

## SUPER-GEOMETRIC QUANTIZATION

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ABSTRACT. Let  $K$  be the complex line bundle where the Kostant-Souriau geometric quantization operators are defined. We discuss possible prolongations of these operators to the linear superspace of the  $K$ -valued differential forms, such that the Poisson bracket is represented by the supercommutator of the corresponding operators. We also discuss the possibility to obtain such super-geometric quantizations by (anti)Hermitian operators on a Hilbert superspace. We apply our general considerations to Kähler manifolds and to cotangent bundles of Riemannian manifolds.

### 1. RECALLING GEOMETRIC QUANTIZATION

In differential geometry, the problem of geometric quantization is a two stage problem which can be stated in the following terms (e.g., [10], [9]).

**Stage 1 — Prequantization.** Let  $M$  be a Poisson manifold with the Poisson bracket

$$(1.1) \quad \{f, g\} = P(df, dg) \quad (f, g \in C^\infty(M)).$$

Find linear representations of the Lie algebra (1.1) on the space  $\Gamma(K)$  of cross sections of a complex line bundle  $K$  over  $M$  by differential operators of order one and symbol equal to the Hamiltonian vector field  $X_f^P$ .

**Stage 2 — Quantization.** Restrict prequantization in such a way as to obtain irreducible anti-Hermitian<sup>1</sup> representations of a subalgebra of  $C^\infty(M)$  with bracket (1.1) on a Hilbert space derived from  $\Gamma(K)$ .

In this paper, we define the problem of **super-geometric quantization** as the problem of prolonging the representations mentioned above to linear and Hilbert superspaces.

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<sup>1</sup>The fact that we use anti-Hermitian operators here is just a technicality. If these operators are multiplied by a purely imaginary constant they become Hermitian operators.

Now, let us be more precise. While more general prequantization representations may exist [5], [9], we consider only the fundamental **Kostant-Souriau representation**. The latter is given by the operators

$$(1.2) \quad \hat{f}\sigma = \nabla_{X_f}\sigma + 2\pi\sqrt{-1}f\sigma \quad (f \in C^\infty(M), \sigma \in \Gamma(K)),$$

where  $\nabla$  is a connection on  $K$  which preserves a Hermitian metric  $h$  of  $K$ .

The condition that (1.2) is a representation means

$$(1.3) \quad \widehat{\{f, g\}} = \hat{f} \circ \hat{g} - \hat{g} \circ \hat{f},$$

and this condition is equivalent to

$$(1.4) \quad \Omega(X_f, X_g) = -2\pi\sqrt{-1}\{f, g\},$$

where  $\Omega$  is the curvature of  $\nabla$ . In particular, (1.4) shows that  $K$ ,  $h$  and  $\nabla$  exist iff  $P$  defines an integral Poisson cohomology class (namely, the image of the integral first Chern class of  $K$ ) [9], and, then, we say that  $(M, P)$  is a **quantizable Poisson manifold**. In the symplectic case, the integrality condition is just that the symplectic form represents an integral cohomology class [10].

Furthermore, let  $\mathcal{D}$  be the bundle of complex valued **halfdensities** of  $M$  (e.g., [6], [7]). Then (1.2) extends to  $\Gamma(K \otimes \mathcal{D})$  by

$$(1.5) \quad \hat{f}(\sigma \otimes \rho) = (\hat{f}\sigma) \otimes \rho + \sigma \otimes L_{X_f}\rho \quad (\sigma \in \Gamma(K), \rho \in \gamma(\mathcal{D}))$$

where  $L$  denotes the Lie derivative, and Stokes' theorem shows that the operators (1.5) are anti-Hermitian on  $\Gamma_c(K \otimes \mathcal{D})$  ( $c$  means "with compact support") endowed with the scalar product

$$(1.6) \quad \langle \sigma_1 \otimes \rho_1, \sigma_2 \otimes \rho_2 \rangle = \int_M h(\sigma_1, \sigma_2) \rho_1 \bar{\rho}_2$$

(the bar means complex conjugation) i.e., we have

$$(1.7) \quad \langle \hat{f}\alpha, \beta \rangle + \langle \alpha, \hat{f}\beta \rangle = 0 \quad (\alpha, \beta \in \Gamma_c(K \otimes \mathcal{D})).$$

Of course, we may complete  $\Gamma_c(K \otimes \mathcal{D})$  to a Hilbert space but, we still remain in the prequantization stage since we do not have irreducibility.

Now, the stage of quantization is based on the notion of a **polarization**, for which we adopt here a new definition that includes the classical definition as a particular case. Let  $\mathcal{F}$  denote the sheaf of Poisson algebras of germs of complex valued  $C^\infty$  functions of  $M$  with the bracket (1.1). Then, a **polarization**  $\mathcal{P}$  of  $(M, P)$  is a subsheaf  $\mathcal{P}$  of  $\mathcal{F}$  whose stalks are abelian subalgebras of the stalks of  $\mathcal{F}$ .

