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Cyclic quadrangles from squares

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Abstract

In this paper we show how to use computers to discover appearence of cyclic quadrangles in geometric configurations based on a fixed square $ABCD$ and a variable point P in the plane. The idea is to consider various central points (like the orthocenters) of the four triangles ABP, BCP, CDP and DAP or their orthogonal projections to the lines AP, BP, CP and DP. This is done in Maple V by describing basic functions for the analytic plane geometry and applying them to these configurations. The figures are realized in The Geometer's Sketchpad, Mathematica, and Maple V.

Keywords: square, triangle, orthocenter, circumcenter, area, Steiner point, cyclic quadrangle

MSC: 51N20, 51M04, 14A25,14Q05

1. Introduction

Consider in the plane a positively oriented (in the counterclockwise sense) square $ABCD$ with the center O and a point P which is not on any of the four lines AB, BC, CD and DA. Let H_a, H_b, H_c and H_d be the orthocenters (i.e., the intersections of altitudes) of the triangles ABP, BCP, CDP and DAP, respectively. Let J_a, J_b, J_c and J_d denote the orthogonal projections of H_a, H_b, H_c and H_d onto the lines AP, BP, CP and DP, respectively. (See Figures 1 and 2.)

In this paper we want to show how one can use computers to explore properties of the quadrangles $H_aH_bH_cH_d$ and $J_aJ_bJ_cJ_d$. The first property of the quadrangle $H_aH_bH_cH_d$ that its diagonals H_aH_c and H_bH_d are perpendicular is obvious because points H_a and H_c are on the perpendicular through P onto lines AB and CD while H_b and H_d are on the perpendicular through P onto lines BC and DA.

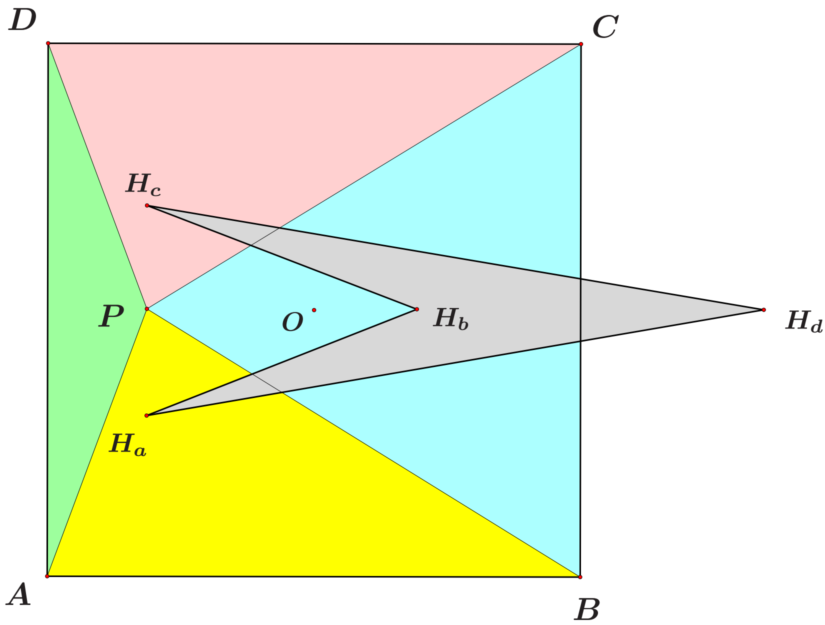

Figure 1: The quadrangle $H_aH_bH_cH_d$ from orthocenters.

The second property is more difficult to establish. We shall do this later using analytic geometry. Ever since Decartes this is the most simple and the most effective method to transfer geometric problems into algebraic setting. The reduction usually leads to equations whose solutions give answers. Since the software for symbolic computation (like Derive, Maple V and Mathematica) excels in solving equations, in this way we get the possibility to use computers in our explorations.

