From the fact that there is at least one vertex, say z, for which $f_2 \prec_P z \prec_P l_1$, there is at least one more component.

Now we prove that A is indeed a component of G'. Assume that there is an edge ab, where

1. $f_1 \prec_P a \prec_P x$ and $b \notin V(P)$. Then the path $(b,a) \cdot \overline{[f_1,a]} \cdot (f_1,s(a)) \cdot [s(a),l_2]$ has length $p+1$, a contradiction.
2. $f_1 \prec_P a \prec_P x$ and $x \prec_P b \prec_P f_2$. Then the cycle $(a,b) \cdot [b,l_2] \cdot (l_2,p(b)) \cdot \overline{[s(a),p(b)]} \cdot (s(a),f_1) \cdot [f_1,a]$ has length $p+1 > c+1$, a contradiction.
3. $f_1 \prec_P a \prec_P x$ and $f_2 \prec_P b \prec_P l_1$. Then the cycle $(a,b) \cdot [b,l_2] \cdot (l_2,f_2) \cdot \overline{[s(a),f_2]} \cdot (s(a),f_1) \cdot [f_1,a]$ has length at least $c+1$, a contradiction.
4. $f_1 \prec_P a \prec_P x$ and $y \prec_P b \prec_P l_2$. Then the cycle $(a,b) \cdot [b,l_2] \cdot (l_2,p(b)) \cdot \overline{[s(a),p(b)]} \cdot (s(a),f_1) \cdot [f_1,a]$ has the length $p+1 > c+1$, again a contradiction.
5. $l_1 \prec_P a \prec_P y$ and $b \notin V(P)$. Then the path $(b,a) \cdot \overline{[f_1,a]} \cdot (f_1,s(a)) \cdot [s(a),l_2]$ has the length $p+1$, a contradiction.
6. $l_1 \prec_P a \prec_P y$ and $x \prec_P b \prec_P f_2$. Then the cycle $(a,b) \cdot \overline{[f_1,b]} \cdot (f_1,s(a)) \cdot [s(a),l_2] \cdot (l_2,s(b)) \cdot [s(b),a]$ has the length $p+1 > c+1$, again a contradiction.
7. $l_1 \prec_P a \prec_P y$ and $f_2 \prec_P b \prec_P l_1$. Then $(a,b) \cdot [b,p(a)] \cdot (p(a),f_1) \cdot [f_1,f_2] \cdot (f_2,l_2) \cdot \overline{[a,l_2]}$ has length at least $c+1$ a contradiction.
8. $l_1 \prec_P a \prec_P y$ and $y \prec_P b \prec_P l_2$. Then the cycle $(a,b) \cdot [b,l_2] \cdot (l_2,p(b)) \cdot \overline{[s(a),p(b)]} \cdot (s(a),f_1) \cdot [f_1,a]$ has length $p+1 > c+1$, which is final contradiction.

Thus A is a component of G'. The fact that B is a component of G' can be proved similarly. The existence of the third component confirms that G is not 4-solid, a contradiction.

(iii) $m \geq 3$. From the minimality of m it holds that $u \in J$ implies $u \prec_P s(f_3)$ and $v \in K$ implies $p(l_{m-2}) \prec_P v$. Choose P_1 and P_m so that $\ell([f_1,l_1])$ and $\ell([f_m,l_m])$

are as small as possible. If m is odd, then the cycle $P_1 \cdot [l_1, f_3] \cdot P_3 \cdot [l_3, f_5] \cdot \ldots \cdot [l_{m-2}, f_m] \cdot P_m \cdot \overline{[l_{m-1}, l_m]} \cdot \overline{P_{m-1}} \cdot \overline{[l_{m-3}, f_{m-1}]} \cdot \overline{P_{m-3}} \cdot \ldots \cdot \overline{P_2} \cdot \overline{[f_1, f_2]}$ has length at least $c+1$. If m is even, then the cycle $P_1 \cdot [l_1, f_3] \cdot P_3 \cdot [l_3, f_5] \cdot \ldots \cdot P_{m-1} \cdot [l_{m-1}, l_m] \cdot \overline{P_m} \cdot \overline{[l_{m-2}, f_m]} \cdot \overline{P_{m-2}} \cdot \ldots \cdot \overline{P_2} \cdot \overline{[f_1, f_2]}$ has length at least $c+1$. Indeed, in both cases these cycles contain all vertices of J and K together with f and l. Moreover, $|J \cap K| \le 1$ and $|J| + |K| \ge c$. This proves the Theorem. □

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