Burkholder’s inequality for multiindex martingales

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Dedicated to the memory of my teacher Péter Kiss

Abstract

Multiindex versions of Khintchine’s and Burkholder’s inequalities are presented.

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1. Introduction and notation

Burkholder’s inequality is a powerful tool of martingale theory. Let \((Z_n, \mathcal{F}_n)\), \(n = 1, 2, \ldots\), be a martingale with difference \(X_n = Z_n - Z_{n-1}\). Let \(p > 1\). There exist finite and positive constants \(C_p\) and \(D_p\) depending only on \(p\) such that

\[
C_p \left[ \mathbb{E} \left( \sum_{k=1}^{n} X_k^2 \right)^{p/2} \right]^{1/p} \leq \left( \mathbb{E} |Z_n|^p \right)^{1/p} \leq D_p \left[ \mathbb{E} \left( \sum_{k=1}^{n} X_k^2 \right)^{p/2} \right]^{1/p}, \tag{1.1}
\]

see Burkholder’s classical paper [1] and the textbook [2]. When the random variables \(X_1, X_2, \ldots\) are independent (1.1) is called the Marcinkiewicz-Zygmund inequality (and in this particular case it is valid also for \(p = 1\)).

Let \(\varepsilon_i(t), i = 1, 2, \ldots\), be the Rademacher system on \([0, 1]\). If \(X_k = \varepsilon_k a_k\), then we obtain Khintchine’s inequality. There exist finite and positive constants \(A_p\) and \(B_p\) depending only on \(p\) such that for any real sequence \(a_k, k = 1, 2, \ldots\),

\[
A_p \left( \sum_{k=1}^{n} a_k^2 \right)^{1/2} \leq \left[ \int_0^1 \left| \sum_{k=1}^{n} \varepsilon_k(t) a_k \right|^p dt \right]^{1/p} \leq B_p \left( \sum_{k=1}^{n} a_k^2 \right)^{1/2}. \tag{1.2}
\]

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This inequality is valid for \( p > 0 \). Actually, the standard proof of (1.1) is based on (1.2), see [1]).

The two-index version of (1.1) is obtained in [8], see also [7].

The aim of this paper is to prove a multiindex version of Burkholder’s inequality. The proof is based on the transform of a single parameter martingale. We also use the multiindex version of Khintchine’s inequality (for the sake of completeness, we prove it).

In [9] the second inequality of (3.2) was presented (without proof) for \( p > 2 \). It was applied to obtain a Brunk-Prokhorov type strong law of large numbers for martingale fields (see [9], Proposition 14). For a recent overview of multiindex random processes see [6]. In [6] a certain version of the Burkholder inequality was presented for continuous parameter random fields without the details of the proof (p. 257, Theorem 4.1.2). We do not use that theorem, we give a simple proof based on well-known one-parameter results.

Our Burkholder type inequality can be used to prove convergence results for multiindex autoregressive type martingales (see [5], for the two-index case see [4]).

We use the following notation. Let \( d \) be a fixed positive integer. Let \( N \) denote the set of positive integers, \( N_0 \) the set of non-negative integers. The multidimensional indices will be denoted by \( k = (k_1, \ldots, k_d), n = (n_1, \ldots, n_d), \cdots \in N_0^d \). Relations \( \leq, \min \) are defined coordinatewise. I.e. \( k \leq n \) means \( k_1 \leq n_1, \ldots, k_d \leq n_d \). Relation \( k < n \) means \( k \leq n \) but \( k \neq n \).

Let \( \|X\|_p = (E|X|^p)^{1/p} \) for \( p > 0 \). Then \( \|X\|_{p_1} \leq \|X\|_{p_2} \) for \( 0 < p_1 \leq p_2 \).

2. Khintchine’s inequality

**Theorem 2.1.** Let \( \varepsilon_i(t), i = 1, 2, \ldots, \) be the Rademacher system on \([0,1] \). Let \( p > 0 \). There exist finite and positive constants \( A_{p,d} \) and \( B_{p,d} \) depending only on \( p \) and \( d \) such that for any \( d \)-index sequence \( a_k, k \in N^d \),

\[
A_{p,d} \left( \sum_{k \leq n} a_k^2 \right)^{1/2} \leq \left[ \int_0^1 \cdots \int_0^1 \left| \sum_{k \leq n} \varepsilon_{k_1}(t_1) \cdots \varepsilon_{k_d}(t_d) a_k \right|^p dt_1 \cdots dt_d \right]^{1/p} \leq B_{p,d} \left( \sum_{k \leq n} a_k^2 \right)^{1/2} . \tag{2.1}
\]

