ON SURJECTIVE KERNELS OF PARTIAL ALGEBRAS

P. ZLATOŠ

Abstract. A partial algebra $A = (A, F)$ is called surjective if each of its elements lies in the range of some of its operations. By a transfinite iteration construction over the class of all ordinals it is proved that in each partial algebra $A$ there exists the largest surjective subalgebra $\text{Skr} A$, called the surjective kernel of $A$. However, what might be found a bit surprising, for each ordinal $\alpha$ there is an algebra $A$ with only finitary operations (even with a single unary operation), such that the described construction stops exactly in $\alpha$ steps. The result is compared with the classical ones on perfect kernels of first countable topological spaces.

We use standard set-theoretical notation and terminology; in particular $Y^X$ denotes the set of all functions from the set $X$ into the set $Y$, each ordinal $\alpha$ is represented as the set of all ordinals $\beta < \alpha$, the least ordinal of cardinality $\aleph_\gamma$ is denoted by $\omega_\gamma$, and $\omega = \omega_0$.

Under the term "partial algebra" we will understand a pair $A = (A, F)$, where $A$ is an arbitrary set and $F$ is a set of partial (finitary or infinitary) operations on $A$ (we do not exclude any of the possibilities $A = \emptyset$ or $F = \emptyset$). For an operation $f \in F$ we denote by $\text{ar}(f)$ the arity and by $\text{D}(f)$ the domain of $f$. This is to say that to each $f \in F$ two sets $\text{ar}(f)$ (in most cases $\text{ar}(f)$ is assumed to be an ordinal) and $\text{D}(f) \subseteq A^{\text{ar}(f)}$ are assigned, such that $f : \text{D}(f) \to A$. A partial algebra $A = (A, F)$ will be called finitary if $\text{ar}(f)$ is finite for each $f \in F$. A will be called a total algebra, or simply an algebra if all its operations are total, i.e., $\text{D}(f) = A^{\text{ar}(f)}$ for each $f \in F$.

Any subset $B \subseteq A$ closed with respect to all operations $f \in F$, i.e. $f(b) \in B$ whenever $b \in \text{D}(f) \cap B^{\text{ar}(f)}$, will be called a subalgebra of $A$ and it will be identified with the corresponding partial algebra $B = (B, F_B)$, where $F_B = \{f_B : f \in F\}$ and $f_B$ denotes the restriction of $f$ to $B$, i.e. $\text{ar}(f_B) = \text{ar}(f)$, $\text{D}(f_B) = \text{D}(f) \cap B^{\text{ar}(f)}$ and $f_B(b) = f(b)$ for $b \in \text{D}(f_B)$. Obviously, every subalgebra of a total algebra is total, as well.

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Let $A = (A, F)$ be a partial algebra, $X \subseteq A$ and $H \subseteq F$. We put
\[ H[X] = \bigcup_{f \in H} f[X^\ast(f)] \]
\[ = \{ f(a) ; f \in H \land a \in D(f) \cap X^\ast(f) \}. \]

The partial algebra $A = (A, F)$ will be called surjective if $A = F[A]$. The largest surjective subalgebra of $A$ (we will prove that it always exists) will be called the **surjective kernel** of $A$ and denoted by $\text{Skr} A$.

Concerning the asymptotic behaviour of a (partial) finitary algebra $A$, it suffices to deal with its surjective kernel $\text{Skr} A$, as the remaining elements of $A$ do not matter at all. It is not our aim to make fully precise the intuitive meaning of the previous sentence in this short note. We expect that the article [Z], devoted to this topic and developing some ideas from [M–Z], will be submitted in the nearest future.

For every partial algebra $A = (A, F)$ and any subset $X \subseteq A$ one can construct a sequence of subsets $F^{(n)}[X] \subseteq A$ by recursion over the set $\omega$ of all natural numbers putting
\[ F^{(0)}[X] = X, \]
\[ F^{(n+1)}[X] = F[F^{(n)}[X]]. \]

If $B$ is a subalgebra of $A$, then obviously, $F^{(n+1)}[B] \subseteq F^{(n)}[B]$ holds for each $n$, all the sets $F^{(n)}[B]$ are subalgebras of $A$, and they are nonempty provided $B$ is. On the other hand, taking for $A$ the algebra with the underlying set $\omega$ and the successor operation, one can see that the intersection $\bigcap_{n<\omega} F^{(n)}[A]$ may well be empty.

But more surprising is (at least for the author was) the fact that the expected and offering “theorem,” asserting
\[ \text{Skr} A = \bigcap_{n<\omega} F^{(n)}[A], \]
is not true even for finitary algebras, as it will be shown within short.

This leads us to prolong the above sequence $F^{(n)}[B]$ over the class $\Omega$ of all ordinals by transfinite recursion. For every partial algebra $A = (A, F)$, any its subalgebra $B$, each ordinal $\alpha$ and each limit ordinal $\lambda > 0$ we put
\[ F^{(0)}[B] = B, \]
\[ F^{(\alpha+1)}[B] = F[F^{(\alpha)}[B]], \]
\[ F^{(\lambda)}[B] = \bigcap_{\beta<\lambda} F^{(\beta)}[B]. \]

We write $f^{(\alpha)}[B]$ instead of $\{ f \}^{(\alpha)}[B]$.