If  $\mathcal{P}$  is given, we may look at the linear space

$$(1.8) \quad \Gamma_0(K) = \{\sigma \in \Gamma(K) / \nabla_{X_\varphi} \sigma = 0, \forall \varphi \in \mathcal{P}\},$$

and we may apply the operators (1.2) to  $\Gamma_0(K)$  if  $\Gamma_0(K) \neq \{0\}$ . It is easy to see that,  $\forall f \in C^\infty(M)$  such that  $\{\varphi, f\} \in \mathcal{P}$  whenever  $\varphi \in \mathcal{P}$ ,  $\hat{f}(\Gamma_0(K)) \subseteq \Gamma_0(K)$ . The set  $\mathcal{Q}(M, \mathcal{P})$  of such functions  $f$  is a Lie subalgebra of  $(C^\infty(M), \{, \})$  which includes all the real global sections  $\psi$  of  $\mathcal{P}$ , and for these  $\psi$  one has  $\hat{\psi}\sigma = 2\pi\sqrt{-1}\psi\sigma$ ,  $\forall \sigma \in \Gamma_0(K)$ , as needed for irreducibility [10].

Furthermore, if  $\Gamma_0(K)$  has nonzero elements with compact support, it may be possible to adapt conveniently the scalar product (1.6), and obtain a Hilbert space where (1.7) holds  $\forall f \in \mathcal{Q}(M, \mathcal{P})$ . Otherwise, the idea is to project the whole configuration onto a lower dimensional quotient manifold  $N$ , if possible, and get a similar scalar product by integration over  $N$  [10], [7], [9].

The basic types of polarizations encountered in applications are as follows (e.g., [10]).

1) Let  $(M^{2n}, \omega)$  ( $d\omega = 0$ ) be a quantizable symplectic manifold, with the Poisson brackets defined by  $\omega$ , and assume that  $M$  has a real Lagrangian foliation  $\mathcal{L}$ . Then, the sheaf  $\mathcal{P}$  of germs of functions which are constant along the leaves of  $\mathcal{L}$  is a polarization called a **real Lagrangian polarization**. An important particular case is that of a cotangent bundle  $M = T^*N$ , where  $\omega = d\theta$ ,  $\theta :=$  the Liouville 1-form of  $T^*N$ , and  $\mathcal{L}$  is the foliation by the fibers of  $T^*N$ . In this case, if  $\sigma \in \Gamma_0(K)$ ,  $\text{supp } \sigma$  is noncompact (it is a union of fibers), and the scalar product will be defined by integration over  $N$  and not over  $T^*N$ .

2) Let  $(M^{2n}, \omega)$  be a quantizable symplectic manifold which admits compatible Kähler metrics. Then, if  $g$  is such a metric, the sheaf  $\mathcal{P}$  of germs of holomorphic functions with respect to the corresponding complex structure is a polarization called a **Kähler polarization**. In this case,  $K$  is a holomorphic line bundle (e.g., [7]), and

$$\mathcal{Q}(M, \mathcal{P}) = \{f \in C^\infty(M) / X_f = X_f^{1,0} + X_f^{0,1}, X_f^{1,0} \text{ holomorphic}\},$$

where the upper indices indicate the complex type. Equivalently, if  $J$  is the tensor of the complex structure, then  $L_{X_f} J = 0$ . We say that  $X$  is an analytic vector field, and we distinguish in this paper between the terms analytic and holomorphic for vector fields i.e.,  $X$  is **analytic** and its component  $X^{1,0}$  is **holomorphic**. Furthermore, we may forget about halfdensities, and make  $\Gamma_{0c}(K)$  into a Hilbert space by the scalar product

$$(1.9) \quad \langle \sigma_1, \sigma_2 \rangle = \int_M h(\sigma_1, \sigma_2) d(\text{vol } g) \quad (\sigma_1, \sigma_2 \in \Gamma_{0c}(K)),$$

and the property (1.7) follows again from Stokes' theorem.

## 2. SUPER-GEOMETRIC PREQUANTIZATION

Now, we proceed to the discussion of super-geometric quantization. We start with a quantizable Poisson manifold  $(M, P)$  and a quantization complex line bundle  $K$ . Let us emphasize that we do not intend to discuss geometric quantization of supermanifolds, as in [3]. Neither do we consider any kind of supermanifolds [1]. But, we shall use the terminology of superalgebra (e.g., [4]).

With  $(M, P, K)$ , we can associate a natural complex linear superspace

$$(2.1) \quad \mathcal{S}(K) = \mathcal{S}^+(K) \oplus \mathcal{S}^-(K).$$

where

$$\mathcal{S}^+ = \bigoplus_{i \geq 0} \wedge^{2i}(M, K), \quad \mathcal{S}^- = \bigoplus_{i \geq 0} \wedge^{2i+1}(M, K),$$

and  $\wedge^h(M, K)$  are the spaces of  $K$ -valued forms on  $M$ , and it is possible to extend the Kostant-Souriau prequantization (1.2), (1.5) to  $\mathcal{S}(K)$ .