Property 2. *The points O, P, J_a, J_b, J_c and J_d lie on a circle. In particular, the quadrangle $J_aJ_bJ_cJ_d$ is cyclic.*

We can say more about the circle m which appears in the Property 2. It is the circumcircle of the negatively oriented square $PONM$ built on the segment $\overline{PO}$. Hence, if $|PO| = \delta$, then the radius of the circle m is $\frac{\delta\sqrt{2}}{2}$ (see Figure 2).

The third property describes the following surprising connection of the quadrangles $H_aH_bH_cH_d$ and $J_aJ_bJ_cJ_d$ (see Figure 3).

Property 3. *The lines H_aJ_a, H_bJ_b, H_cJ_c and H_dJ_d intersect in the point N and go through the points B, C, D and A, respectively.*

Now one can wonder when is the quadrangle $H_aH_bH_cH_d$ cyclic and when will the quadrangle $J_aJ_bJ_cJ_d$ have perpendicular diagonals J_aJ_c and J_bJ_d. The answers give the following two theorems.

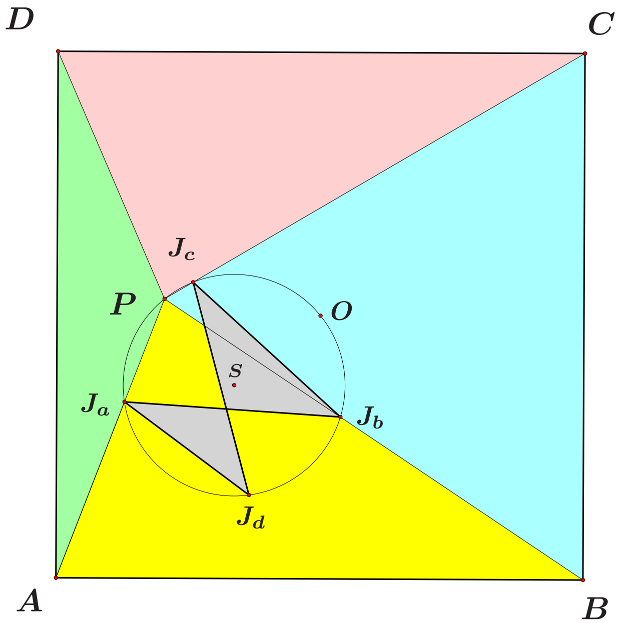

Figure 2: The cyclic quadrangle $J_aJ_bJ_cJ_d$.

Theorem 1.1. *The quadrangle $H_aH_bH_cH_d$ is cyclic if and only if the point P is either on the line AC, on the line BD or on the circumcircle k of the square $ABCD$.*

Theorem 1.2. *In the quadrangle $J_aJ_bJ_cJ_d$ the lines J_aJ_c and J_bJ_d are perpendicular if and only if the point P is either on the line AC, on the line BD or on the circumcircle k of the square $ABCD$.*

More precisely, $H_a = H_c$ and/or $H_b = H_d$ if and only if P is either on the line AC or on the line BD. Hence, the first two parts of the locus from Theorems 1.1 and 1.2 correspond to the case when the quadrangle $H_aH_bH_cH_d$ degenerates to a segment or a point and either $J_a = J_c$ or $J_b = J_d$. The role of the third part (the circumcircle k) is explained better by the following statement: If P is on the circumcircle k, then

(a) $H_aH_bH_cH_d$ is a square of side equal to the diagonals of $ABCD$ with P as the center whose diagonals H_aH_c and H_bH_d are parallel to the lines BC and AB,

(b) $J_aJ_bJ_cJ_d$ is also a square of side equal to the half of the diagonals of $ABCD$,

(c) $J_aJ_bJ_cJ_d$ and $H_aH_bH_cH_d$ are related by the homothety $h(N, 2)$ where N is the vertex of the square on the segment PO (see Figure 4).

