MINIMAL SIZE OF A GRAPH WITH DIAMETER 2
AND GIVEN MAXIMAL DEGREE, II

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Abstract. Let $F_2(n, \lfloor pn \rfloor)$ be the minimal size of a graph on $n$ vertices with diameter 2 and maximal degree $\lfloor pn \rfloor$. The asymptotic behaviour of $F_2(n, \lfloor pn \rfloor)$ is considered for $2/5 < p < 5/12$.

1. Introduction

Denote by $H_2(n, \lfloor pn \rfloor)$ the family of undirected graphs of order $n$, diameter 2 and maximal degree $\lfloor pn \rfloor$ $(0 < p < 1)$ and put

$$F_2(n, \lfloor pn \rfloor) = \min_{G \in H_2(n, \lfloor pn \rfloor)} e(G)$$

where $e(G)$ is the size of $G$. Denote further by

$$f(p) = \lim_{n \to \infty} F_2(n, \lfloor pn \rfloor).$$

The function $f(p)$ was introduced in [1] and in [5] the existence of the limit (conjectured in [2]) was proved for all values of $p$ except of a sequence tending to 0. It is also showed in [5] that for a given $p$, $f(p)$ can be determined using linear programming. However, this procedure is too slow to enable us to solve the problem even for relatively large values of $p$.

In [2] the values of $f(p)$ for $p > 1/2$ were determined. Further, in [4] it was shown that if a projective plane of order $t$ exists, then $f(p) = t + 1$ for $(t + 1)/(t^2 + t + 1) < p < 1/t$, hence putting $t = 2$ we get $f(p) = 3$ if $3/7 < p < 1/2$. In [6] $f(p)$ is determined for $5/12 < p < 3/7$. Thus for all $p > 5/12$ the values of $f(p)$ are known.

In this paper we determine $f(p)$ for smaller $p$. In fact, we prove here the following result conjectured in [5].

**Theorem.**

(1) \[ f(p) = 8 - 11p \quad \text{for} \quad 2/5 < p < 5/12. \]

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We shall use here very often methods and results of [5]. In those cases when our assertions can be proved by a slight modification of those in [5], the proofs will be omitted here. On the other hand, the following lemma is used here in exactly the same way as in [5].

**Lemma 1 ([5]).** Let a set \( U \) with \( |U| \geq 8 \) be given. Let \( Z \) be a system of triples of distinct elements of \( U \). If every element of \( U \) is contained in some triple of \( Z \) and any two triples of \( Z \) intersect then there exist \( x, y \in U \) such that every triple of \( Z \) contains at least one of \( x, y \) (we say that \( x, y \) cover \( Z \)).

2. The Main Inequality

We prove now that if \( p \) fulfills (1) and \( n \) is sufficiently large then for every \( G \in H_2(n, \lfloor pn \rfloor) \) we have

\[
e(G) \geq 8n - 11|pn| - 8 \left\{ 4 \left( \frac{192}{3} \right) + 128 \right\} \sqrt{n}.
\]

Let \( I = 4 \left( \frac{192}{3} \right) + 128 \). We shall proceed indirectly: suppose there exists a graph \( G_0 \in H_2(n, \lfloor pn \rfloor) \) with

\[
e(G_0) < 8n - 11|pn| - 8I \sqrt{n}.
\]

Denote by \( V \) the set of all vertices of \( G_0 \), and by \( A \) the set of vertices of degree at least \( \sqrt{n} \). According to (3) we have

\[
|A| < 16 \sqrt{n}.
\]

Denote further by \( B \) the set of vertices of degree at most 7 adjacent to 3 vertices of \( A \) (due to (1), no vertex of degree less than 3 exists in \( G_0 \)), and by \( C \) the set of such vertices adjacent to at least 4 vertices of \( A \), and finally, let \( D = V - A - B - C \). If \( x \in D \) then \( 8 \leq \deg x \leq \sqrt{n} \).

The proof of the following lemma is straightforward (and very similar to that of Lemma 2 in [5]).

**Lemma 2.**

\[
2e(G_0) \geq 8n - 2|B| - 128 \sqrt{n}.
\]

Let now \( E \) be the set of vertices of degree at least \( n/12 \). By (3) we have

\[
|E| < 192.
\]

According to (1) every vertex of \( B \) is adjacent to 3 vertices of \( E \). Let \( abc \) be the set of vertices of \( B \) adjacent to vertices \( a, b, c \in E \) (a set of the form \( abc \) will be called a triple-set). \( B \) consists of at most \( \left( \frac{192}{3} \right) \) triple-sets. Let \( F \) be the union of triple-sets from \( B \) with cardinalities at least \( 4 \sqrt{n} \). Then

\[
|B| < |F| + 4 \left( \frac{192}{3} \right) \sqrt{n}.
\]

Denote by \( T \) the system of neighbours of triple-sets in \( F \) and by \( W \) the set of vertices occurring in \( T \). Now we shall state some lemmas.
Lemma 3. Any two triples of $T$ have a common element.