**Proof.** First we remark that for \( d = 1 \) inequality (2.1) is the original Khintchine’s inequality.

Denote by \( \varepsilon_{i,n_i}, n_i = 1, 2, \ldots, i = 1, 2, \ldots, d, \) independent sequences of independent Bernoulli random variables with \( P(\varepsilon_{i,n_i} = 1) = P(\varepsilon_{i,n_i} = -1) = 1/2 \) for each \( i \) and \( n_i \). Let \( s_n = \left( \sum_{k \leq n} a_k^2 \right)^{1/2} \) and \( S_n = \sum_{k \leq n} \varepsilon_{1,k_1} \cdots \varepsilon_{d,k_d} a_k \). Then, by the Fubini theorem, inequality (2.1) is equivalent to

\[
A_{p,d} s_n \leq \|S_n\|_p \leq B_{p,d} s_n . \tag{2.2}
\]

Now we prove that the products \( \varepsilon_{1,k_1} \cdots \varepsilon_{d,k_d}, (k_1, \ldots, k_d) \in N^d, \) are pairwise independent Bernoulli variables. By induction, it is enough to prove that \( \varepsilon_{1,k_1} \varepsilon_{2,k_2}, \ldots \)
(k_1, k_2) \in \mathbb{N}^2$, are pairwise independent Bernoulli variables if \( \varepsilon_{1,k_1}, k_1 \in \mathbb{N} \), and \( \varepsilon_{2,k_2}, k_2 \in \mathbb{N} \), are independent sequences of pairwise independent Bernoulli variables. Indeed, if \( \varepsilon_1 \) and \( \varepsilon_2 \) are independent Bernoulli variables then their product is Bernoulli: \( P(\varepsilon_1 \varepsilon_2 = \pm 1) = 1/2 \). Now turn to the independence. It is obvious that the independence of \( \varepsilon_1 \), \( \varepsilon_2 \), \( \varepsilon_3 \), and \( \varepsilon_4 \) implies the independence of \( \varepsilon_1 \varepsilon_2 \) and \( \varepsilon_3 \varepsilon_4 \). Moreover, the independence of \( \varepsilon_1 \), \( \varepsilon_2 \) and \( \varepsilon_3 \) implies the independence of \( \varepsilon_1 \varepsilon_3 \) and \( \varepsilon_2 \varepsilon_4 \):

\[
P(\varepsilon_1 \varepsilon_3 = \pm 1, \varepsilon_2 \varepsilon_3 = \pm 1) = \frac{1}{4} = P(\varepsilon_1 \varepsilon_3 = \pm 1)P(\varepsilon_2 \varepsilon_3 = \pm 1).
\]

Therefore \( \|S_n\|^2_p \) is the variance of the sum of pairwise independent random variables, so we have \( s_n = \|S_n\|_2 \). In particular, (2.2) is true for \( p = 2 \).

Now we show that the products \( \varepsilon_{1,k_1} \cdots \varepsilon_{d,k_d} \), \( (k_1, \ldots, k_d) \in \mathbb{N}^d \), are not (completely) independent. Indeed, if \( \varepsilon_1 \), \( \varepsilon_2 \), \( \varepsilon_3 \), and \( \varepsilon_4 \) are independent Bernoulli variables, then \( \varepsilon_1 \varepsilon_3 \), \( \varepsilon_2 \varepsilon_3 \), \( \varepsilon_1 \varepsilon_4 \), and \( \varepsilon_2 \varepsilon_4 \) are not independent:

\[
P(\varepsilon_1 \varepsilon_3 = 1, \varepsilon_2 \varepsilon_3 = 1, \varepsilon_1 \varepsilon_4 = 1, \varepsilon_2 \varepsilon_4 = 1) = 1/8 \neq 1/16 = P(\varepsilon_1 \varepsilon_3 = 1)P(\varepsilon_2 \varepsilon_3 = 1)P(\varepsilon_1 \varepsilon_4 = 1)P(\varepsilon_2 \varepsilon_4 = 1).
\]

So relation (2.2) is really different from its one-index version.

