

## REFINEMENT OF AN INEQUALITY OF E. LANDAU

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ABSTRACT. We prove: Let  $P(z) = \sum_{k=0}^n a_k z^k$  be a complex polynomial with  $n \geq 1$  and  $a_0 a_n \neq 0$ . If  $z$  is a zero of  $P$ , then we have for all real numbers  $t > 0$ :

$$(*) \quad |z| > \frac{|a_0|t}{|a_0| + K_n(t)}$$

with

$$\begin{aligned} K_n(t) &= \frac{1}{1 - \alpha_n(t)^n} \min_{1 \leq m \leq n} \left[ (\alpha_n(t)^m - \alpha_n(t)^n) \max_{m \leq p \leq n} A_p(t) \right. \\ &\quad \left. + (1 - \alpha_n(t)^m) \max_{1 \leq p \leq n} A_p(t) \right], \\ \alpha_n(t) &= \frac{|a_0|}{|a_0| + \max_{1 \leq p \leq n} A_p(t)}, \\ A_p(t) &= \frac{1}{p} \sum_{k=1}^p |a_k| t^k. \end{aligned}$$

Inequality (\*) sharpens a result of E. Landau.

In this paper we denote by  $P$  the polynomial  $P(z) = \sum_{k=0}^n a_k z^k$ , where  $z$  and the  $a_k$ 's are complex numbers. Moreover, we assume  $n \geq 1$  and  $a_0 a_n \neq 0$ . In 1914 E. Landau [2] presented the following lower bound for the moduli of the zeros of  $P$ :

**Theorem A.** *If  $z$  is a zero of  $P$  and  $t$  is any positive real number, then*

$$(1) \quad |z| \geq \frac{|a_0|t}{|a_0| + \max_{1 \leq p \leq n} |a_p| t^p}.$$

The same result was also obtained by J. Karamata [1] and D. Markovitch [3]. In 1967 D. M. Simeunović [5] proved an interesting refinement of inequality (1) (see also [4, pp. 222–223]):

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**Theorem B.** *If  $z$  is a zero of  $P$  and  $t$  is any positive real number, then*

$$(2) \quad |z| \geq \frac{|a_0|t}{|a_0| + \max_{1 \leq p \leq n} A_p(t)}$$

with

$$A_p(t) = \frac{1}{p} \sum_{k=1}^p |a_k| t^k.$$

It is the aim of this note to establish an inequality which sharpens (1) as well as (2).

**Theorem.** *Let  $z$  be a zero of  $P$ . Then we have for all real numbers  $t > 0$ :*

$$(3) \quad |z| > \frac{|a_0|t}{|a_0| + K_n(t)}$$

with

$$K_n(t) = \frac{1}{1 - \alpha_n(t)^n} \min_{1 \leq m \leq n} \left[ (\alpha_n(t)^m - \alpha_n(t)^n) \max_{m \leq p \leq n} A_p(t) + (1 - \alpha_n(t)^m) \max_{1 \leq p \leq n} A_p(t) \right]$$

and

$$\alpha_n(t) = \frac{|a_0|}{|a_0| + \max_{1 \leq p \leq n} A_p(t)}.$$

**Remark.** Since  $\alpha_n(t) < 1$  we have

$$(4) \quad K_n(t) \leq \max_{1 \leq p \leq n} A_p(t).$$

This implies that inequality (3) improves the results of Landau and Simeunović. It is easy to find conditions such that inequality (4) is strict. For example, if  $n > 2$  and  $\frac{1}{p-1} \sum_{k=2}^p |a_k| t^{k-1} < |a_1|$  for all  $p \in \{2, \dots, n\}$ , then (4) holds with “ $<$ ” instead of “ $\leq$ ”.

*Proof of the Theorem.* Let  $|z| = r$ ,  $R = \frac{|a_0|t}{|a_0| + K_n(t)}$  and  $D = \{w \in \mathbb{C} \mid |w| \leq R\}$ . We assume that  $z$  lies on the disk  $D$ . Because of

$$|a_0| = \left| \sum_{k=1}^n a_k z^k \right| \leq \sum_{k=1}^n |a_k| r^k$$

and  $0 < \frac{r}{t} \leq \frac{R}{t} < 1$  we obtain

$$(5) \quad \frac{\sum_{k=1}^n |a_k| t^k}{\sum_{k=1}^n (r/t)^k} \geq \frac{|a_0|}{\sum_{k=1}^n (r/t)^k} > \frac{t-r}{r} |a_0| \geq \frac{t-R}{R} |a_0| = K_n(t).$$

Let  $m \in \{1, \dots, n\}$ ; using Abel's identity we get

$$\begin{aligned} \sum_{k=1}^n |a_k| r^k &= \sum_{k=1}^n |a_k| t^k (r/t)^k \\ &= \sum_{k=1}^{m-1} k A_k(t) [(r/t)^k - (r/t)^{k+1}] + \sum_{k=m}^{n-1} k A_k(t) [(r/t)^k - (r/t)^{k+1}] \\ &\quad + n A_n(t) (r/t)^n \\ &\leq \max_{1 \leq p \leq n} A_p(t) \sum_{k=1}^{m-1} k [(r/t)^k - (r/t)^{k+1}] \\ &\quad + \max_{m \leq p \leq n} A_p(t) \sum_{k=m}^{n-1} k [(r/t)^k - (r/t)^{k+1}] + \max_{m \leq p \leq n} A_p(t) n (r/t)^n \\ &= \max_{m \leq p \leq n} A_p(t) \sum_{k=1}^n (r/t)^k \\ &\quad + \left( \max_{1 \leq p \leq n} A_p(t) - \max_{m \leq p \leq n} A_p(t) \right) \left( \sum_{k=1}^m (r/t)^k - m (r/t)^m \right) \\ &\leq \max_{m \leq p \leq n} A_p(t) \sum_{k=1}^n (r/t)^k + \left( \max_{1 \leq p \leq n} A_p(t) - \max_{m \leq p \leq n} A_p(t) \right) \sum_{k=1}^m (r/t)^k. \end{aligned}$$

This implies

$$(6) \quad \frac{\sum_{k=1}^n |a_k| r^k}{\sum_{k=1}^n (r/t)^k} \leq \max_{m \leq p \leq n} A_p(t) + \left( \max_{1 \leq p \leq n} A_p(t) - \max_{m \leq p \leq n} A_p(t) \right) \frac{1 - (r/t)^m}{1 - (r/t)^n}.$$

From inequality (2) we obtain  $\frac{r}{t} \geq \alpha_n(t)$ . Since the function  $x \mapsto \frac{1-x^m}{1-x^n}$  is decreas-

ing on  $(0, 1)$  we conclude from (6):

$$\frac{\sum_{k=1}^n |a_k| r^k}{\sum_{k=1}^n (r/t)^k} \leq \min_{1 \leq m \leq n} \left[ \max_{m \leq p \leq n} A_p(t) + \left( \max_{1 \leq p \leq n} A_p(t) - \max_{m \leq p \leq n} A_p(t) \right) \frac{1 - \alpha_n(t)^m}{1 - \alpha_n(t)^n} \right]$$

$$= K_n(t),$$

which contradicts inequality (5). Therefore,  $z$  lies outside the disk  $D$ .  $\square$

### References

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