

ON OSCILLATION OF LIMIT FUNCTIONS

J. BORSÍK

In the paper [1] the set of all continuity points of the limit function of a functional sequence by using the oscillation of these functions is investigated. In the present paper we investigate the oscillation of a limit function.

Let X be a topological space and let (Y, d) be a metric space. Let $f: X \rightarrow Y$ be a function. The function $\omega_f: X \rightarrow \mathbb{R} \cup \{\infty\}$ (\mathbb{R} is the set of all real numbers), given by the formula $\omega_f(x) = \inf\{d(f(U)): U \text{ is neighbourhood of } x\}$ (where $d(A) = \sup\{d(x, y): x, y \in A\}$) is said to be the oscillation of the function f . It is well-known that f is continuous at x if and only if $\omega_f(x) = 0$ ([2]). The symbol $C(f)$ denotes the set of all continuity points of f , the letters \mathbb{N} and \mathbb{Q} stand for the set of all natural and rational numbers, respectively.

Let $f_n, f: X \rightarrow Y$ ($n = 1, 2, \dots$) be functions. It is easy to see that if the sequence (f_n) uniformly converges to f on an open set G , then for each $x \in G$ we have

$$(1) \quad \lim_{n \rightarrow \infty} \omega_{f_n}(x) = \omega_f(x).$$

Therefore, if X is a locally compact topological space, then the uniform on compacta convergence implies (1) on X . In general the uniform on compacta convergence does not imply (1).

Example 1.1 in [1] shows that (1) is not true for pointwise or quasiuniform convergence. We recall that a sequence $(f_n), f_n: X \rightarrow Y$ quasiuniformly converges to $f: X \rightarrow Y$ (see [2]) if the sequence (f_n) pointwise converges to f and

$$\forall \varepsilon > 0 \forall m \in \mathbb{N} \exists p \in \mathbb{N} \forall x \in X : \\ \min\{d(f_{m+1}(x), f(x)), \dots, d(f_{m+p}(x), f(x))\} < \varepsilon.$$

We shall show that for the quasiuniform convergence we have

$$\{x \in X : \limsup_{n \rightarrow \infty} \omega_{f_n}(x) = 0\} \subset C(f).$$

As corollary we obtain that the quasiuniform limit of continuous functions is continuous function.

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Theorem 1. *Let X be a topological space and let (Y, d) be a metric space. Let $f, f_n: X \rightarrow Y$ ($n = 1, 2, \dots$) and let (f_n) converges quasiuniformly to f . Then for each $x \in X$ we have*

$$(2) \quad \omega_f(x) \leq 2 \limsup_{n \rightarrow \infty} \omega_{f_n}(x).$$

Proof. Suppose that there is $x \in X$ such that $\omega_f(x) > 2 \limsup_{n \rightarrow \infty} \omega_{f_n}(x)$. Then there are α, β such that

$$(3) \quad 2 \limsup_{n \rightarrow \infty} \omega_{f_n}(x) < \alpha < \beta < \omega_f(x).$$

Since $\limsup_{n \rightarrow \infty} \omega_{f_n}(x) < \frac{\alpha}{2}$, there is $n_1 \in \mathbb{N}$ such that for each $n \geq n_1$ we have $\omega_{f_n} < \frac{\alpha}{2}$. Then for each $n \geq n_1$ there is a neighbourhood U_n of x such that

$$(4) \quad d(f_n(U_n)) < \frac{\alpha}{2}.$$

Since (f_n) pointwise converges to f , there is $n_2 \in \mathbb{N}$ such that for $n \geq n_2$ we have

$$(5) \quad d(f_n(x), f(x)) < \frac{\beta - \alpha}{4}.$$

Denote $m = \max\{n_1, n_2\}$. Then there is $p \in \mathbb{N}$ such that for each $y \in X$ we have

$$\min\{d(f_{m+1}(y), f(y)), \dots, d(f_{m+p}(y), f(y))\} < \frac{\beta - \alpha}{4}.$$

Denote $U = \bigcap_{i=1}^p U_{m+i}$. Then U is a neighbourhood of x . Let $z \in U$. Then there is $j \in \{1, 2, \dots, p\}$ such that

$$(6) \quad d(f_{m+j}(z), f(z)) < \frac{\beta - \alpha}{4}.$$

Then according to (4), (5) and (6) we obtain

$$\begin{aligned} d(f(z), f(x)) &\leq d(f(z), f_{m+j}(z)) + d(f_{m+j}(z), f_{m+j}(x)) + d(f_{m+j}(x), f(x)) \\ &< \frac{\beta - \alpha}{4} + \frac{\alpha}{2} + \frac{\beta - \alpha}{4} = \frac{\beta}{2}. \end{aligned}$$