The proof is straightforward and very similar to that of Lemma 3 in [5].

Lemma 4. Let $J$ be the set of vertices $x$ having the following property: there is some triple-set $abc$ in $F$ such that $x$ is not adjacent to any of the vertices $a$, $b$, $c$. Then $|J| \leq 4^{192}/3\sqrt{n}$.

The proof follows from the obvious fact that to any triple-set $abc$ there exist at most $4\sqrt{n}$ vertices adjacent to neither $a$, $b$, $c$ (see also the proof of Lemma 4 in [5]).

Lemma 5. $T$ is covered by two vertices.

Proof. Assume the opposite. Then by Lemma 4 each of at least $n - 4^{192}/3\sqrt{n}$ vertices must be adjacent to at least 3 vertices of $W$. However, by Lemma 1 and Lemma 2 we get $|W| \leq 7$, a contradiction with (1).

In what follows we shall use the notation $\lfloor pn \rfloor = k$.

First of all, from (3), (6) and Lemma 2 follows

\[
8n - 2|F| - 2I\sqrt{n} \leq 16n - 22k - 16I\sqrt{n}, \quad \text{thus} \quad |F| \geq 11k - 4n + 7I\sqrt{n}.
\]

Let now $M = V - A - J - H$ where $H$ is a set of cardinality less than $\sqrt{n}$ which will be specified later. From (4) and Lemma 4 we have

\[
m = |M| > n - I\sqrt{n}.
\]

Thus (7) can be rewritten in the form

\[
|F| \geq 11k - 4m + 3I\sqrt{n}.
\]

Further considerations will be restricted to the vertices of $M$. The following two lemmas will be of some use later.

Lemma 6. If $a$, $b$, $c_1$, ..., $c_r$ are distinct vertices of $W$, and there exist $p$ vertices of $M$ adjacent to both $a$ and $b$ then

\[
|abc_1 \cup \cdots \cup abc_r| + rp \leq r(3k - m).
\]

Proof. For every $i \in \{1, \ldots, r\}$ the number of edges incident to vertices $a$, $b$, $c_i$ is at least $m + p + abc_i$ ($\leq 3k$), and thus the assertion follows.

From the obvious inequality $|abc_i| \leq p$ we get, by taking $r = 1$:

Corollary.

\[
|abc_i| \leq (3k - m)/2.
\]
Lemma 7. There exist less than \((5m - 11k)/2\) vertices of \(M\) having at least 5 neighbours in \(V - M\).

Proof. If it is not the case then \(e(G_0) \geq 3m + 2(5m - 11k)/2 = 8m - 11k\), a contradiction with (3).

Now we shall prove that (3) leads to a contradiction. First of all, if all triples of \(T\) contain a fixed vertex then we have \(|F| \leq k\), a contradiction with (9) for \(k \geq 2n/5\) (see (1)). Assume now that \(T\) is covered by two vertices \(x, y\) (see Lemma 5). Denote by \(X, Y\) and \(Z\), respectively, the union of all triple-sets of \(F\) adjacent only to \(x\), only to \(y\), and to both \(x\) and \(y\), respectively.

We have to distinguish several cases depending on the form of the sets \(X, Y\) and \(Z\).

Case 1. \(xuv, xwt \subset X\) and \(yuw, yvt \subset Y\) (or \(yuv, ywt \subset Y\), or \(yut, yvw \subset Y\)) where all included vertices are distinct.

Let \(K = \{x, y\}, L = \{u, v, w, t\}\). Further let \(S, Q, R\), respectively, be the set of vertices of \(M\) adjacent to exactly one vertex, exactly two vertices, and no vertex, respectively, of \(K\). We can easily derive the following inequalities:

1. \(2m + |S| \leq 6k\), i.e. \(|S| \leq 6k - 2m\);
2. \(2|Q| + |S| \leq 2k\), i.e. \(|Q| \leq m - 2k\);
3. \(|R| = m - |S| - |Q| \geq 2m - 4k\).