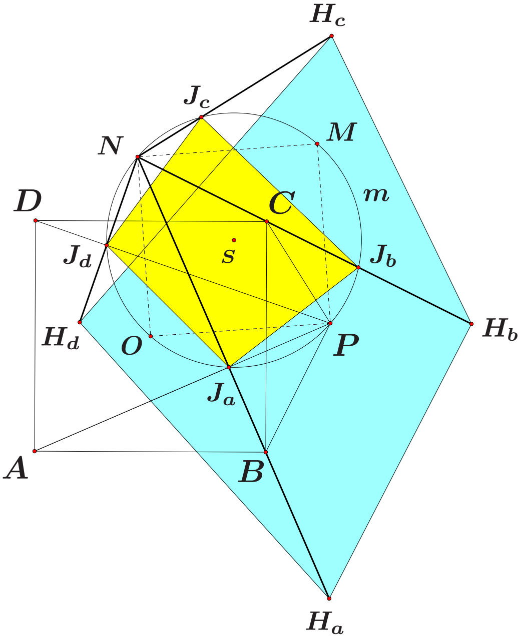

Figure 3: The lines H_aJ_a, H_bJ_b, H_cJ_c and H_dJ_d concur in the point N.

An interesting path is to explore how the areas of the quadrangles $H_aH_bH_cH_d$ and $J_aJ_bJ_cJ_d$ compare to the area Ω of the square $ABCD$. We define the area $|WXYZ|$ of the quadrangle $WXYZ$ as the sum $|WXY| + |WYZ|$ of (oriented) areas of the triangles WXY and WYZ.

For every real number m let P_m and Q_m denote the loci of all points P such that $|H_aH_bH_cH_d| = m\,\Omega$ and $|J_aJ_bJ_cJ_d| = m\,\Omega$, respectively.

Theorem 1.3. *For $m < 0$, $m = 0$, $0 < m < 2$, $m = 2$ and $m > 2$ the set P_m is the union of two hyperbolas, the union of lines AC and BD, the empty set, the circumcircle k of the square $ABCD$, and the union of two ellipsis, respectively. If the axes of one conic are φ and ψ then the axes of the other are ψ and φ.*

The Figures 5 and 6 show the sets P_m for $m = -1$ and $m = 3$ and the set $Q_{\frac{1}{2}}$ together with the square $ABCD$. Notice that $Q_{\frac{1}{2}}$ is the union of the circumcircle k of the square $ABCD$ and a symmetric curve of order six that touches k in the vertices of the square.

Let us conclude this description of our results with some comments on what else one can do with this approach. Instead of orthocenters we can consider other central points of the triangle (like the centroid, the circumcenter, the center of the nine-point circle – see the references [3] and [4] for the list of more than thou-

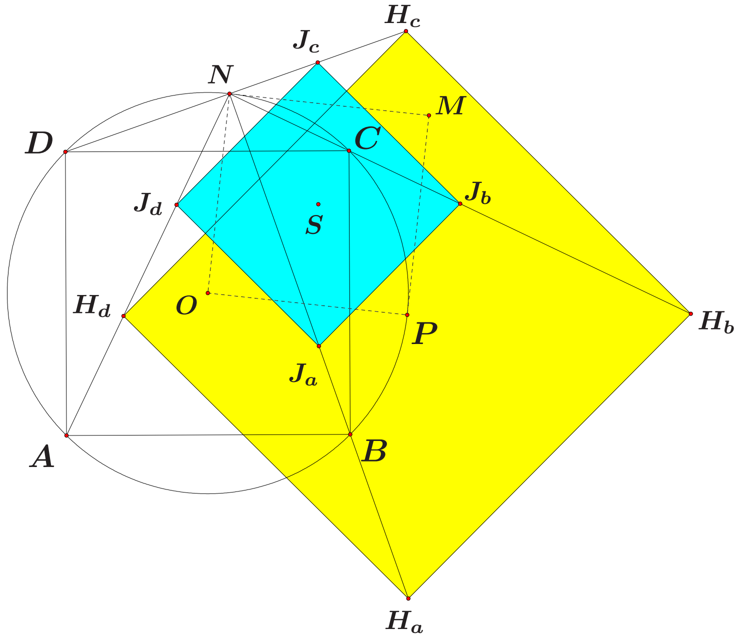

Figure 4: The squares $H_aH_bH_cH_d$ and $J_aJ_bJ_cJ_d$ when the point P is on the circumcircle of $ABCD$.

sand such points). For example, we have the following analogue of Property 2 for circumcenters.