Now we prove the second part of (2.2). We start with the case of \( d = 1 \) it is the original Khintchine’s inequality. Assume (2.2) for \( d - 1 \). Let

\[
I_{k_1,n_2,\ldots,n_d}(t_2,\ldots,t_d) = \sum_{k_2=1}^{n_2} \cdots \sum_{k_d=1}^{n_d} \varepsilon_{k_1}(t_2) \cdots \varepsilon_{k_d}(t_d)a_1,k_2,\ldots,k_d.
\]

Then, by the original Khintchine’s inequality,

\[
\int_0^1 \left| \sum_{k_1=1}^{n_1} \varepsilon_{k_1}(t_1)I_{k_1,n_2,\ldots,n_d}(t_2,\ldots,t_d) \right|^p dt_1 \leq B^p_{p,1} \left( \sum_{k_1=1}^{n_1} I^2_{k_1,n_2,\ldots,n_d}(t_2,\ldots,t_d) \right)^{p/2}.
\]

From here

\[
\|S_n\|_p^p \leq B^p_{p,1} \int_0^1 \cdots \int_0^1 \left( \sum_{k_1=1}^{n_1} I^2_{k_1,n_2,\ldots,n_d}(t_2,\ldots,t_d) \right)^{p/2} dt_2 \cdots dt_d \leq \left\{ \sum_{k_1=1}^{n_1} \left[ \int_0^1 \cdots \int_0^1 \left( I^2_{k_1,n_2,\ldots,n_d}(t_2,\ldots,t_d) \right)^{p/2} dt_2 \cdots dt_d \right]^{2/p} \right\}^p \\
\leq B^p_{p,1} \left\{ \sum_{k_1=1}^{n_1} \left[ B_{p,d-1} \left( \sum_{k_2=1}^{n_2} \cdots \sum_{k_d=1}^{n_d} a^2_{k_1,k_2,\ldots,k_d} \right)^{1/2} \right]^{2/p} \right\} \\
\leq B^p_{p,1} \left\{ \sum_{k_1=1}^{n_1} \left[ B_{p,d-1} \left( \sum_{k_2=1}^{n_2} \cdots \sum_{k_d=1}^{n_d} a^2_{k_1,k_2,\ldots,k_d} \right)^{1/2} \right]^{2/p} \right\}.
Let \( \mathbf{Z} \) be an increasing sequence of \( \mathcal{F}_n \) \( \mathcal{G}_n \) \( \mathcal{H}_n \).

We follow the lines of [\ref{ref:Burkholder}] and (2.2) for \( p = 2 \). We can prove the second part of (2.2) for \( p \geq 2 \).

As \( \| \mathbf{Z} \|_p \leq \| \mathbf{Z} \|_2 \) for \( 0 < p \leq 2 \), the second part of (2.2) is true for \( 0 < p \).

Now turn to the first part of (2.2). We see that

\[
\mathbb{E}\left[ \| \mathbf{Z} \|_p^{pr} \| \mathbf{Z} \|^{2pr}_2 \right] \leq \| \mathbf{Z} \|_p^{pr} \| \mathbf{Z} \|_2^{2pr}.
\]

Therefore it is enough to prove the inequality for \( 0 < p < 2 \). We follow the lines of [\ref{ref:Fazekas}], p. 367.

Let \( 0 < p < 2 \). Choose \( r_1, r_2 > 0 \), \( r_1 + r_2 = 1 \), \( pr_1 + 4r_2 = 2 \). By Holder’s inequality and the second part of (2.2), we have

\[
s_n^2 = \| \mathbf{Z} \|_2^2 \leq \| \mathbf{Z} \|_p^{pr_1} \| \mathbf{Z} \|_2^{2pr_2} \leq \| \mathbf{Z} \|_p^{pr_1} B s_n^{4r_2}.
\]

From here

\[
\| \mathbf{Z} \|_p^{pr_1} \geq (1/B)s_n^{2-4r_2} = (1/B)s_n^{pr_1}.
\]

Therefore the first part of (2.2) is true for \( 0 < p < 2 \). \( \square \)

3. Burkholder’s inequality

Let \((X_n, \mathcal{F}_n), \, n \in \mathbb{N}^d\), be a martingale difference. It means that \( \mathcal{F}_n, \, n \in \mathbb{N}^d\), is an increasing sequence of \( \sigma \)-algebras, i.e. \( \mathcal{F}_k \subseteq \mathcal{F}_n \) if \( k \leq n \); \( X_n \) is \( \mathcal{F}_n \)-measurable and integrable; \( \mathbb{E}(X_n | \mathcal{F}_k) = 0 \) if \( k < n \).

To obtain Burkholder’s inequality, we shall assume the so called condition (F4).