Suppose \(Z = xyz_1 \cup \cdots \cup xyz_i\) and let \((xyz_1) [(xyz_2)]\) be the set of all vertices of \(M\) adjacent to \(x, y, x_1\) (to \(x, y, x_2\), respectively). The vertices of \(R\) are adjacent to every \(z_j\), hence by (12) we get

4. \((xyz_j) + 2m - 4k \leq k\), i.e. \((xyz_j) \leq 5k - 2m\).

Now we need to consider several subcases.

Case 1a. If there exist at least 3 vertices \(z_j \notin L\) then a vertex of \(R\) is adjacent to 2 vertices of \(L\), to at least three vertices \(z_j\) but this is by (12) a contradiction to Lemma 7.

Case 1b. There exists at most one \(z_j \notin L\). Because all the remaining triple-sets of \(F\) contain 3 vertices of \(L \cup K\), we have \(|F - xyz_i| \leq 6k - 2m\), and by (13) we get a contradiction with (9). (In this case for \(|F| = 11k - 4n - 3\) we obtain the extremal graph – see Section 3.)

Case 1c. Now we have the most complicated case when \(Z\) contains exactly 2 triple-sets \(xyz_1\) and \(xyz_2\) such that \(z_1, z_2 \notin K \cup L\). Denote by \(N, O, P\), respectively, the set of all vertices of \(M\) adjacent to at least two vertices, to one vertex, and to no vertex, respectively, of \(L\). Then

5. \(2|N| + |O| \leq 4k\),
6. \(|O| + |P| \leq |Q|\),
7. \(-2(|N| + |O| + |P|) = -2m\).
Adding these inequalities, and taking into account (11), we have

$$|P| \geq 2m - 4k - |Q| \geq m - 2k.$$  

Hence, by (13)

$$|P| - |(xyz_1)| - |(xyz_2)| \geq 5m - 12k.$$  

Now put $p = 5/12 - \varepsilon$, $0 < \varepsilon < 1/60$ (see (1)). Then, from (8), $5m - 12k \geq 5(n - I\sqrt{n}) - 12(5/12 - \varepsilon)n = 12\varepsilon n - 5I\sqrt{n}$ which for sufficiently large $n$ is greater than $10\varepsilon n$. Hence there exist at least $10\varepsilon n$ vertices of $M$ not adjacent to any vertex of $L$, thus adjacent to both $x$ and $y$, but not adjacent to $z_1$, $z_2$. By (3), among these vertices there exists a vertex $b$ of degree less than $16/(10\varepsilon)$. Let $N(b)$ be the set of neighbours of $b$, and put $H = N(b)$ ($H$ is the above-mentioned set.) Now, each vertex of $R$ is adjacent to at least two vertices of $L$, to the vertices $z_1, z_2$ and to some vertex of $H$. Thus each vertex of $R$ is adjacent to at least 5 vertices of $V - M$ which by (1) and (12) contradicts Lemma 7.

Thus if the conditions $xuv, xut \subseteq X, yuv, yet \subseteq Y$ (or $yuv, ywt \subseteq Y$, or $yet, yuw \subseteq Y$) are satisfied, then we always get a contradiction. Assume now that these conditions are not satisfied. Say, $X$ does not contain triple-sets of the form $xuv, xut$. Then the following cases can occur:

(a) $X = xuv \cup xuw \cup xvw$,
(b) all the triple-sets of $X$ are adjacent to a further fixed vertex of $W$,
(c) $X = xuv$.

Now consider these cases.

**Case 2.** Suppose $X = xuv \cup xuw \cup xvw$. The vertices of $M$ not adjacent to $x$ are adjacent to at least two vertices of the set $\{u, v, w\}$. Hence $2(m - k) + 2|X| \leq 3k$, i.e. $|X| \leq (5k - 2m)/2$, a contradiction with (9), because all the remaining vertices of $F$ are adjacent to $y$.

**Case 3.** Suppose $X = xtx_1 \cup \cdots \cup xtx_q, q \geq 2$. Again we have to distinguish several subcases.

**Case 3a.** $y_1x_j \subseteq Y$ for some $i, j \in \{1, \ldots, q\}$. Then obviously $q = 2$. Let $U = xtx_1 \cup xtx_2 \cup yx_1x_2$. Then every vertex of $M$ is adjacent to at least two vertices of the set $\{x, t, y, x_1, x_2\} = W_0$ and every vertex of $U$ is adjacent to 3 vertices of $W_0$. Thus $2m + |U| \leq 5k$, i.e. $|U| \leq 5k - 2m$ which is a contradiction to (9) because all remaining vertices of $F$ are adjacent to $Y$.