Let O_a, O_b, O_c and O_d be the circumcenters of the triangles ABP, BCP, CDP and DAP, respectively. Let N_a, N_b, N_c and N_d denote the orthogonal projections of O_a, O_b, O_c and O_d onto the lines AP, BP, CP and DP, respectively.

Property 4. *The points N_a, N_b, N_c and N_d are vertices of the square which is related to the square $ABCD$ by the homothety $h(P, \frac{1}{2})$.*

A computer search reveals that the Steiner point gives the following result also similar to the Property 2.

Recall that the Steiner point is denoted as $X(99)$ in [3] and on page 120 of [1] it is noted that the Steiner point of a triangle is the center of mass of the system obtained by suspending at each vertex a mass equal to the magnitude of the exterior angle at that vertex. It is also the intersection of the circumcircle with the Steiner ellipse and the point of concurrency of parallels through the vertices to the corresponding sides of its first Brocard triangle (see [2]).

Let S_a, S_b, S_c and S_d be the Steiner points of the triangles ABP, BCP, CDP and DAP, respectively.

Property 5. *The points P, S_a, S_b, S_c and S_d lie on a circle whose center is*

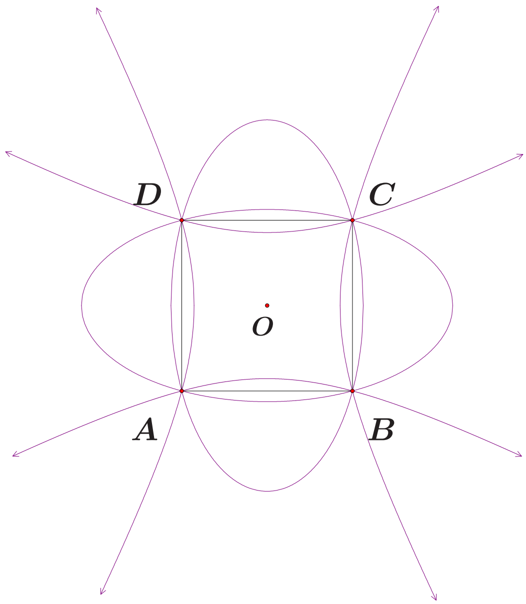

Figure 5: The loci P_{-1} (hyperbolas) and P_3 (ellipses).

on the line PO. In particular, the quadrangle $S_aS_bS_cS_d$ is cyclic.

In this article we took the square $ABCD$ as the underlying figure. Of course, it is possible to take instead any quadrangle or any triangle and perform similar constructions. The possibilities are numerous here but it remains to explore which of these choices give interesting results.

2. Primer on analytic plane geometry in Maple V

The key idea of the analytic geometry is to associate algebraic entities with geometric objects and then investigate them using algebraic methods.

The input of points on the plane in Maple V is quite simple because they are just ordered pairs of real numbers (their rectangular coordinates). For example, the input

```
tA:=[2, 3]:  tB:=[5, 7]:  tC:=[-2, 0]:  tT:=[x, y]:
```

defines four points on the plane $A(2,3)$, $B(5,7)$, $C(-2,0)$, $T(x,y)$.

We shall now list definitions of basic functions in Maple V for analytic geometry in the plane in rectangular coordinates. The first group of these functions are `FS`