I. e.

\[
\mathbb{E}\{\mathbb{E}(\eta | \mathcal{F}_m) | \mathcal{F}_n \} = \mathbb{E}\{\eta | \mathcal{F}_{\min(m,n)} \}
\]

for each integrable random variable \( \eta \) and for each \( m, n \in \mathbb{N}^d \) (see, e.g., [\ref{ref:Schloegl} and \ref{ref:Fazekas}]).

Denote by \((Z_n, \mathcal{F}_n), \, n \in \mathbb{N}^d\), the martingale corresponding to the difference \((X_n, \mathcal{F}_n), \, n \in \mathbb{N}^d\). More precisely, let \( Z_n = 0 \) and \( \mathcal{F}_n = \{\emptyset, \Omega\} \) if \( n \in \mathbb{N}^d \setminus \mathbb{N}^d \) and \( Z_n = \sum_{k \leq n} X_k, \, n \in \mathbb{N}^d \).

**Theorem 3.1.** Let \((Z_n, \mathcal{F}_n), \, n \in \mathbb{N}^d\), be a martingale and \((X_n, \mathcal{F}_n), \, n \in \mathbb{N}^d\), the martingale difference corresponding to it. Assume that (3.1) is satisfied. Let \( p > 1 \). There exist finite and positive constants \( C_{p,d} \) and \( D_{p,d} \) depending only on \( p \) and \( d \) such that

\[
C_{p,d} \left[ \mathbb{E}\left( \sum_{k \leq n} X_k^2 \right)^{p/2} \right]^{1/p} \leq (\mathbb{E}|Z_n|^p)^{1/p} \leq D_{p,d} \left[ \mathbb{E}\left( \sum_{k \leq n} X_k^2 \right)^{p/2} \right]^{1/p}.
\]

**Proof.** We follow the lines of [\ref{ref:Burkholder}]. Let \( u_{i,n_i} \in \{0, 1\}, \, n_i = 1, 2, \ldots, i = 1, 2, \ldots, d \).

Let

\[
T_n = \sum_{k \leq n} u_{1,k_1} \cdots u_{d,k_d} X_k = \sum_{k_1=1}^{\alpha_1} u_{1,k_1} Y_{k_1},
\]
where
\[ Y_{k_1} = Y_{k_1,n_2,...,n_d} = \sum_{k_2=1}^{n_2} \cdots \sum_{k_d=1}^{n_d} u_{2,k_2} \cdots u_{d,k_d} X_{k_1,k_2,...,k_d}. \]

First we show that
\[ E[Z_n]^p \leq M_d E[T_n]^p. \tag{3.3} \]

We use induction. For \( d = 1 \) (3.3) is included in [1], p. 1502 (because \( T_n \) is a transform of the martingale \( Z_n \) and vice versa). Now we assume that (3.3) is true for \( d = 1 \). Let \( n_2, \ldots, n_d \) be fixed, \( F_{k_1} = F_{k_1,n_2,...,n_d} \). Then, using (3.1), we can show that \( (Y_{k_1}, F_{k_1}) \), \( k_1 = 1, 2, \ldots \), is a martingale difference. As the martingale \( \sum_{k_1=1}^{n_1} Y_{k_1} = \sum_{k_1=1}^{n_1} u_{1,k_1}(u_{1,k_1}Y_{k_1}) \) is a transform of the martingale \( \sum_{k_1=1}^{n_1}(u_{1,k_1}Y_{k_1}) \), by [1], p. 1502,
\[ E \left| \sum_{k_1=1}^{n_1} Y_{k_1} \right|^p \leq M_1 E \left| \sum_{k_1=1}^{n_1} (u_{1,k_1}Y_{k_1}) \right|^p. \tag{3.4} \]

Now, using (3.1), we can show that for any fixed \( n_1 \) the \((d-1)\)-index sequence \( \{ \sum_{k_1=1}^{n_1} X_{k_1,k_2,...,k_d}, F_{n_1,k_2,...,k_d} \} \), \( (k_2, \ldots, k_d) \in \mathbb{N}^{d-1} \), is a martingale difference. Therefore, using (3.3) for \( d - 1 \), we obtain
\[ E[Z_n]^p = E \left| \sum_{k_2=1}^{n_2} \cdots \sum_{k_d=1}^{n_d} \left[ \sum_{k_1=1}^{n_1} X_{k_1,k_2,...,k_d} \right] \right|^p \leq M_{d-1} E \left| \sum_{k_2=1}^{n_2} \cdots \sum_{k_d=1}^{n_d} u_{2,k_2} \cdots u_{d,k_d} \left[ \sum_{k_1=1}^{n_1} X_{k_1,k_2,...,k_d} \right] \right|^p = M_{d-1} E \left| \sum_{k_1=1}^{n_1} Y_{k_1} \right|^p \leq M_d E \left| \sum_{k_1=1}^{n_1} (u_{1,k_1}Y_{k_1}) \right|^p = (3.5) \]

In (3.5) we applied (3.4). So we proved (3.3).