We did this in [8] as follows. Since  $\wedge^h(M, K) = \Gamma(\wedge^h T^*M \otimes K)$ , one has the well-known **covariant exterior differential**

$$(2.2) \quad D(\alpha \otimes \sigma) = (d\alpha) \otimes \sigma + (-1)^{\deg \alpha} \alpha \wedge \nabla \sigma,$$

and the **covariant Lie derivative**

$$(2.3) \quad L_X^\nabla(\alpha \otimes \sigma) = (L_X \alpha) \otimes \sigma + \alpha \otimes \nabla_X \sigma,$$

where  $\alpha \in \wedge^h M$ ,  $\sigma \in \Gamma(K)$ , and  $X$  is a vector field on  $M$ . These operators have the same global expressions as  $d$  and  $L_X$ , except for the fact that the action of  $X$  on functions is replaced by the action of  $\nabla_X$  on sections of  $K$ . Notice also the formula

$$(2.4) \quad L_X^\nabla = Di(X) + i(X)D$$

which follows from (2.2) and (2.3).

Now, if (1.2) is extended to  $\mathcal{S}(K)$  by

$$(2.5) \quad \hat{f}A = L_{X_f}^\nabla A + 2\pi\sqrt{-1}fA \quad (A \in \mathcal{S}(K)),$$

it follows from (1.4) that the commutator condition (1.3) is still valid. Indeed [8], using (2.3) we get

$$(2.6) \quad L_X^\nabla L_Y^\nabla A - L_Y^\nabla L_X^\nabla A - L_{[X, Y]}^\nabla A = \Omega(X, Y)A,$$

where  $\Omega$  is the curvature of  $\nabla$ , and then, (1.4) is obtained by a straightforward computation.

The operators  $\hat{f}$  preserve the degree of a form. Thus, if we want to give a role to the structure (2.1), it is natural to define a **super-geometric prequantization** of  $M$  on  $K$  as a prolongation of (2.5) of the form

$$(2.7) \quad \tilde{f}A = \hat{f}A + 2\pi\sqrt{-1}l(f)(A) \quad (A \in \mathcal{S}(K)),$$

where  $l(f)$  is an odd endomorphism of  $\mathcal{S}(K)$ , such that the following commutation condition holds

$$(2.8) \quad \widetilde{\{f, g\}} = {}^s[\tilde{f}, \tilde{g}].$$

In the right hand side of (2.8), one has the supercommutator [4] of the operators  $\tilde{f}$ ,  $\tilde{g}$ , and we denoted it by the index  $s$ . Brackets without this index will denote usual commutators.

**Proposition 2.1.** *The operation  $*$  defined by*

$$(2.9) \quad f * \theta = [\hat{f}, \theta] = \hat{f}\theta - \theta\hat{f},$$

*( $f \in C^\infty(M)$ ,  $\theta \in \text{End}\mathcal{S}(K)$ ) is a representation of the Lie algebra  $(C^\infty(M), \{ , \})$  on  $\text{End}\mathcal{S}(K)$  which leaves  $\text{End}_-\mathcal{S}(K)$  invariant, and (2.7) is a super-geometric prequantization iff  $l$  is a 1-cocycle with values in  $\text{End}_-\mathcal{S}(K)$ , and with respect to the representation (2.9), such that*

$$(2.10) \quad l^2(f) = 0, \quad \forall f \in C^\infty(M).$$

*Proof.* The results are rather straightforward since, in view of (1.3), (2.8) is equivalent to

$$(2.11) \quad l(\{f, g\}) = [\hat{f}, l(g)] + [l(f), \hat{g}], \quad l(f)l(g) + l(g)l(f) = 0,$$

$\forall f, g \in C^\infty(M)$ . □

**Corollary 2.2.**  *$\forall c \in \text{End}_-\mathcal{S}(K)$  such that  $[\hat{f}, c]^2 = 0$ ,  $\forall f \in C^\infty(M)$ , the operators*

$$(2.12) \quad \tilde{f}_c(A) = L_{\hat{X}}^\nabla A + 2\pi\sqrt{-1}fA + 2\pi\sqrt{-1}[\hat{f}, c](A)$$

*( $A \in \mathcal{S}(K)$ ) define a super-geometric prequantization.*

*Proof.*  $[\hat{f}, c]$  is the coboundary of  $c$  in the Lie algebra cohomology mentioned in Proposition 2.1. □

We note some important particular cases in

**Proposition 2.3.** *Let  $\theta$  be a complex valued 1-form, and  $V$  be a complex vector field on the Poisson manifold  $(M, P)$ . Then, (2.7) is a super-geometric prequantization for each of the following choices of  $l$ :*

$$(2.13) \quad \begin{aligned} l_1(f) &= e(L_{X_f}\theta), & l_2(f) &= i([X_f, V]), \\ l_3(f) &= l_1(f) + l_2(f). \end{aligned}$$

*Proof.* In (2.13),  $e$  means “exterior product by”, and  $i$  means “interior product by”.  $l_1$  is obtained by using Corollary (2.2) for  $c = e(\theta)$ , and  $l_2$  is obtained for  $c = i(V)$ .  $\square$

**Remark 2.4.** If we take  $c = D$ , then, using (2.4) and the well known fact that  $D^2 = e(\Omega)$ , we get

$$l(f) = e(i(X_f)\Omega - 2\pi\sqrt{-1}df),$$

which is 0 in the symplectic case because of (1.4).