**Case 3b.** Suppose $Y = yty_1 \cup \cdots \cup yty_r$ and put $Z = yx_1 \cup \cdots \cup yx_1z_s$. Assume that among the vertices $x_1, \ldots, x_p, y_1, \ldots, y_r, z_1, \ldots, z_s$ there exist $u$ distinct vertices $w_1, \ldots, w_u$ different from $x$, $y$, $t$, and put $W_1 = \{x, y, t, w_1, \ldots, w_u\}$. Since $F$ is not covered by a single vertex, every vertex of $M$ is adjacent to at least two vertices of $W_1$ and the vertices of $F$ to 3 vertices of $W_1$. Thus $2m + |F| \leq (3 + u)k$ which contradicts (9) if $u \leq 3$. Hence assume $u > 3$. 

Denote by \( L \) the set of all vertices of \( M \) not adjacent to any vertex of the set \( \{x, y, t\} \). Every vertex of \( F \) is adjacent to two vertices of this set, hence by (9) we have

\[
(14) \quad 3k + |L| \geq m + |F|, \quad \text{i.e.} \quad |L| \geq m - 3k + |F| \geq 8k - 3m.
\]

The vertices of \( L \) are adjacent to all vertices \( w_i \), thus for \( u = 4 \) we have

\[
2m + |F| + 2(m - 3k + |F|) \leq 7k, \quad \text{i.e.} \quad |F| \leq (13k - 4m)/3
\]

which is, by (1), a contradiction to (9).

For \( u \geq 5 \) all vertices of \( L \) are adjacent to at least \( 5 \) vertices of \( W_1 \) which by (14) contradicts Lemma 7.

**Case 4.** The last case is \( X = xuv \). If \( Z = \emptyset \) then \( F \) is covered also by \( y \) and \( u \), and we may proceed as in the case 3a. Similarly, if all vertices of \( F \) are adjacent to \( u \) (or \( v \)) then we get again the case 3b. Hence we may assume

\[
F = xuv \cup xyx_1 \cup \cdots \cup xyx_i \cup uyu_1 \cup \cdots \cup uyu_j \cup vyu_1 \cup \cdots \cup vyv_q
\]

where \( i, j, q \) are different from \( 0 \). Now we show that all the vertices \( x_u, u_b, v_c \) are distinct. Indeed, suppose, for example, that \( x_1 = u_1 \) and consider \( U = xuv \cup xyx_1\cup uyx_1 \). Every vertex of \( M \) is adjacent to two vertices of \( W_3 = \{x, y, u, v, x_1\} \) and every vertex of \( U \) is adjacent to \( 3 \) vertices of \( W_3 \), thus \( |U| \leq 5k - 2m \). However, all remaining vertices of \( F \) are adjacent to \( y \), and so we get a contradiction with (9). Now, if \( i = j = q = 1 \) then by Corollary of Lemma 6 we get a contradiction to (9). Hence suppose \( i > 1 \) (as \( F \) is covered also by the couples \( u, y \) and \( v, y \), we may proceed in the remaining cases similarly). Distinguish now two subcases.

**Case 4a.** Suppose \( j = q = 1 \). Consider first the case \( i \geq 3 \). By (9) and by Corollary to Lemma 6 we have

\[
|x_yx_1 \cup \cdots \cup xyx_i| \geq 11k - 4m - (9k - 3m)/2.
\]

Thus the number of vertices adjacent neither to \( x \) nor to \( y \) is at least \( m - 2k + [11k - 4m - (9k - 3m)/2] = (9k - 3m)/2 \). However, each such vertex is adjacent to at least \( 5 \) vertices of \( V - M \), a contradiction to Lemma 7 (see (1)).

Assume now \( i = 2 \). Let \( s \) be the number of vertices adjacent to \( y \) but not to \( x_1 \) nor to \( x_2 \). Each such vertex is adjacent to at least one of vertices \( x, u, v \). So let \( s_1, s_2, \) and \( s_3, \) respectively, be the number of such vertices adjacent to \( x, u, v \), respectively. Obviously,

\[
(15) \quad s_1 + s_2 + s_3 \geq s.
\]

Now according to Lemma 6 we have

\[
(16) \quad |uyu_1| \leq (3k - m - s_2)/2, \quad |vyv_1| \leq (3k - m - s_3)/2.
\]
Consider now the number of vertices adjacent to $x$. By (16),

$$|xuv| + |xyx_1| + |xyx_2| \geq |F| - 3k + m + (s_2 + s_3)/2.$$  

Each vertex of $M - F$ must be adjacent to at least one of vertices $x$, $y$, $x_1$. However, there exist at most $k - |xyx_1|$ such vertices adjacent to $x_1$ and $s - s_1$ such vertices adjacent to $y$ but not to $x$. So, by (17), the total number of vertices adjacent to $x$ is at least

$$|[F] - 3k + m + (s_2 + s_3)/2] + [m - |F| - k + |xyx_1| - s + s_1] \leq k.$$

Thus by (15) we get

$$|xyx_1| \leq 5k - 2m + (s_1 + s_2)/2.$$

Hence, according to (16), we get

$$|uyu_1| + |vyv_1| + |xyx_1| \leq 8k - 3m,$$

thus, by Corollary of Lemma 6, $|F| \leq 11k - 4m$, a contradiction to (9).

