A NOTE ON THE HAMILTONIAN GENUS
OF A COMPLETE BIPARTITE GRAPH

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Abstract. The Hamiltonian genus of a graph $G$ (denoted by $\gamma_H(G)$) is the smallest number $g$ such that the graph $G$ is embeddable in the orientable surface with genus $g$ and there is some face-boundary $b$ which is a Hamiltonian cycle of $G$. In this paper we show that

$$[(n-2)(n-1)/4] \leq \gamma_H(K_{n,n}) \leq \lfloor n/2 \rfloor^2 + \lceil n/2 \rceil.$$

Introduction

The Hamiltonian genus of a graph $G$ (denoted $\gamma_H(G)$) is the smallest number such that the graph $G$ is embeddable in the orientable surface with genus $g$ and there is some face-boundary $b$ such that $b$ is a Hamiltonian cycle of $G$. Such an embedding is called a Hamiltonian embedding. D. Bénard [1] proved the equality

$$\gamma_H(K_n) = \gamma(K_{n+1}) = \lfloor (n-2)(n-3)/12 \rfloor.$$  

This is a corollary of the following result whose proof can be found in [1].

For every integer $n \geq 3$ there is an embedding of $K_n$ into an orientable surface of genus $\gamma(K_n)$ such that at least one vertex of $K_n$ is incident to triangular faces only.

In our paper we extend the known results on the Hamiltonian genus of graphs by investigating the Hamiltonian genus of complete bipartite graphs. For the sake of comparison it may be of interest to mention that $K_{n,n}$ admits an orientable embedding in which each face is bounded by a Hamiltonian cycle [4, 5]. By the Euler-Poincaré formula, its genus is $(n-1)(n-2)/2$.

Main Result

Theorem. Let $K_{n,n}$ be a complete bipartite graph. Then

$$[(n-2)(n-1)/4] \leq \gamma_H(K_{n,n}) \leq \lfloor n/2 \rfloor^2 + \lceil n/2 \rceil.$$  

The proof will be performed in the following two lemmas.
Lemma 1. \( \gamma_H(K_{n,n}) \geq [(n - 2)(n - 1)/4] \).

Proof. Let \( f_H : K_{n,n} \leftrightarrow S \) be a minimum Hamiltonian embedding of the graph \( K_{n,n} \) into a surface \( S \). Add one vertex and the corresponding edges into the face \( f \) whose boundary \( \partial f \) is the Hamiltonian cycle of \( G \) to obtain the graph \( K_{n,n+1} \). From the construction of \( K_{n,n+1} \) is clear, that it is still embedded into the surface \( S \). Therefore

\[
\gamma(K_{n,n+1}) = [(n - 2)(n - 1)/4] \leq \gamma_H(K_{n,n}).
\]

Lemma 2. \( \gamma_H(K_{n,n}) \leq [n/2]^2 - [n/2] \), i.e.

\[
\gamma_H(K_{n,n}) \leq n(n - 2)/4 \text{ if } n \text{ is even and }
\gamma_H(K_{n,n}) \leq (n + 1)(n - 1)/4 \text{ if } n \text{ is odd}.
\]

Proof. Let \( m = [n/2] \) and let

\[
f_H : K_{m,m} \leftrightarrow S,
\]

\[
g_H : K_{m,m} \leftrightarrow T
\]

be minimum Hamiltonian embeddings of \( K_{m,m} \) such that \( g_H \) is a mirror image of \( f_H \) (i.e. if \( v_1v_2 \ldots v_{m-1}v_1 \) is the face-boundary in \( f_H \), then \( v_1v'_1 \ldots v_{m}v'_1 \) is the face-boundary in \( g_H \)). We express the number of faces of \( f_H \) (denoted \( F(f_H) \)) using the well-known Euler-Poincaré formula

\[
F(G) = |E(G)| - |V(G)| + 2 - 2\gamma(G).
\]

For the embedding \( f_H \) the number of faces is

\[
(1) \quad F(f_H) = [n/2]^2 - 2[n/2] + 2 - 2\gamma_H(K_{m,m}).
\]