Therefore, for $s, t \in U$ we have

$$d(f(s), f(t)) \leq d(f(s), f(x)) + d(f(x), f(t)) < \frac{\beta}{2} + \frac{\beta}{2} = \beta.$$

From this we get $\omega_f(x) < \beta$, which contradicts to [3]. □

Evidently, [2] is not true for the pointwise convergence. However, we have

Theorem 2. *Let X be a Baire space and let (Y, d) be a separable metric space. Let $f_n, f: X \rightarrow Y$ ($n = 1, 2, \dots$) be functions and let (f_n) converge pointwise to f . Let the function f be locally bounded and let $M > 2$. Then $\{x \in X: \omega_f(x) \leq M \cdot \liminf_{n \rightarrow \infty} \omega_{f_n}(x)\}$ is dense in X .*

If (Y, d) is an arbitrary metric space then $\{x \in X: \omega_f(x) \leq M \cdot \liminf_{n \rightarrow \infty} \omega_{f_n}(x)\}$ is dense in X for each $M > 3$.

First we shall prove the following

Lemma 1. *Let X be a Baire space and let (Y, d) be a separable metric space ((Y, d) be arbitrary metric space). Let G be an open set in X , let $M > 2$ ($M > 3$) and let $0 < S < \infty$. Let $f_n, f: X \rightarrow Y$ ($n = 1, 2, \dots$) be functions and let $\lim_{n \rightarrow \infty} f_n = f$. Let $\liminf_{n \rightarrow \infty} \omega_{f_n}(x) \leq S$ for each $x \in G$. Then $\{x \in G: \omega_f(x) \geq M \cdot S\}$ is a nowhere dense set in X .*

Proof. Denote $A = \{x \in G: \omega_f(x) \geq M \cdot S\}$. Suppose that A is not nowhere dense in X . Then there is a nonempty open set $H \subset G$ such that A is dense in H . We shall show that $H \subset A$. Let $z \in H - A$. Then $\omega_f(z) < MS$. Since ω_f is upper semi-continuous ([2]), there is a neighbourhood U of z such that $\omega_f(x) < MS$ for each $x \in U$, a contradiction with the density of A . Hence

$$(7) \quad \forall x \in H: \omega_f(x) \geq M \cdot S.$$

I. (Y, d) is a separable metric space and $M > 2$.

Let $\{u_1, u_2, u_3, \dots\}$ be a countable dense subset of Y . Then $Y = \bigcup_{n=1}^{\infty} S(u_n, \frac{S}{24}(M-2))$ (where $S(u, \varepsilon)$ is the open sphere of radius $\varepsilon > 0$ about u). Since X is a Baire space, there is $j \in \mathbb{N}$ such that the set $H \cap f^{-1}(S(u_j, \frac{S}{24}(M-2)))$ is not of the first category. Denote

$$B = H \cap f^{-1}\left(S(u_j, \frac{S}{24}(M-2))\right),$$

$$D = H \cap f^{-1}\left(S(u_j, \frac{S}{4}(M+2))\right),$$

$$A_k = \left\{x \in H: \forall n \geq k; d(f_n(x), f(x)) < \frac{S}{24}(M-2)\right\}$$

for $k \in \mathbb{N}$.

Then evidently $B \subset D$, $A_k \subset A_{k+1}$ for each $k \in \mathbb{N}$ and $H = \bigcup_{k=1}^{\infty} A_k$. Then there is $i \in \mathbb{N}$ such that $B \cap A_i$ is not nowhere dense. Therefore there is a nonempty open set $J \subset H$ such that $B \cap A_i$ is dense in J . Then $B \cap A_n$ is dense in J for each $n \geq i$.

Now we shall prove that $J - D$ is a nonempty set. If namely $J - D = \emptyset$, then $J \subset D$ and hence $f(J) \subset S(u_j, \frac{S}{4}(M+2))$. From this we have $d(f(J)) \leq \frac{S}{2}(M+2)$ and $\omega_f(x) \leq \frac{S}{2}(M+2) < MS$ for each $x \in J$, a contradiction with (7).

Let $z \in J - D$. Let $p \in \mathbb{N}$ be such that $z \in A_p$. Since $\liminf_{n \rightarrow \infty} \omega_{f_n}(z) \leq S < \frac{S}{8}(M+6)$, there is $m \geq \max\{i, p\}$ such that

$$(8) \quad \omega_{f_m}(z) < \frac{S}{8}(M+6).$$

Let U be arbitrary neighbourhood of z . Then there is $v \in B \cap A_m \cap U \cap J$. We have

$$\begin{aligned} \frac{S}{4}(M+2) &\leq d(f(z), u_j) \\ &\leq d(f(z), f_m(z)) + d(f_m(z), f_m(v)) + d(f_m(v), f(v)) + d(f(v), u_j) \\ &< \frac{S}{24}(M-2) + d(f_m(z), f_m(v)) + \frac{S}{24}(M-2) + \frac{S}{24}(M-2). \end{aligned}$$

From this we get $d(f_m(z), f_m(v)) > \frac{S}{8}(M+6)$ and hence $d(f_m(U)) > \frac{S}{8}(M+6)$. Since this is true for each neighbourhood of z , we have $\omega_{f_m}(z) \geq \frac{S}{8}(M+6)$, a contradiction with (8).

