Acta Math. Univ. Comenianae
Vol. LX, 1(1991), pp. 19–21

ACCRETIVE METRIC PROJECTIONS

L. VESELÝ

Abstract. In this note we prove that all metric projections onto closed subsets of a normed linear space X are accretive if and only if X is an inner-product space. Instead of all closed sets it suffices to consider more special classes of sets in X.

Introduction. Let X be a real normed linear space and let 2^X be the set of all its subsets. A multivalued mapping $A\colon X \to 2^X$ is termed **accretive** if $\|x-y+t(\bar{x}-\bar{y})\| \geq \|x-y\|$ whenever $t > 0, \bar{x} \in A(x), \bar{y} \in A(y)$. Accretive mappings have been intensively studied in connections with semi-groups of nonexpansive mappings and with differential equations and inclusions in Banach spaces. In Hilbert spaces, accretive operators coincide with monotone operators. We refer the reader to [**3**], [**4**] for basic facts about accretive operators and their applications.

For a set $F \subset X$ we define $P_F(x) = \{\bar{x} \in F : \|x - \bar{x}\| = \operatorname{dist}(x, F)\}$. The mapping $P_F\colon X \to 2^F \subset 2^X$ is called **metric projection onto** F. We put $P_F^{-1}(y) = \{x \in X : y \in P_F(x)\}$ for any $y \in X$.

If X is an inner product space, it is easy to prove that both P_F and P_F^{-1} are accretive for any $F \subset X$. It is natural to ask whether this property extends to more general spaces. H. Berens and U. Westphal [**2**] proved that the accretivity of all P_F^{-1} is equivalent to the existence of an inner product generating the norm of X. Our aim is to prove that a similar situation appears for metric projections themselves. Clearly we can confine ourselves to metric projections onto closed sets, since $P_{\bar{F}}(x) \supset P_F(x)$ for any $x \in X$ and any $F \subset X$. We shall show that it is possible to consider all two-points sets or, if $dim(X) \geq 3$, all lines only.

Results. We need two well-known characterizations of inner product spaces in terms of orthogonality. For $x, y \in X$ let us write

$x \# y$ if $\|x + y\| = \|x - y\|$ (**James orthogonality**), and
$x \perp y$ if $\|x + ty\| \geq \|x\|$ for any $t \in \mathbb{R}$ (**Birkhoff orthogonality**).

Theorem 1 (cf. [**1**, (4.1) and (12.11)]).
(a) *If the implication $x \# y \Longrightarrow x \perp y$ holds in X, then X is an inner-product space.*

Received December 12, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 41A65, 47H06; Secondary 46B99, 46C10.

(b) *If* $dim(X) \geq 3$ *and the Birkhoff orthogonality is left-additive (i.e.* $(x+y) \perp v$ *whenever* $x \perp y$ *and* $y \perp v$*), then* X *is an inner-product space.*

Theorem 2. *For a normed linear space* X *the following assertions are equivalent:*

(i) X *is an inner product space.*
(ii) P_F *is accretive for any closed* $F \subset X$.
(iii) P_Q *is accretive for any* $Q = \{a, b\} \subset X$.

Proof. (i) $\Longrightarrow$ (ii). Let $\bar{x} \in P_F(x)$, $\bar{y} \in P_F(y)$, $t > 0$. Then $\|x - \bar{x}\| \leq \|x - \bar{y}\|$ and $\|y - \bar{y}\| \leq \|y - \bar{x}\|$. Hence $\|x - y + t(\bar{x} - \bar{y})\|^2 \geq \|x - y\|^2 + 2t<x - y, \bar{x} - \bar{y}> = \|x - y\|^2 + t(\|x - \bar{y}\|^2 - \|x - \bar{x}\|^2 + \|y - \bar{x}\|^2 - \|y - \bar{y}\|^2) \geq \|x - y\|^2$. The implication (ii) $\Longrightarrow$ (iii) is obvious.

We shall use Theorem 1(a) for the proof of (iii) $\Longrightarrow$ (i). Let $x, y \in X$, $x \# y, Q = \{-y, y\}$. Then $P_Q(x) = P_Q(0) = Q$. For any $t > 0$ the defintion of accretivity implies $\|x - 2ty\| \geq \|x\|$ (because $-y \in P_Q(x)$ and $y \in P_Q(0)$) and $\|x + 2ty\| \geq \|x\|$ (because $y \in P_Q(x)$ and $-y \in P_Q(O)$). Hence $x \perp y$ and the proof is complete. □

Now let us consider various classes of convex sets. We begin with hyperplanes.