Now, as in Section 1, we can relate super-geometric prequantization with a scalar product. Namely, we consider again the complex line bundle  $\mathcal{D}$  of halfdensities over  $M$ , and use the bundle  $K \otimes \mathcal{D}$  instead of  $K$ . Then, instead of (2.1), we have

$$(2.14) \quad \tilde{\mathcal{S}}(K) := \mathcal{S}(K \otimes \mathcal{D}) := \tilde{\mathcal{S}}^+(K) \oplus \tilde{\mathcal{S}}^-(K),$$

which consists of forms with values in  $K \otimes \mathcal{D}$  organized as those in (2.1).

Furthermore, we put on  $M$  a Riemannian metric  $g$ , and define a scalar product of  $\wedge_c^p(M, K \otimes \mathcal{D})$  (i.e., forms with a compact support) by

$$(2.15) \quad \langle \alpha_1 \otimes \sigma_1 \otimes \rho_1, \alpha_2 \otimes \sigma_2 \otimes \rho_2 \rangle = \int_M g(\alpha_1, \alpha_2) h(\sigma_1, \sigma_2) \rho_1 \bar{\rho}_2,$$

where  $\alpha_i \in \wedge^p(M)$ ,  $\sigma_i \in \Gamma(K)$ ,  $\rho_i \in \Gamma(\mathcal{D})$  ( $i = 1, 2$ ). Then, we get

**Proposition 2.5.** *Assume that (2.7) is a super-geometric prequantization where the odd cocycle  $l$  is Hermitian with respect to  $gh$ . Then, the extension of (2.7) defined by*

$$(2.16) \quad \tilde{f}(A \otimes \rho) = (\tilde{f}A) \otimes \rho + A \otimes L_{X_f}\rho$$

*satisfies the commutator property (2.8), and, if  $X_f$  is a Killing vector field for  $g$ ,  $\tilde{f}$  is anti-Hermitian with respect to (2.15).*

*Proof.* That  $\tilde{f}$  of (2.16) also satisfies (2.8) follows by a straightforward calculation. (Notice that  $l(f)$  extends to  $\tilde{\mathcal{S}}(K)$  by  $l(f)(A \otimes \rho) = (l(f)A) \otimes \rho$ .) Furthermore, by the metric  $gh$  we mean

$$gh(\alpha_1 \otimes \sigma_1, \alpha_2 \otimes \sigma_2) = g(\alpha_1, \alpha_2) h(\sigma_1, \sigma_2),$$

and  $l(f)$  are supposed to be  $gh$ -Hermitian. The anti-Hermitian character (1.7) of the present situation follows by using Stokes' theorem under the form (e.g., [6])

$$\int_M L_{X_f}(g(\alpha_1, \alpha_2)h(\sigma_1, \sigma_2)\rho_1\bar{\rho}_2) = 0.$$

□

### 3. SUPER-GEOMETRIC QUANTIZATION

Now, we combine super-geometric prequantization with a polarization and this process is **super-geometric quantization**.

Let  $(M, P)$  be a Poisson manifold endowed with the prequantization (2.7), (2.16), and the scalar product (2.15), and let  $\mathcal{P}$  be a polarization of  $M$ . Then, we shall define the linear superspace

$$(3.1) \quad \mathcal{S}_0(K) = \{A \in \mathcal{S}(K) / L_{X_\varphi}^\nabla A = 0, i(X_\varphi)A = 0, \forall \varphi \in \mathcal{P}\}.$$

Using (2.5), (2.6) and (1.4), we see easily that  $\forall A \in \mathcal{S}_0(K), \forall f \in \mathcal{Q}(M, \mathcal{P})$ , one has  $\tilde{f}A \in \mathcal{S}_0(K)$ . We recall that (Section 1)

$$\mathcal{Q}(M, \mathcal{P}) = \{f \in C^\infty(M) / \{\varphi, f\} \in \mathcal{P}, \forall \varphi \in \mathcal{P}\}.$$

Furthermore, in order to deal with the odd part of (2.7), we restrict ourselves to  $\mathcal{Q}'(M, \mathcal{P}) \subseteq \mathcal{Q}(M, \mathcal{P})$ , where we define that  $f \in \mathcal{Q}'(M, \mathcal{P})$  if it satisfies the following supplementary conditions

$$(3.2) \quad [L_{X_\varphi}^\nabla, l(f)] = 0, \quad {}^s[i(X_\varphi), l(f)] = 0, \quad \forall \varphi \in \mathcal{P}.$$