II. (Y, d) is arbitrary metric space and $M > 3$.

Denote $A_k = \{x \in H : \forall n \geq k; d(f_n(x), f(x)) < \frac{S}{24}(M-3)\}$ for $k \in \mathbb{N}$. Then there is $i \in \mathbb{N}$ such that A_i is not nowhere dense. Therefore there is a nonempty open set $J \subset H$ such that A_i is dense in J .

Let $x \in J$. Let $p \in \mathbb{N}$ be such that $x \in A_p$. Since $\liminf_{n \rightarrow \infty} \omega_{f_n}(x) \leq S < \frac{S}{12}(M+9)$, there is $m \geq \max\{i, p\}$ such that $\omega_{f_m}(x) < \frac{S}{12}(M+9)$. Therefore there is a neighbourhood U_x of x such that $d(f_m(U_x)) < \frac{S}{12}(M+9)$.

Let $y \in U_x \cap A_i$. Then

$$(9) \quad \begin{aligned} d(f(y), f(x)) &\leq d(f(y), f_m(y)) + d(f_m(y), f_m(x)) + d(f_m(x), f(x)) \\ &< \frac{S}{24}(M-3) + \frac{S}{12}(M+9) + \frac{S}{24}(M-3) = \frac{S}{6}(M+3). \end{aligned}$$

Let $y, z \in U_x \cap A_i$. Then similarly

$$(10) \quad d(f(y), f(z)) < \frac{S}{6}(M+3).$$

Now let $r \in J$. Let $u, v \in U_r \cap J$. Then there are $s \in U_r \cap U_u \cap A_i$ and $t \in U_r \cap U_v \cap A_i$. According to (9) and (10) we have

$$d(f(u), f(v)) \leq d(f(u), f(s)) + d(f(s), f(t)) + d(f(t), f(v)) < \frac{S}{2}(M+3).$$

Therefore $d(f(U_r \cap J)) \leq \frac{S}{2}(M + 3)$ and hence $\omega_f(r) < M \cdot S$, a contradiction with (7). \square

Proof of Theorem 2. Since f is locally bounded, we have $\omega_f(x) < \infty$ for each $x \in X$. Suppose that this theorem is not true. Then there is an open nonempty set G in X such that

$$(11) \quad \forall x \in G: \infty > \omega_f(x) > M \cdot \liminf_{n \rightarrow \infty} \omega_{f_n}(x).$$

Define functions $h, g: G \rightarrow \mathbb{R}$ as

$$h(x) = \liminf_{n \rightarrow \infty} \omega_{f_n}(x),$$

$$g(x) = \inf\{\sup\{h(y) : y \in U\} : U \text{ is a neighbourhood of } x\}.$$

Then g is a nonnegative upper semi-continuous function. Let K be such that $M > K > 2$ ($M > K > 3$ if (Y, d) is not separable). Then we have $h(x) < \frac{1}{M}\omega_f(x) < \frac{1}{K}\omega_f(x)$ for each $x \in G$. We observe that

$$g(x) \leq \frac{1}{M}\omega_f(x) < \frac{1}{K}\omega_f(x).$$

Since X is a Baire space, so there is $z \in G \cap C(\omega_f)$. Let α be such that $g(z) < \alpha < \frac{1}{K}\omega_f(z)$. Then there is an open neighbourhood U of z such that

$$(12) \quad h(x) \leq g(x) < \alpha < \frac{1}{K}\omega_f(x) \quad \text{for each } x \in U.$$

Hence according to Lemma 1 the set $\{x \in U : \omega_f(x) \geq K\alpha\}$ is nowhere dense, a contradiction with (12). \square

The following example shows that the assumption “ f is locally bounded” cannot be omitted.

Example 1. Let $X = Y = \mathbb{R}$, let $\mathbb{Q} = \{q_1, q_2, q_3, \dots\}$ (one-to-one sequence). Define functions $f_n, f: \mathbb{R} \rightarrow \mathbb{R}$ as

$$f_n(x) = \begin{cases} k, & \text{if } x = q_k \text{ and } k \leq n, \\ 0, & \text{otherwise;} \end{cases}$$

$$f(x) = \begin{cases} k, & \text{if } x = q_k, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\lim_{n \rightarrow \infty} f_n = f$, $\omega_f(x) = \infty$ for each $x \in X$ and $\liminf_{n \rightarrow \infty} \omega_{f_n}(x) = f(x)$ for each $x \in X$.