**Case 4b.** Assume that at least two of the numbers $i$, $j$, $q$ are greater than 1, say, $i \geq 2$, $j \geq 2$ (in the remaining cases we may proceed similarly). Now we introduce some notation. Let $W_3 = \{x, u, v, y\}$, and let $X_1$, $U_1$, and $V_1$ be the set of all vertices of $M$ adjacent only to $x$, only to $u$, and only to $v$ (and to no other vertices of $W_3$), respectively. Let further $XU$, $XV$, and $UV$ be the set of vertices adjacent only to $x$ and $u$, only to $x$ and $v$, and only to $u$ and $v$ (and to no other vertices of $W_3$), respectively. Finally, let $XUV$ be the set of vertices adjacent to $x$, $u$, and $v$ but not to $y$. Then

$$|X_1| + |U_1| + |V_1| + |XU| + |XV| + |UV| + |XUV| \geq m - k.$$  

The number of vertices adjacent to $x$ is

$$k \geq |XUV| + |XU| + |XV| + |X_1| + |xyx_1| + \cdots + |xyx_i|.$$  

The number of vertices adjacent to $u$ is

$$k \geq |XUV| + |XU| + |UV| + |U_1| + |uyu_1| + \cdots + |uyu_j|.$$  

Adding (18), (19) and (20) and using Lemma 6 gives

$$|V_1| \geq m - 3k + |XU| + |F| - (|vyv_1| + \cdots + |vyv_p|)$$

$$\geq m - 3k + |F| - (3k - m) = 2m - 6k + |F|.$$

However, all vertices of $V_1$ are adjacent to vertices $v$, $x_1$, $x_2$, $u_1$, $u_2$, thus $e(G) \geq 4m - |F| + (2m - 6k + |F|) = 6m - 6k$ which, by (1), contradicts (3).

We have seen that (3) leads to a contradiction in all cases, and so for any graph $G \in H_2(n, \lfloor pn \rfloor)$ we have

$$e(G) \geq 8n - 11\lfloor pn \rfloor - 8I\sqrt{n}.$$
3. Proof of the Theorem

Consider the graph $G_1$ consisting of

(a) nine vertices $a, b, c, d, e, f, g, h, i$ of high degrees;
(b) the edges $ab, ag, af, bg, bd, ce, cg, ch, df, di, eg, eh, fi, gi, gh$;
(c) triple-sets $acd, aef, abg, bcf, bed$;
(d) groups $abih, dfgh, cefg$ of vertices of degree 4 adjacent to vertices involved in these 4-tuples.

In case $3k - n$ is even, the cardinalities of these triple-sets and groups are (in other case proceed similarly):

\[ |acd| = |aef| = |bcf| = |bed| = (3k - n)/2, \]
\[ |abg| = 5k - 2n - 3, \]
\[ |abih| = 3n - 7k, \]
\[ |cefg| = |dfgh| = n - 2k - 3. \]

Then the vertices $a, b, c, d, e, f$ are of degree $k$, $g$ is of degree $k - 3$ and $\deg h = \deg i = 4n - 9k$ which is less than $k$ for $p > 2/5$ and $n$ sufficiently large. It is easy to check that $G_1 \in H_2(n, \lfloor pn \rfloor)$ for such $p$ and $n$ and $e(G_1) = 8n - 11k - 18$. Hence, by (21), if $G \in H_2(n, \lfloor pn \rfloor)$ where $p$ satisfies (1) and $n$ is sufficiently large, we get

\[ 8n - 11|pn| - 8I\sqrt{n} \leq F_2(n, \lfloor pn \rfloor) \leq 8n - 11|pn| - 18, \]

and the assertion of Theorem follows.

Remark. The structure of extremal graphs in a similar problem for graphs of diameter 3 was determined in [3]. The author hopes to find a characterization of extremal graphs in general for our case in a future paper. The “kernel system” of the extremal graph is a uniquely determined hypergraph in general.

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