Then, we get

**Proposition 3.1.**  *$\forall f \in \mathcal{Q}'(M, \mathcal{P})$  and  $\forall A \in \mathcal{S}_0(K)$ , we have  $\tilde{f}A \in \mathcal{S}_0(K)$ , for  $\tilde{f}$  defined by (2.7). In particular, if the 1-form  $\theta$  and the vector field  $V$  of  $M$  are such that  $L_{X_\varphi}\theta = i(X_\varphi)\theta = 0, [X_\varphi, V] = 0, \forall \varphi \in \mathcal{P}$ , the prequantizations of Proposition 2.3 induce quantization formulas on  $\mathcal{S}_0(K), \forall f \in \mathcal{Q}(M, \mathcal{P})$ .*

*Proof.* The first assertion follows straightforwardly from the definitions. For the second assertion, we check that (3.2) holds for the cocycles  $l_1$  and  $l_2$  of (2.13), and  $\forall B \in \mathcal{S}(K)$ :

$$\begin{aligned} L_{X_\varphi}^\nabla((L_{X_f}\theta) \wedge B) - (L_{X_f}\theta) \wedge L_{X_\varphi}^\nabla B &= (L_{X_\varphi}L_{X_f}\theta) \wedge B \\ &= (L_{X_f}L_{X_\varphi}\theta + L_{X_{\{\varphi, f\}}}\theta) \wedge B = 0, \\ i(X_\varphi)((L_{X_f}\theta) \wedge B) + (L_{X_f}\theta) \wedge (i(X_\varphi)B) &= (i(X_\varphi)L_{X_f}\theta)B \\ &= (L_{X_f}i(X_\varphi)\theta + i(X_{\{\varphi, f\}})\theta)B = 0, \\ L_{X_\varphi}^\nabla i([X_f, V])B - i([X_f, V])L_{X_\varphi}^\nabla B &= i([X_\varphi, [X_f, V]])B \\ &= i([X_{\{\varphi, f\}}, V])B + i([X_f, [X_\varphi, V]])B = 0, \\ i(X_\varphi)i([X_f, V])B + i([X_f, V])i(X_\varphi)B &= 0. \end{aligned}$$

□

Furthermore, if we want a good scalar product, we may try to adapt conveniently formula (2.15), but, since we know from Proposition 2.5 that we shall need to ask  $X_f$  to be a Killing vector field for  $g$ , it is simpler to look at the subspace  $\mathcal{S}_{0c}(K)$  of the elements of  $\mathcal{S}_0(K)$  which have a compact support, and put

$$(3.3) \quad \langle \alpha_1 \otimes \sigma_1, \alpha_2 \otimes \sigma_2 \rangle = \int_M g(\alpha_1, \alpha_2) h(\sigma_1, \sigma_2) d(\text{vol } g)$$

(while, of course,  $M$  is assumed to be oriented). This scalar product vanishes on forms of different degrees. Hence, it makes  $\mathcal{S}_{0c}(K)$  into a pre-Hilbert superspace, which, afterwards, will be completed to a Hilbert superspace. Then, just as for Proposition 2.5, we deduce

**Proposition 3.2.** *Assume that the cocycle  $l$  is Hermitian with respect to the metric  $gh$ , and put*

$$(3.4) \quad \mathcal{Q}''(M, \mathcal{P}) = \{f \in \mathcal{Q}'(M, \mathcal{P}) / L_{X_f} g = 0\}.$$

*Then, the operator  $\tilde{f}$  of (2.7), associated with any  $f \in \mathcal{Q}''(M, \mathcal{P})$  is anti-Hermitian with respect to the metric (3.3).*

*Proof.* Make explicit the Lie derivative in the Stokes' formula

$$\int_M L_{X_f} (g(\alpha_1, \alpha_2) h(\sigma_1, \sigma_2) d(\text{vol } g)) = 0.$$

□

**Corollary 3.3.** *Assume that,  $\forall \varphi \in \mathcal{P}$ ,  $L_{X_\varphi} g = 0$ . Assume that there exists a 1-form  $\theta$  on  $M$  such that  $\forall \varphi \in \mathcal{P}$  one has  $L_{X_\varphi} \theta = 0$ ,  $i(X_\varphi) \theta = 0$ , and define  $V = \sharp_g \theta$ . Then, the cocycle  $l_3$  of (2.13) is  $g$ -selfadjoint, and  $\forall f \in \mathcal{Q}(M, \mathcal{P})$  such that  $X_f$  is a Killing vector field for  $g$  the superquantization  $\tilde{f}$  of (2.7) with  $l = l_3$  is defined on  $\mathcal{S}_0(K)$ , and it is anti-Hermitian with respect to (3.3).*

*Proof.* By the definition of  $V$ , we have  $g(V, Z) = \theta(Z)$  for any vector field  $Z$  of  $M$ , and the hypotheses of Proposition 3.1 are satisfied. Furthermore, we also see that  $\sharp_g(L_{X_f} \theta) = L_{X_f} V = [X_f, V]$ . Hence, the  $g$ -adjoint of  $e(L_{X_f} \theta)$  is  $i([X_f, V])$ , and the result follows. □

#### 4. KÄHLER AND LAGRANGIAN POLARIZATIONS

Now, we shall apply the general Propositions of Section 3 to the two basic examples mentioned in Section 1 i.e., where  $M$  is a symplectic manifold and  $\mathcal{P}$  is either a Kähler or a real Lagrangian polarization of  $M$ .