Clearly, for \( g_H \), \( F(g_H) = F(f_H) \). Now, we will create the embedding

\[
h : K_{2m,2m} \leftrightarrow R
\]

from the embeddings \( f_H \) and \( g_H \). First we cut the disks in all faces of \( f_H \) and \( g_H \) and glue the corresponding holes. By this operation we amalgamate two surfaces (\( S \) and \( T \)), each of genus \( \gamma_H(K_{m,m}) \) with \( F(f_H) \) handles. Therefore by (1) the genus of the resulting surface \( R \) is

\[
(2) \quad \gamma(R) = 2\gamma_H(K_{m,m}) + F(f_H) - 1 = m^2 - 2m + 1.
\]

Let \( v_1v_2v_3 \ldots v_{m-1}v_1 \) be some face-boundary of the embedding \( f_H \) (see Fig. 1) and let \( v'_1v'_2 \ldots v'_m v'_1 \) be the corresponding face-boundary of \( g_H \). By amalgamating these two faces we obtain a non-cellular face (a handle) with two boundary
cycles $v_1v_2v_3 \ldots v_{t-1}v_t$ and $v'_1v'_2v'_3 \ldots v'_{t-1}v'_t$ (as shown in Fig. 2). Add edges $v_1v'_2$, $v_2v'_3$, ..., $v_{t-1}v'_t$, $v_tv'_1$ into this face. It is easy to see that the orientability of the surfaces $S$ and $T$ guarantees that the addition of the relevant edges is possible. The necessary considerations can also be made using the rotation schemes. Fig. 3 illustrates the result of this operation.
Routine calculations show that by adding the appropriate edges into each face of $R$ (precisely in the sense of the rotation) we obtain the requested embedding $h$ of $K_{2m, 2m}$. A similar construction is described in Pisanski [3] and Nedela and Škoviera [2].

Let $(u_1 u_2 \cdots u_{m-1} u_m)$ be the permutation of the vertices $\{v_1, v_2, \ldots, v_m\}$ such that $u_1 u_2 \cdots u_{m-1} u_m$ is the Hamiltonian cycle in $f_H$ and let $(u'_1 u'_2 \cdots u'_m)$ be the permutation of the vertices $\{v'_1, v'_2, \ldots, v'_m\}$ such that $u'_1 u'_2 \cdots u'_m$ is the Hamiltonian cycle in $g_H$. From the method described above it follows that since the embeddings $f_H$ and $g_H$ have the Hamiltonian cycles $u_1 u_2 \cdots u_{m-1} u_m$ and $u'_1 u'_2 \cdots u'_m$ as the boundaries of some faces $s$ and $s'$, respectively, then in the embedding $h$ of $K_{2m, 2m}$ there are faces $r_1, r_2, \ldots, r_m$ with boundaries $u_1 u_2 u_3 u'_2$, $u_2 u_3 u'_2 u'_3$, $u_{m-1} u_m u'_1 u'_2$, $u_m u_1 u'_2 u'_1$, respectively.

Without loss of generality, select one from these faces, say $r_1$. Join the other faces $r_2, \ldots, r_m$ by deleting edges $u_3 u'_1$, $u_4 u'_2$, $u_{m-1} u'_m$, $u_m u'_1$ to one face $t$. It is easy to see that the boundary $u_2 u_3 u_4 u'_2 u'_3 u'_1$ of $t$ is a Hamiltonian cycle of a new graph. Moreover by adding at most $m - 1$ handles we can add all deleted edges back to the graph to obtain the complete bipartite graph $K_{2m, 2m}$ (as shown in Fig. 4).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure4.png}
\caption{The Hamiltonian cycle and adding the deleted edges.}
\end{figure}

Thus we obtain a new surface $R'$ from the surface $R$, and the genus of $R'$ is

\begin{equation}
\gamma(R') = \gamma(R) + \lfloor n/2 \rfloor - 1.
\end{equation}

The statement of the lemma now follows immediately from (2) and (3). $\square$

References


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