Next example shows that Theorem 2 (and Lemma 1) does not hold for $M = 2$.

Example 2. Let $X = Y = \mathbb{R}$, $\mathbb{Q} = \{q_1, q_2, \dots\}$ (one-to-one sequence) and $\mathbb{Q} = A \cup B$, where A and B are dense disjoint sets. Define $f_n, f: \mathbb{R} \rightarrow \mathbb{R}$ as

$$f_n(x) = \begin{cases} 1 - \frac{1}{k}, & \text{if } x = q_k, x \in A \text{ and } k \leq n, \\ \frac{1}{k} - 1, & \text{if } x = q_k, x \in B \text{ and } k \leq n, \\ 0, & \text{otherwise;} \end{cases}$$

$$f(x) = \begin{cases} 1 - \frac{1}{k}, & \text{if } x = q_k \text{ and } x \in A, \\ \frac{1}{k} - 1, & \text{if } x = q_k \text{ and } x \in B, \\ 0, & \text{otherwise.} \end{cases}$$

Then $\lim_{n \rightarrow \infty} f_n = f$, $\omega_f(x) = 2$ for each $x \in X$ and

$$\liminf_{n \rightarrow \infty} \omega_{f_n}(x) = \begin{cases} 1 - \frac{1}{k}, & \text{if } x = q_k, \\ 0, & \text{otherwise.} \end{cases}$$

This example shows also that the set $\{x \in X : \omega_f(x) \leq M \cdot \liminf_{n \rightarrow \infty} \omega_{f_n}(x)\}$ (where $M > 2$) may be not residual.

Since every function $f: \mathbb{Q} \rightarrow \mathbb{R}$ is in the first Baire class the assumption on X to be a Baire space (in Theorem 2) cannot be omitted. Next example shows that Theorem 2 does not hold for arbitrary metric space (Y, d) with $M > 2$.

Example 3. Let $\{B_n: n \in \mathbb{N}\}$ be a countable base in \mathbb{R} . We choose two different points $a_1, b_1 \in B_1$ and for $n > 1$ we choose two different points $a_n, b_n \in B_n - \{a_1, b_1, \dots, a_{n-1}, b_{n-1}\}$. Denote $P = \mathbb{R} - \{a_1, b_1, \dots, a_n, b_n, \dots\}$.

Let $X = \mathbb{R}$ with the usual topology and let $Y = \mathbb{R}$ with the following metric d :

$$d(y, x) = d(x, y) = \begin{cases} 0, & \text{if } x = y, \\ 1, & \text{if } x = a_n, y \in P \text{ and } |y - a_n| \leq |y - b_n|, \\ & \text{or } x = b_n, y \in P \text{ and } |y - b_n| < |y - a_n|, \\ & \text{or } x, y \in P, x \neq y, \\ 3, & \text{if } x = a_n, y = b_n, \\ 2, & \text{otherwise.} \end{cases}$$

Further denote for $n \in \mathbb{N}$

$$U_n = \left(a_n - \frac{|a_n - b_n|}{2}, a_n + \frac{|a_n - b_n|}{2}\right),$$

$$V_n = \left(b_n - \frac{|a_n - b_n|}{2}, b_n + \frac{|a_n - b_n|}{2}\right),$$

and for $j > k$ denote

$$D_k^j = \begin{cases} U_k, & \text{if } a_j \in U_k, \\ V_k, & \text{if } a_j \in V_k, \\ \mathbb{R}, & \text{if } a_j \notin U_k \cup V_k; \end{cases}$$

$$C_k^j = \begin{cases} U_k, & \text{if } b_j \in U_k, \\ V_k, & \text{if } b_j \in V_k, \\ \mathbb{R}, & \text{if } b_j \notin U_k \cup V_k. \end{cases}$$

Now for $j > n$ choose $p_n^j \in P \cap D_1^j \cap D_2^j \cap \dots \cap D_n^j$ and $q_n^j \in P \cap C_1^j \cap C_2^j \cap \dots \cap C_n^j$.

Define $f_n, f: X \rightarrow Y$ as

$$f_n(x) = \begin{cases} x, & \text{if } x \in P \cup \{a_1, b_1, \dots, a_n, b_n\}, \\ p_n^j, & \text{if } x = a_j \text{ and } j > n, \\ q_n^j, & \text{if } x = b_j \text{ and } j > n; \end{cases}$$

$$f(x) = x \quad \text{for each } x \in X.$$

Then for each $x \in X$ we have $\lim_{n \rightarrow \infty} f_n(x) = f(x)$, $\omega_f(x) = 3$ and $\liminf_{n \rightarrow \infty} \omega_{f_n}(x) = 1$.

References

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J. Borsík, Matematický ústav SAV, Grešákova 6, 040 01 Košice, Czechoslovakia