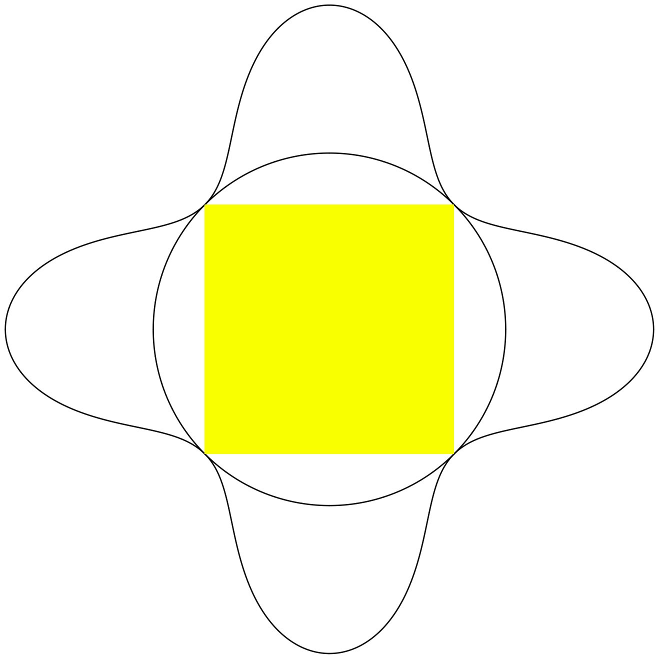

Figure 6: The locus $Q_{\frac{1}{2}}$.

(shortcut for composition of commands `simplify` and `factor`), `dis` (the distance of two points), `mid` (the midpoint of two points), `rat` (the point which divides two given points in given ratio $k \neq -1$), `rat2` (the point which divides two given points in given ratio $\frac{m}{n}$ where $m+n \neq 0$).

```
FS:=a->factor(simplify(a)):
dis:=(a,b)->FS(sqrt((a[1]-b[1])^2+(a[2]-b[2])^2)):
mid:=(a,b)->FS([(a[1]+b[1])/2,(a[2]+b[2])/2]):
rat:=(a,b,k)->FS([(a[1]+k*b[1])/(1+k),
                              (a[2]+k*b[2])/(1+k)]):
rat2:=(a,b,m,n)->FS([(n*a[1]+m*b[1])/(m+n),
                              (n*a[2]+m*b[2])/(m+n)]):
```

The lines in the program Maple V are represented as ordered triples $[a,\, b,\, c]$ of coefficients of their linear equations. For example, the input

```
pX:=[1,0,0]:  pY:=[0,1,0]:  pD:=[1,-1,0]:  pG:=[-1,2,2]:
```

define the y-axis, the x-axis, the bisector of the first and the third quadrant and the line $-x+2\,y+2=0$.

We continue with functions `li1` (for a line through a given point with a given slope), `li2` (for a line through two given different points), `olQ` (to test if a point is on a line), `clQ` (to test if three given points are collinear), and `ins` (the intersection of two lines or the information that they are parallel).

```
li1:=(a,k)->FS([k,-1,a[2]-k*a[1]]):
li2:=(a,b)->FS([a[2]-b[2],b[1]-a[1],a[1]*b[2]-b[1]*a[2]]):
olQ:=(t,p)->FS(t[1]*p[1]+t[2]*p[2]+p[3]):
clQ:=(a,b,c)->FS(a[1]*b[2]-a[1]*c[2]-b[1]*a[2]+
                    b[1]*c[2]+c[1]*a[2]-c[1]*b[2]):

ins:=(p,q)->FS([(q[3]*p[2]-q[2]*p[3])/(q[2]*p[1]-q[1]*p[2]),
                (q[1]*p[3]-q[3]*p[1])/(q[2]*p[1]-q[1]*p[2])]):
```

Functions `par` and `per` for the parallel and the perpendicular through a point to a line and tests `paQ` and `peQ` if two lines are parallel or perpendicular and the test `ccQ` for concurrency of three lines (i.e., whether they are parallel or intersect in a point) are next.

```
par:=(t,p)->FS([p[1],p[2],-t[1]*p[1]-t[2]*p[2]]):
per:=(t,p)->FS([p[2],-p[1],t[2]*p[1]-t[1]*p[2]]):
paQ:=(p,q)->FS(q[1]*p[2]-p[1]*q[2]):
peQ:=(p,q)->FS(p[1]*q[1]+p[2]*q[2]):

ccQ:=(a,b,c)->FS(a[1]*b[2]*c[3]-a[1]*b[3]*c[2]-b[1]*a[2]*
        c[3]+b[1]*a[3]*c[2]+c[1]*a[2]*b[3]-c[1]*a[3]*b[2]):
```