Because \( Z_n \) and \( T_n \) are each other’s transforms, (3.3) implies
\[ N_d E[T_n]^p \leq E[Z_n]^p \leq M_d E[T_n]^p. \tag{3.6} \]

Now we prove the first part of (3.2). By (2.1),
\[
E \left( \sum_{k \leq n} X_k \right)^{p/2} \leq \frac{1}{A_{p,d}} E \left[ \int_0^1 \cdots \int_0^1 \left| \sum_{k \leq n} \varepsilon_{k_1}(t_1) \cdots \varepsilon_{k_d}(t_d)X_k \right| \, dt_1 \cdots dt_d \right] \\
= \frac{1}{A_{p,d}} \int_0^1 \cdots \int_0^1 E \left[ \left| \sum_{k \leq n} \varepsilon_{k_1}(t_1) \cdots \varepsilon_{k_d}(t_d)X_k \right| \right] \, dt_1 \cdots dt_d \\
\leq \frac{1}{A_{p,d}} \int_0^1 \cdots \int_0^1 \frac{1}{N_d} E \left[ \left| \sum_{k \leq n} X_k \right| \right] \, dt_1 \cdots dt_d \\
= \frac{1}{A_{p,d}} \frac{1}{N_d} E[Z_n]^p.
\]
In the third step we applied (3.6).

We turn to the second part of (3.2). By (3.6),
\[ E|Z_n|^p \leq M_d E \left[ \left| \sum_{k \leq n} \varepsilon_{k_1}(t_1) \cdots \varepsilon_{k_d}(t_d) X_k \right|^p \right]. \]

From here, using (2.1),
\[ E|Z_n|^p \leq M_d \int_0^1 \cdots \int_0^1 E \left[ \left| \sum_{k \leq n} \varepsilon_{k_1}(t_1) \cdots \varepsilon_{k_d}(t_d) X_k \right|^p \right] dt_1 \cdots dt_d \]
\[ \leq M_d B_{p,d}^p E \left( \sum_{k \leq n} X_k^2 \right)^{p/2}. \]

The proof is complete. \(\Box\)

4. Final comments

Burkholder’s inequality is valid for martingales with values in \(\mathbb{R}^t\) (\(t\) is a fixed positive integer). For \(p > 0\) and \(x = (x_1, \ldots, x_t) \in \mathbb{R}^t\) let \(\|x\|_p = \left( \sum_{i=1}^t |x_i|^p \right)^{1/p}\).

Let \((X_n, \mathcal{F}_n), n \in \mathbb{N}^d\), be a martingale difference with values in \(\mathbb{R}^t\). Assume that condition (F4) is satisfied. Let \((Z_n, \mathcal{F}_n), n \in \mathbb{N}^d\), be the martingale corresponding to the difference \((X_n, \mathcal{F}_n), n \in \mathbb{N}^d\).

**Theorem 4.1.** Let \((Z_n, \mathcal{F}_n), n \in \mathbb{N}^d\), be a martingale with values in \(\mathbb{R}^t\) and \((X_n, \mathcal{F}_n), n \in \mathbb{N}^d\), the martingale difference corresponding to it. Assume that (3.1) is satisfied. Let \(p > 1\). There exist finite and positive constants \(C\) and \(D\) depending only on \(t\), \(p\) and \(d\) such that

\[ C \left[ E \left( \sum_{k \leq n} \|X_k\|_2^2 \right)^{p/2} \right]^{1/p} \leq \left( E\|Z_n\|_p^p \right)^{1/p} \leq D \left[ E \left( \sum_{k \leq n} \|X_k\|_2^2 \right)^{p/2} \right]^{1/p}. \] (4.1)

**Proof.** It is known that for any \(p, q > 0\) there exist \(0 < c, d < \infty\) such that \(c\|x\|_p \leq \|x\|_q \leq d\|x\|_p\) for all \(x \in \mathbb{R}^t\). Applying this observation and (3.2) we obtain (4.1). \(\Box\)

Using this theorem we can prove limit theorems for autoregressive type martingale fields. For details see [5] and [4] including the \(d\)-index case and the two-index case, respectively.

**References**

Burkholder’s inequality for multi-index martingales


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