Again, each $F^{(\alpha)}[B]$ is a subalgebra of $A$ and $F^{(\beta)}[B] \subseteq F^{(\alpha)}[B]$ for all ordinals $\alpha \leq \beta$. Also, if $C$ is another subalgebra of $A$ and $B \subseteq C$, then $F^{(\alpha)}_B[B] \subseteq F^{(\alpha)}_C[C]$ holds for each $\alpha \in \Omega$. 
**Proposition.** For every partial algebra \( A = (A, F) \) there is an ordinal number \( \vartheta \) such that
\[
\Skr A = F^{(\vartheta)}[A] = \bigcap_{\alpha \in \Omega} F^{(\alpha)}[A].
\]

**Proof.** As \( A \) is a set, the sequence \( \{F^{(\alpha)}[A]\}_{\alpha \in \Omega} \) cannot be strictly decreasing. Let us denote \( \vartheta \) the least ordinal such that \( F^{(\vartheta)}[A] = F^{(\vartheta+1)}[A] \). Then obviously
\[
F^{(\vartheta)}[A] = \bigcap_{\alpha \in \Omega} F^{(\alpha)}[A],
\]
and it is a surjective subalgebra of \( A \). On the other hand, if \( B \) is any surjective subalgebra of \( A \), then for each ordinal \( \alpha \) we have \( B = F^{(\alpha)}[B] \). In particular,
\[
B = F^{(\vartheta)}[B] \subseteq F^{(\vartheta)}[A].
\]
Hence \( F^{(\vartheta)}[A] = \Skr A \) is the largest surjective subalgebra of \( A \). \( \square \)

The least ordinal \( \vartheta \) such that \( F^{(\vartheta)}[A] = F^{(\vartheta+1)}[A] \) will be called the **depth** of \( A \) and denoted by \( \vartheta_A \). Thus
\[
\Skr A = F^{(\vartheta_A)}[A].
\]

If \( A \) is finite, then obviously \( \vartheta_A \leq \text{card}(A) - 1 \). If \( \text{card}(A) = \aleph_\gamma \), say, then, as it will be shown during the proof of the next Theorem, one cannot prove more than the obvious inequality \( \vartheta_A < \omega_{\gamma+1} \).

Given a partial algebra \( A = (A, F) \), we will introduce the surjectivity rank function on \( A \) putting
\[
\text{rank}_A[x] = \begin{cases} 
\alpha & \text{if } x \in F^{(\alpha)}[A] \setminus F^{(\alpha+1)}[A], \\
\Omega & \text{if } x \in \Skr A
\end{cases}
\]
for \( x \in A \).

**Theorem.** For each ordinal \( \alpha \) there exists an algebra \( A = (A, f) \) with a single unary operation, such that \( \vartheta_A = \alpha \).

**Proof.** We will construct a sequence of algebras \( T_\alpha = (T_\alpha, f_\alpha) \) with a single unary operation by transfinite recursion over \( \Omega \). Each \( T_\alpha \) in fact will be a tree with finite branches only, and \( f_\alpha \) will be the tree-predecessor operation along the branches. Let \( t \) be any element, distinct from any finite sequence of ordinals (hence from all the nodes to be added during the construction); it will be the root of each of the trees \( T_\alpha \).

We start with the trivial tree, i.e.,
\[
T_0 = \{t\} \quad \text{and} \quad f_0(t) = t.
\]
Then for each $\alpha \in \Omega$ we put

$$T_{\alpha+1} = T_\alpha \cup \{\alpha\} \quad \text{and} \quad f_{\alpha+1}(x) = \begin{cases} f_\alpha(x), & \text{for } x \in T_\alpha, \ x \neq t \neq f_\alpha(x), \\ \alpha, & \text{for } x \in T_\alpha, \ x \neq t = f_\alpha(x), \\ t, & \text{for } x = \alpha \text{ or } x = t. \end{cases}$$

Thus $T_{\alpha+1}$ is the tree obtained by inserting a new node $\alpha$ between the root $t$ and the rest of the tree.

Further, for each limit ordinal $\lambda > 0$ we define

$$T_\lambda = \{t\} \cup \bigcup_{\alpha < \lambda} (T_\alpha \setminus \{t\}) \times \{\alpha\},$$

$$f_\lambda(t) = t \quad \text{and} \quad f_\lambda(x, \alpha) = \begin{cases} (f_\alpha(x), \alpha) & \text{if } f_\alpha(x) \neq t, \\ t & \text{if } f_\alpha(x) = t, \end{cases}$$

whenever $\alpha < \lambda$, $x \in T_\alpha$, $x \neq t$. In other words, $T_\lambda$ is the tree obtained by identifying the roots (but no other nodes) of all the preceding trees $T_\alpha$, $\alpha < \lambda$.