We conclude with the functions `pro` and `ar` for the orthogonal projection of a point onto a line and for the oriented area of a triangle on three given points.

```
pro:=(a,p)->FS([
  (p[2]*(a[1]*p[2]-a[2]*p[1])+p[1]*p[3])/(p[1]^2+p[2]^2),
  (p[1]*(a[2]*p[1]-a[1]*p[2])-p[2]*p[3])/(p[1]^2+p[2]^2)]):

ar:=(a,b,c)->FS((a[2]*c[1]-b[1]*a[2]-a[1]*c[2]+
                    a[1]*b[2]+b[1]*c[2]-c[1]*b[2])/2):
```

3. Central points functions

In this continuation of the previous section we shall describe functions for the central points that are mentioned in the introduction: the circumcenter, the orthocenter, and the Steiner point. On the way to define the Steiner point we also need functions for the symmedian point and the vertices of the first Brocard triangle.

First define the functions for the perpendicular bisector of a segment and the triangle circumcenter and orthocenter.

```
bis:=(a,b)->per(mid(a,b),li2(a,b)):
O_:=(a,b,c)->ins(bis(a,b),bis(a,c)):
H_:=(a,b,c)->ins(per(a,li2(b,c)),per(b,li2(c,a))):
```

The following function for the symmedian point is using the fact (see [1, p. 60g]) that symmedians bisect sides of the triangle from the projections A_h, B_h, C_h of vertices on opposite sidelines.

```
Ah_:=(a,b,c)->pro(a,li2(b,c)):
K_:=(a,b,c)->ins(li2(a,mid(Ah_(b,c,a),Ah_(c,a,b))),
                 li2(b,mid(Ah_(c,a,b),Ah_(a,b,c)))):
```

The vertices A_b, B_b, C_b of the first Brocard triangle are the orthogonal projections of the symmedian point onto the perpendicular bisectors of sides (see [1, p. 110] and Figure 7).

```
Ab_:=(a,b,c)->pro(K_(a,b,c),bis(b,c)):
```

Hence, the Steiner point is defined as follows (see Figure 8):

```
S_:=(a,b,c)->ins(par(a,li2(Ab_(b,c,a),Ab_(c,a,b))),
                 par(b,li2(Ab_(c,a,b),Ab_(a,b,c)))):
```

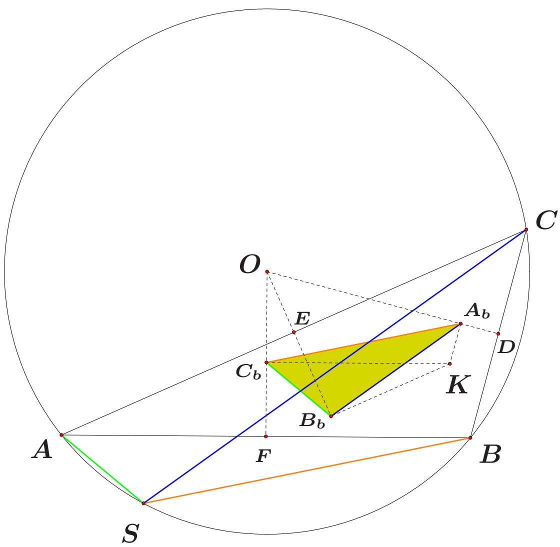

Figure 7: The Steiner point S is the intersection of parallels through vertices to sides of the first Brocard triangle $A_bB_bC_b$.

4. Verification of results

We shall now show how to prove most of the claims from the introduction with the help of computers in Maple V.