Theorem 3. *For a normed linear space* X *the following two assertions are equivalent:*

(i) X *is strictly convex (i.e. the unit sphere does not contain any nontrivial line segment).*
(ii) P_H *is accretive for any closed hyperplane* $H \subset X$.

Proof. (i) $\Longrightarrow$ (ii). Let X be strictly convex and $H \subset X$ be a closed hyperplane containing the origin. Then either H is a Chebyshev hyperplane or $P_H(x) = \emptyset$ for all $x \in X \setminus H$, and in both cases P_H is singlevalued and linear on $D(P_H) = \{x \in X | P_H(x) \neq \emptyset\}$, [**5**]. For any $x, y \in D(P_H)$ and any $t > 0$ we have

$$\|x - y + t(P_H(x) - P_H(y))\| = \|x - y + tP_H(x - y)\|$$
$$\geq (1+t)\|x-y\| - t\|(x-y) - P_H(x-y)\| \geq (1+t)\|x-y\| - t\|(x-y) - 0\| = \|x-y\|.$$

Consequently P_H is accretive.
(ii)$\Longrightarrow$(i). Let $x, v \in X$ be such that $\|x\| = \|x - v\| = \|x + v\| = 1$. Take a nonzero functional $f \in X^*$ such that $f(x) = \|f\|$ and denote $H = f^{-1}(0)$. Then $v \in P_H(x)$ and $0 \in P_H(x+v)$. Consequently $\|v\| = \|(x+v) - x\| \leq \|(x+v) - x + (0 - v)\| = 0$, since P_H is accretive by (ii). This implies (i). □

Corollary. *Let* $\dim(X) = 2$. *Then the following are equivalent:*

(i) X *is strictly convex.*
(ii) P_M *is accretive for any subspace* $M \subset X$.

The following theorem shows that for spaces of dimension greater than 2 the Corollary does not hold.

Theorem 4. *Let X be a normed linear space with* $\dim(X) \geq 3$. *Then the following assertions are equivalent:*

(i) X *is an inner-product space.*
(ii) P_C *is accretive for any closed convex* $C \subset X$.
(iii) P_M *is accretive for any closed subspace* $M \subset X$.
(iv) P_L *is accretive for any 1-dimensional subspace* $L \subset X$.

Proof. (i) $\Longrightarrow$ (ii) follows from Theorem 2. The implications (ii) $\Longrightarrow$(iii)$\Longrightarrow$(iv) are obvious. We shall prove (iv) $\Longrightarrow$(i) using Theorem 1(b).
Let $x, y, v \in X, x \perp v$ and $y \perp v$. If $v = 0$ then $(x + y) \perp v$ holds trivially. Let $v \neq 0$, $L = \operatorname{span}\{v\}$. Then the definition of the Birkhoff orthogonality implies $0 \in P_L(-x)$ and $tv \in P_L(y + tv)$ for any $t \in \mathbb{R}$. The accretivity of P_L implies $\|y + tv + x\| \leq \|y + tv + x + stv\|$ for any $t \in \mathbb{R}$ and $s > 0$. Introducing the substitution $r = st$ we get

$$\|y + (r/s)v + x\| \leq \|y + (r/s)v + x + rv\| \qquad \text{whenever } r \in \mathbb{R},\ s > 0.$$

After passing $s \to \infty$ we obtain $(x + y) \perp v$ and the proof is complete by Theorem 1(b). □

References

1. Amir D., *Characterizations of Inner Product Spaces*, Operator Theory: Advances and Appl. 20, Birkhäuser Basel, 1986.
2. Berens H. and Westphal U., *Kodissipative metrische Projektionen in normierten linearen Räumen*, Linear Spaces and Approximation (P. L. Butzer and B. Sz.-Nagy, eds.), ISNM 40, Birkhäuser Basel, 1978, pp. 120–130.
3. Brézis H., *Opérateurs Maximaux Monotones et Semi-groupes de Contractions dans les Espaces de Hilbert*, Math. Studies 5, North-Holland, Amsterdam, 1973.
4. Deimling K., *Nonlinear Functional Analysis*, Springer-Verlag, 1985.
5. Holmes R. B., *A Course on Optimization and Best Approximation*, Lecture Notes in Math. 257, Springer-Verlag, 1972.

L. Veselý, Charles University, Faculty of Math. and Phys., KMA, Sokolovská 83, 186 00 Praha 8, Czechoslovakia