Some of the trees $T_\alpha$ are in the following picture:
Now, following the described construction, one can easily prove by transfinite induction that every tree $T_\alpha$ has indeed finite branches only, and that
\[ f^{(\alpha)}_\alpha[T_\alpha] = \{ t \} = f^{(\alpha+1)}_\alpha[T_\alpha] \quad \text{and} \quad \operatorname{rank}_{T_\alpha}[x] < \alpha \]
for each $\alpha \in \Omega$ and each $x \in T_\alpha$, $x \neq t$. Consequently
\[ \text{Skr}_{T_\alpha} = \{ t \} \quad \text{and} \quad \vartheta_{T_\alpha} = \alpha \]
for each $\alpha \in \Omega$.

Any first countable topological space $(M, \mathcal{O})$, in particular any metric space $(M, d)$, raises to a partial algebra $M = (M, \lim)$ where $\lim$ denotes the partial $\omega$-ary operation of taking limits of convergent sequences $a \in M^\omega$. It is clear that a set $C \subseteq M$ is a subalgebra of $M$ if and only if it is closed in the corresponding topological space $(M, \mathcal{O})$. However, the above construction applied to the partial algebra $M$, though it reminds of the Cantor's derivative, leads to trivial results, only. For each set $X \subseteq M$, $\lim[X]$ is namely the closure of $X$, and $\lim[C] = C$ for each closed set $C \subseteq M$. Nevertheless our algebraic construction can be modified to include the Cantor's derivative in the following way.

For every partial algebra $A = (A, F)$ and all $X \subseteq A$, $H \subseteq F$ we put
\[ H\langle X \rangle = \{ f(a); \ f \in H \ \& \ a \in D(f) \cap X^{\ar(f)} \ \& \ (\forall p \in \ar(f))(a(p) \neq f(a)) \}. \]

Obviously, $\lim(X)$ is the set of all accumulation points of the set $X \subseteq M$ in the first countable topological space $(M, \mathcal{O})$, or if you like, the Cantor's derivative of $X$.

Now, a partial algebra $A = (A, F)$ can be be called perfect if $A = F\langle A \rangle$.

Similarly as in the previous case, iterating this new construction for a given subalgebra $B$ of $A$, one can produce a transfinite sequence $\{ F^{(\alpha)}\langle B \rangle \}_{\alpha \in \Omega}$ of subalgebras of $A$, isote in $B$ and antitone in $\alpha$. Then one can show that for each partial algebra $A$ the sequence $\{ F^{(\alpha)}\langle A \rangle \}_{\alpha \in \Omega}$ stabilizes starting from some ordinal $\tau$ and the respective sequence item is the largest perfect subalgebra of $A$ which can be called the perfect kernel of $A$ and denoted by $\text{Pkr}_A$. Also the correspondent characteristic $\tau_A = \tau$, called the order of $A$, and the rank function $\operatorname{rank}_A\langle x \rangle$ on elements of $A$ can be introduced in the obvious way.

It is a well known result that for each second countable topological space $(M, \mathcal{O})$ it holds that $\tau_M < \omega_1$, and each ordinal $< \omega_1$ can occur. More generally, for each ordinal $\alpha$ there is a metric space $(M, d)$ satisfying $\tau_M = \alpha$. At a glance the fact that infinitely many iterations of the operation $\lim(X)$ are needed, seems to be caused by the $\omega$-arity of the partial operation $\lim$. However, introducing a slight modification of the algebras $T_\alpha$ we will show that this is not the reason.
Let $s$ be an arbitrary element not belonging to any of the sets $T_\alpha$. For every $\alpha \in \Omega$ we put
\[ S_\alpha = T_\alpha \cup \{s\} \quad \text{and} \quad g_\alpha(x) = \begin{cases} f_\alpha(x) & \text{if } t \neq x \in T_\alpha, \\ s & \text{if } x = t, \\ t & \text{if } x = s. \end{cases} \]

Now, substituting the algebras $S_\alpha = (S_\alpha, g_\alpha)$ in the places of the algebras $T_\alpha$ and inspecting once more the proof of the Theorem, one can find that
\[ g_\alpha^{(\beta)}(S_\alpha) = g_\alpha^{(\beta)}[S_\alpha] \]
holds for all $\alpha, \beta \in \Omega$. With this fact in mind it can be easily seen that
\[ g_\alpha^{(\alpha)}(S_\alpha) = \{s, t\} = g_\alpha^{(\alpha+1)}(S_\alpha) \quad \text{and} \quad \text{rank}_{S_\alpha}(x) < \alpha \]
for each $\alpha \in \Omega$ and each $x \in S_\alpha \setminus \{s, t\}$. Hence
\[ \text{Pkr } S_\alpha = \{s, t\} \quad \text{and} \quad \tau_{S_\alpha} = \alpha \]
for each $\alpha \in \Omega$. Thus also in this case every ordinal can occur as the order of a finitary algebra.

References


P. Zlatos, Department of Algebra and Number Theory, Faculty of Mathematics and Physics, Comenius University, 842 15 Bratislava, Czechoslovakia