We first define the points A, B, C, D, O and P. They will be denoted with small letters because `D` is reserved in Maple V.

```
a:=[-1,-1]:b:=[1,-1]:c:=[1,1]:d:=[-1,1]:o:=[0,0]:p:=[x,y]:
```

Then we give the points U_t for $U = H$, J, O, N, S and $t = a$, b, c, d (all points mentioned in the introduction).

```
Ha:=H_(a,b,p):Hb:=H_(b,c,p):Hc:=H_(c,d,p):Hd:=H_(d,a,p):
Ja:=pro(Ha,li2(a,p)):Jb:=pro(Hb,li2(b,p)):
Jc:=pro(Hc,li2(c,p)):Jd:=pro(Hd,li2(d,p)):
Oa:=O_(a,b,p):Ob:=O_(b,c,p):Oc:=O_(c,d,p):Od:=O_(d,a,p):
Na:=pro(Oa,li2(a,p)):Nb:=pro(Ob,li2(b,p)):
Nc:=pro(Oc,li2(c,p)):Nd:=pro(Od,li2(d,p)):
Sa:=S_(a,b,p):Sb:=S_(b,c,p):Sc:=S_(c,d,p):Sd:=S_(d,a,p):
```

The proof of the Property 2 is accomplished with the following input.

```
s:=O_(o,p,Ja):FS(dis(s,o)-dis(s,Jb));
FS(dis(s,o)-dis(s,Jc));FS(dis(s,o)-dis(s,Jc));
```

Since the output is three times number 0 (zero), we conclude that the points O, P, J_a, J_b, J_c and J_d are on a circle. Its center S gives for the input

```
peQ(li2(o,s),li2(p,s));
```

the output 0, so that S is the center of the negatively oriented square $PONM$. Notice that the points N and M are `n:=rat(p,s,-2):` and `m:=rat(o,s,-2):`.

The proof of the Property 3 requires to see that we get zero as the output of each of the following eight commands.

```
clQ(n,Ha,Ja); clQ(n,Hb,Jb); clQ(n,Hc,Jc); clQ(n,Hd,Jd);
clQ(b,Ha,Ja); clQ(c,Hb,Jb); clQ(d,Hc,Jc); clQ(a,Hd,Jd);
```

Proof of Theorem 1.1. The quadrangle $H_aH_bH_cH_d$ is cyclic if and only if the distance between the circumcenters of the triangles $H_aH_bH_c$ and $H_aH_bH_d$ is equal to zero.

```
dis(O_(Ha,Hb,Hc),O_(Ha,Hb,Hd))^2;
```

This square of distance is equal to

$$\frac{T\,(x-y)^2\,(x+y)^2\,\left(x^2+y^2-2\right)^2\,\left(x^2+y^2+2\right)^2}{4\,\left(x^2-2\,x+y^2\right)^2\,(y-1)^2\,(y+1)^2\,(x-1)^2\,(x+1)^2\,\left(x^2+2\,y+y^2\right)^2},$$

where T is the polynomial

$$x^6+3\,x^4y^2+3\,x^2y^4+y^6-2\,x^5+6\,x^4y-8\,x^3y^2+8\,x^2y^3-6\,xy^4+2\,y^5+2\,x^4$$
$$-\,16\,x^3y+12\,x^2y^2-16\,xy^3+2\,y^4-4\,x^3+12\,x^2y-12\,xy^2+4\,y^3+4\,x^2+4\,y^2.$$

Since $x^2+y^2+2>0$ for all real numbers x and y and $x-y=0$, $x+y=0$ and $x^2+y^2-2=0$ are equations of the lines AC and BD and of the circumcircle k of the square $ABCD$, the claim of Theorem 1.1 follows provided we prove that the polynomial T is equal to zero only for P from the subset of the union $W = AC \cup BD \cup k$.

Let $x = r\cos\theta$ and $y = r\sin\theta$ for $r \geqslant 0$ and $0 \leqslant \theta < 2\pi$. Let u and v denote $3(\sin\theta - \cos\theta) + \sin 3\theta + \cos 3\theta$ and $3 - 8\sin 2\theta - \cos 4\theta$. Then $T = r^2 U$ with $U = r^4 + u\,r^3 + v\,r^2 + 2\,u\,r + 4$. Hence, it remains to show that the real roots of the equation **(*)** $U = 0$ give points from the set W.

Note that $u^2 - 4v + 16 = 4(1+\sin 2\theta)(4-(\sin 2\theta - 1)^2)$ is always positive except for $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$ when it is equal to zero. Let w denote $\sqrt{u^2-4v+16}$.