In the case of a Kähler polarization we get

**Proposition 4.1.** *Let  $(M, \omega)$  be a quantizable symplectic manifold, and  $\mathcal{P}$  a Kähler polarization of  $M$ , with the corresponding complex structure  $J$  and metric  $g$ . Then  $K$  is a holomorphic line bundle,  $\mathcal{S}_0(K)$  is the linear superspace of the  $K$ -valued holomorphic forms of  $(M, J)$ , and,  $\forall f \in \mathcal{Q}(M, \mathcal{P})$ , the Hamiltonian vector field  $X_f$  is Killing. Furthermore, if  $\theta$  is a holomorphic 1-form on  $M$ , (2.7) with  $l(f) = e(L_{X_f}\theta)$  is a super-geometric quantization on  $\mathcal{S}_0(K)$ ,  $\forall f \in \mathcal{Q}(M, \mathcal{P})$ . Moreover, if*

$$\mathcal{Q}_0(M, \mathcal{P}) := \{f \in \mathcal{Q}(M, \mathcal{P}) \mid \sharp_g L_{X_f} \bar{\theta} \text{ is holomorphic}\},$$

then (2.7) with

$$(4.1) \quad l(f) = e(L_{X_f}\theta) + i([X_f, \sharp_g \bar{\theta}])$$

is an anti-Hermitian super-geometric quantization of  $\mathcal{Q}_0(M, \mathcal{P})$  on  $\mathcal{S}_0(K)$  seen as a Hilbert superspace with the scalar product (3.3).

*Proof.* We already recalled in Section 1 that  $K$  is holomorphic and that,  $\forall f \in \mathcal{Q}(M, \mathcal{P})$ ,  $X_f$  is analytic ( $L_{X_f}J = 0$ ). Since, of course,  $L_{X_f}\omega = 0$ , we also have  $L_{X_f}g = 0$ . The assertion about the super-geometric quantization with the odd cocycle  $e(L_{X_f}\theta)$  follows from Proposition 3.1.

Finally, we claim that,  $\forall f \in \mathcal{Q}_0(M, \mathcal{P})$ , the conditions (3.2) are also satisfied for the cocycle  $l(f) = i([X_f, \sharp_g \bar{\theta}])$ . Indeed, the second condition (3.2) is well known, and, as shown during the proof of Proposition 3.1, the first condition (3.2) is satisfied if

$$(4.2) \quad [X_\varphi, [X_f, \sharp_g \bar{\theta}]] = 0, \quad \forall \varphi \in \mathcal{P}.$$

By taking  $\varphi$  equal to the local complex coordinates  $z^i$  of  $(M, J)$ , we see that the antiholomorphic tangent bundle  $T_{0,1}M$  of  $(M, J)$  has local bases of the form  $\{X_{\varphi_i}\}$ , for some  $\varphi_i \in \mathcal{P}$ . Hence, (4.2) means that  $[X_f, \sharp_g \bar{\theta}]$  preserves  $T_{0,1}M$ . On the other hand, since  $X_f$  is Killing, we have

$$(4.3) \quad [X_f, \sharp_g \bar{\theta}] = \sharp_g(L_{X_f}\bar{\theta}),$$

and this is a vector field of the complex type  $(1, 0)$ . Accordingly, (4.2) holds iff  $[X_f, \sharp_g \bar{\theta}]$  is a holomorphic vector field, as claimed.

Now, if we use again Proposition 3.1, and the argument of Corollary 3.3, namely, that the adjoint of  $e(\theta)$  is  $i(\bar{\theta})$ , we obtain the last assertion of Proposition 4.1.  $\square$

We shall also add a few more results about the space  $\mathcal{Q}(M, \mathcal{P})$  of a Kähler polarization.

**Proposition 4.2.** *i) For a Kähler polarization  $\mathcal{P}$ ,  $f \in \mathcal{Q}(M, \mathcal{P})$  iff*

$$(4.4) \quad \nabla_{\bar{i}} \left( \frac{\partial f}{\partial \bar{z}^j} \right) = 0,$$

where  $(z^i)$  are complex coordinates and  $\nabla$  is the Riemannian connection of the Kähler manifold  $(M, g, J)$ .