Applying the basic command `solve` to (*) we see that its roots are

$$r_{1,2} = \frac{-u+w\pm\sqrt{H}}{4}, \qquad r_{3,4} = \frac{-u-w\pm\sqrt{K}}{4},$$

where $H = L - 2uw$ and $K = L + 2uw$ with $L = u^2 + w^2 - 32$. Note that L can be written as $16\left(2(\cos\theta)^2 - 1\right)^2(\sin\theta\,\cos\theta - 1)$. Hence, L is always negative except for $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ when it is equal to zero and (*) has roots $\pm\sqrt{2}$.

Since $L^2 - (2uw)^2 = 64(2(\cos\theta)^2-1)^4$ is always positive except for the values $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ and L is there always negative it follows that both H and K are negative so that the four roots above are not real unless they are $\pm\sqrt{2}$. □

Proof of Theorem 1.2. The output of the command

```
peQ(li2(Ja,Jc),li2(Jb,Jd));
```

is $\frac{8(x-y)\left(x^2+y^2+2\right)(x+y)\left(x^2+y^2-2\right)\left(x^2+y^2\right)}{(2+2\,y+y^2+2\,x+x^2)(2-2\,y+y^2-2\,x+x^2)(2+2\,y+y^2-2\,x+x^2)(2-2\,y+y^2+2\,x+x^2)}$. This is equal to zero (i.e., the quadrangle $J_aJ_bJ_cJ_d$ has perpendicular diagonals) if and only if the point P is in the set W. □

Proof of Theorem 1.3. The area Ω of the square $ABCD$ is 4. The output of `FS(ar(Ha,Hb,Hc)+ar(Ha,Hc,Hd)-4*m);` is $\frac{-2\,T}{(1+y)(1-x)(1-y)(1+x)}$, where T denotes the polynomial

$$2\,(1+y)\,(1-x)\,(1-y)\,(1+x)\,m + (x+y)^2\,(x-y)^2\,.$$

For $m \neq 0$, let $\alpha = -\frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{2}{m}}$ and $\beta = \frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{2}{m}}$. Then

$$T = -2m(\alpha\,x^2 - \beta\,y^2 + 1)(\beta\,x^2 - \alpha\,y^2 - 1).$$

For $m < 0$, $\alpha > 0$ and $\beta > 1$ so that P_m is the union of two hyperbolas. For $m = 0$, $T = (x-y)^2(x+y)^2$ and P_m is the union of the lines AC and BD. For $0 < m < 2$, the discriminant $4\,m\,(y-1)^2\,(y+1)^2\,(m-2)$ of T considered as a quadratic trinomial in x^2 is negative so that $T > 0$ and P_m is empty. For $m = 2$, $T = \left(x^2+y^2-2\right)^2$ and P_m is the circumcircle k. Finally, for $m > 2$, $\alpha < 0$ and $\beta > 0$ so that P_m is the union of two ellipsis. □

Verification of Property 4. It suffices to note that the output for each of the commands

```
dis(Na,rat(p,a,1)); dis(Nb,rat(p,b,1));
dis(Nc,rat(p,c,1)); dis(Nd,rat(p,d,1));
```

is equal to zero. □

Verification of Property 5. It suffices to note that the output for the last three commands

```
t:=O_(Sa,Sb,Sc):  FS(dis(t,Sa)-dis(t,p));
FS(dis(t,Sa)-dis(t,Sd)); clQ(o,p,t);
```

is equal to zero. □

References

[1] Ross Honsberger, Episodes in nineteenth and twentieth century Euclidean geometry, The Mathematical Association of America, New Mathematical Library no. 37, Washington, 1995.

[2] R. A. Johnson, Advanced Euclidean Geometry, Dover Publications, New York, 1960.

[3] Clark Kimberling, Triangle Centers and Central Triangles, volume 129 of Congressus Numerantium, Utilitas Mathematica Publishing, Winnipeg, 1999.

[4] Clark Kimberling, Encyclopedia of Triangle Centers, 2000. (Internet address: http://cedar.evansville.edu/~ck6/encyclopedia/).

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