*ii) If the Kähler manifold  $M$  is compact,  $f \in \mathcal{Q}(M, \mathcal{P})$  iff*

$$(4.5) \quad \Delta df - 2\sharp_r^{-1} \sharp_g df = 0,$$

where  $\Delta$  is the Laplace operator and  $r$  is the Ricci tensor of  $g$ .

*Proof.* i) The condition (4.4) follows immediately from the local coordinate expression of a Hamiltonian vector field  $X_f$ .

ii) In (4.5) the definition of  $\sharp_r^{-1}$  is similar to that of  $\sharp_g^{-1}$ , but  $\sharp_r$  may not exist. It is well known that, if  $M$  is compact,  $X_f$  is analytic iff

$$\Delta(\sharp_g^{-1} X_f) - 2\sharp_r^{-1} X_f = 0$$

(e.g., see Proposition 2.140 in [2]). But, it follows easily that  $\sharp_g^{-1} X_f = -df \circ J$ , and, using the known properties of  $\Delta$  in the Kähler case, the previous relation becomes

$$(\Delta df) \circ J + 2\sharp_r^{-1} J \sharp_g df = 0.$$

If this equality is composed by  $J$ , and if we remember that  $r$  is compatible with  $J$ , (4.5) follows.  $\square$

**Remark 4.3.** 1) If  $M$  is a compact connected Kähler-Einstein manifold, (4.5) becomes

$$(4.6) \quad \Delta f - 2\kappa f = \text{const.},$$

where  $\kappa$  is the (constant) scalar curvature of  $g$ .

2) If the Ricci curvature of the compact Kähler manifold  $M$  is negative definite,  $\mathcal{Q}(M, \mathcal{P}) = \mathbf{R}$ . Indeed, in this case  $M$  has no non zero analytic vector fields (e.g., Proposition 2.138 in [2]).

In order to exemplify the case of a Lagrangian polarization, we consider the basic situation of a cotangent bundle  $M = T^*N$  with the symplectic form

$$(4.7) \quad \omega = -d\theta + p^*F,$$

where  $\theta$  is the Liouville form,  $p: T^*N \rightarrow N$  is the natural projection, and  $F$  is an exact 2-form  $F = d\lambda$  of  $N$  (the **electromagnetic term**). Thus, if  $q^i$  are local

coordinates on  $N$ , and  $p_i$  are covector coordinates, we have (with the Einstein summation convention)

$$(4.8) \quad \theta = p_i dq^i, \quad \lambda = \lambda_i(q) dq^i.$$

Then,  $K$  may be taken trivial, the  $K$ -valued forms are just complex valued forms, the connection  $\nabla$  can be defined by the global, flat connection form  $2\pi\sqrt{-1}(\theta - \lambda)$ , and the prequantization formula (2.5) becomes

$$(4.9) \quad \hat{f}A = L_{X_f}A + 2\pi\sqrt{-1}(\theta(X_f) - \lambda(X_f) + f)A \quad (A \in \wedge M \otimes \mathbf{C}).$$

Furthermore, the polarization  $\mathcal{P}$  is defined as the sheaf of germs of lifts to  $T^*N$  of functions on  $N$  (i.e., functions of the  $(q^i)$  alone), and

$$(4.10) \quad \mathcal{Q}(M, \mathcal{P}) = \{f \in C^\infty(M) / f = \mu(Y) + \varphi\},$$

where  $Y$  is a tangent vector field of  $N$ ,  $\mu(Y)$  is its **momentum**  $\mu(Y) = p_i Y^i$  ( $Y = Y^i(\partial/\partial q^i)$ ), and  $\varphi \in \mathcal{P}$  (e.g., [10]).

We shall use the notions of complete and vertical lift as defined, for instance, in [Y]. Then, it is easy to obtain

$$(4.11) \quad X_\varphi = \text{vertical lift of } d\varphi = \frac{\partial\varphi}{\partial q^i} \frac{\partial}{\partial p_i}, \quad \forall \varphi \in \mathcal{P},$$

and, for a vector field  $Y$  of  $N$

$$(4.12) \quad \begin{aligned} X_{\mu(Y)} &= \text{--complete lift of } Y \text{ -- vertical lift of } i(Y)F \\ &= -Y^i \frac{\partial}{\partial q^i} + \text{vertical part} \end{aligned}$$

(**vertical** means tangent to the fibers of  $T^*M$ ).

From (4.11) we see easily that  $\mathcal{S}_0(K)$  can be identified with the linear superspace of the complex valued differential forms of the base manifold  $N$ .

An odd cocycle  $l$  is provided by the Liouville form  $\theta$  and, as we know, it is  $l(f) = e(L_{X_f}\theta)$  ( $f \in C^\infty(T^*N)$ ). In particular, using (4.11) and (4.12), we get for  $f = \mu(Y) + \varphi \in \mathcal{Q}(M, \mathcal{P})$

$$(4.13) \quad l(\mu(Y) + \varphi) = e(-i(Y)F + d\varphi),$$

which is a 1-form on  $N$ . Hence, this cocycle  $l$  defines a super-geometric quantization of  $\mathcal{Q}(M, \mathcal{P})$  on  $\mathcal{S}_0(K)$ . Moreover, we can prove

**Proposition 4.4.** *With the notation above, and with respect to a fixed Riemannian metric  $g$  on the base manifold  $N$ , the formula*

$$(4.14) \quad \begin{aligned} \hat{f}A &= -L_Y A + 2\pi\sqrt{-1}(\varphi + \lambda(Y))A + 2\pi\sqrt{-1}(-i(Y)F + d\varphi) \wedge A \\ &\quad + 2\pi\sqrt{-1}i(-i(Y)F + d\varphi)A \quad (A \in \wedge^* N \otimes \mathbf{C}) \end{aligned}$$

defines a super-geometric quantization of the observables  $f = \mu(Y) + \varphi \in \mathcal{Q}(M, \mathcal{P})$ , such that  $Y$  is a  $g$ -Killing vector field of  $N$ , on the linear superspace  $\wedge_c^* N \otimes \mathbf{C}$  ( $c$  means “with compact support”) with the odd-even grading. This quantization is by anti-Hermitian operators with respect to the scalar product defined by  $g$  on the forms of  $N$ .

*Proof.* In the right hand side of (4.14), the first two terms are  $\hat{f}A$  (as one can see by using (4.9), (4.11), (4.12)), and the third term is the odd cocycle (4.13). Moreover, the operator of the fourth term is the  $g$ -adjoint of the operator of the third term. Therefore, we must only check that this fourth term behaves like a superquantization 1-cocycle i.e., it satisfies the conditions (2.11),  $\forall f = \mu(Y) + \varphi$ ,  $g = \mu(Z) + \psi$ , where  $Y, Z$  are  $g$ -Killing vector fields of  $N$ ,  $\varphi, \psi \in C^\infty(N)$ . The second condition (2.11) is obvious, and, for the first, we compute the corresponding expressions for  $l(f) = i(-i(Y)F + d\varphi)$ , and in the following cases.

a)  $f = \varphi$ ,  $g = \psi$ . Then, with (4.11),  $\{f, g\} = 0$ , and  $l(\{f, g\}) = 0$ . Furthermore,  $[\hat{\varphi}, l(\psi)] = 0$ ,  $[l(\varphi), \hat{\psi}] = 0$ .

b)  $f = \varphi$ ,  $g = \mu(Z)$ . Then,  $\{f, g\} = X_f g = Zf$ , and  $l(\{f, g\}) = i(dZ\varphi)$ . Furthermore, we obtain

$$[\hat{\varphi}, l(g)] + [l(\varphi), \hat{g}] = i([Z, \sharp_g d\varphi]) = i(\sharp_g dZ\varphi) = i(dZ\varphi).$$

We used that  $\forall \alpha \in \wedge^1(M)$ ,  $i(\alpha) := i(\sharp_g \alpha)$ , and that  $Z$  is Killing i.e.,  $L_Z \sharp_g = 0$ .

c)  $f = \mu(Y)$ ,  $g = \mu(Z)$ . Then

$$(4.15) \quad \{f, g\} = X_{\mu(Y)}\mu(Z) \stackrel{(4.12)}{=} -\mu([Y, Z]) - F(Y, Z),$$

and, since  $dF = 0$ ,

$$(4.16) \quad \begin{aligned} l(\{f, g\}) &= i(i([Y, Z])F - d(F(Y, Z))) \\ &= -i(i(Z)L_Y F - L_Y i(Z)F + d(F(Y, Z))) \\ &= -i(i(Z)di(Y)F - i(Y)di(Z)F + 2d(F(Y, Z))). \end{aligned}$$

Furthermore, using again the general relations that exist among  $L_X$ ,  $i(X)$ ,  $d$  for any vector field  $X$ , we get

$$(4.17) \quad \begin{aligned} [\hat{f}, l(g)] + [l(f), \hat{g}] &= i([Z, \sharp_g i(Y)F] - [Y, \sharp_g i(Z)F]) \\ &= i(\sharp_g L_Z i(Y)F - \sharp_g L_Y i(Z)F) \end{aligned}$$

(because  $Y, Z$  are Killing vector fields), and the final result will be the same as in (4.16).  $\square$

**Remark 4.5.** If  $\lambda$  is used instead of  $\theta$ , the same results as in Proposition 4.4, can be proven in the same way for

$$(4.18) \quad \begin{aligned} \tilde{f}A = & -L_Y A + 2\pi\sqrt{-1}(\varphi + \lambda(Y))A - \\ & - 2\pi\sqrt{-1}(L_Y \lambda) \wedge A - 2\pi\sqrt{-1}i([Y, \sharp_g \lambda])A. \end{aligned}$$

In (4.18), the notation and the hypotheses are the same as in Proposition 4.4.

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