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# Common Domain of Asymptotic Stability of a Family of Difference Equations *

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**Abstract**

A necessary and sufficient condition is obtained for each difference equation in a family to be asymptotically stable.

The following difference equation (see e.g. [1,2] for its importance)

$$u_n = au_{n-\tau} + bu_{n-\sigma},\ n = 0, 1, 2, ... \tag{1}$$

where $a, b$ are nontrivial real numbers and $\tau, \sigma$ are positive integers, is said to be (globally) asymptotically stable if each of its solutions tends to zero.

When the delays $\tau$ and $\sigma$ are given, whether the corresponding equation (1) is asymptotically stable clearly depends on the coefficients $a$ and $b$. For this reason, we denote the set of all pairs $(x, y)$ such that the equation

$$u_n = xu_{n-\tau} + yu_{n-\sigma},\ n = 0, 1, 2, ... \tag{2}$$

is asymptotically stable by $D(x, y|\tau, \sigma)$. It is well known that equation (1) is asymptotically stable if, and only if, all the roots of its characteristic equation

$$\lambda^n = a\lambda^{n-\tau} + b\lambda^{n-\sigma},$$

are inside the open unit disk [3]. Since the latter statement holds if, and only if, all the roots of the equation

$$1 = a\lambda^{-\tau} + b\lambda^{-\sigma} \tag{3}$$

are inside the open unit disk, the set $D(x, y|\tau, \sigma)$ is also the set of pairs $(x, y)$ such that all the roots of

$$1 = x\lambda^{-\tau} + b\lambda^{-\sigma} \tag{4}$$

has magnitude less than one.

By means of commercial software such as the MATLAB, it is not difficult to generate domains $D(x, y|\tau, \sigma)$ in the $x, y$-plane for different values of the delays $\tau$ and $\sigma$. It is interesting to observe that the set $\{(x, y)| |x| + |y| < 1\}$ is included in all of these computer generated domains. This motivates the following theorem.

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THEOREM 1. Let $D(x,y|\tau,\sigma)$ be the set of all pairs of the form $(x,y)$ such that equation (2) is asymptotically stable. Then we have

$$\bigcap_{\tau,\sigma\in N} D(x,y|\tau,\sigma)=\{(x,y)|\,|x|+|y|<1\}\,,$$

where $N$ is the set of all positive integers.

One part of the proof is easy. Let $\mu$ be a nonzero root of equation (3). If $|a|+|b|<1$, then since

$$|a|+|b|<1\le |a|\,|\mu|^{-\tau}+|b|\,|\mu|^{-\sigma}\,,$$

we see that

$$|a|<|a|\,|\mu|^{-\tau}$$

or

$$|b|<|b|\,|\mu|^{-\sigma}\,.$$

But then $|\mu|^{\tau}<1$ or $|\mu|^{\sigma}<1$. In other words, $|\mu|<1$.

In order to complete our proof, we need the following preparatory lemma.

LEMMA 1 (cf. [4, Lemma 2.1]). Suppose $a,b$ are real numbers such that $|a|+|b|\neq 0$, and $\tau$ and $\sigma$ are two positive integers. Then the equation

$$|a|\,x^{-\tau}+|b|\,x^{-\sigma}=1,\ x>0$$

has a unique solution in $(0,\infty)$.

PROOF. Consider the function

$$f(x)=|a|\,x^{-\tau}+|b|\,x^{-\sigma},\ x>0.$$

Since $f$ is continuous on $(0,\infty)$, $\lim_{x\to 0^+}f(x)=\infty$, $\lim_{x\to\infty}f(x)=0$ and

$$f'(x)=-\left(|a|\,\tau x^{-\tau-1}+|b|\,\sigma x^{-\sigma-1}\right)<0,\ x>0,$$

thus our proof follows from the intermediate value theorem.

Now if $(a,b)$ belongs to $\bigcap_{\tau,\sigma\in N}D(x,y|\tau,\sigma)$, then for each pair $(\tau,\sigma)$ of integers, each root $\mu$ of equation (3) satisfies $|\mu|<1$. Let us write $\mu=re^{i\theta}$ and write (3) in the form

$$ar^{-\tau}\cos\tau\theta+br^{-\sigma}\cos\sigma\theta=1, \tag{5}$$

$$ar^{-\tau}\sin\tau\theta+br^{-\sigma}\sin\sigma\theta=0. \tag{6}$$

There are several cases to consider: (i) $a=0$ or $b=0$; (ii) $a>0,b>0$; (iii) $a<0,b<0$; (iv) $a<0,b>0$; and (v) $a>0,b<0$. The first case is easily dealt with. In the second case, since the equation

$$ax^{-\tau}+bx^{-\sigma}=1$$

has a positive root $\rho_1$, thus $(r, \theta) = (\rho_1, 0)$ is a solution of equations (5)-(6). This implies that $\rho_1 = r = |\mu| < 1$. But then

$$1 = a\rho_1^{-\tau} + b\rho_1^{-\sigma} > a + b = |a| + |b|\,.$$

In the third case, since the equation

$$-ax^{-\tau} - bx^{-\sigma} = 1$$

has a positive root $\rho_2$, if we pick $\tau = 1$ and $\sigma = 3$, then $(r, \theta) = (\rho_2, \pi)$ is a solution of equations (5)-(6). This implies $\rho_2 = |\mu| < 1$. But then

$$1 = -a\rho_2^{-\tau} - b\rho_2^{-\sigma} > -a - b = |a| + |b|\,.$$

In the fourth case, since the equation

$$-ax^{-\tau} + bx^{-\sigma} = 1$$

has a positive root $\rho_3$, if we pick $\tau = 1$ and $\sigma = 2$, then $(r, \theta) = (\rho_3, \pi)$ is a solution of equations (5)-(6). This implies $\rho_3 = |\mu| < 1$. But then

$$1 = -a\rho_3^{-\tau} + b\rho_3^{-\sigma} > -a + b = |a| + |b|\,.$$

In the final case, since the equation

$$ax^{-\tau} - bx^{-\sigma} = 1$$

has a positive root $\rho_4$, if we pick $\tau = 2$ and $\sigma = 3$, then $(r, \theta) = (\rho_4, \pi)$ is a solution of equations (5-6). This implies $\rho_4 < 1$ and consequently

$$1 \geq a\rho_4^{-\tau} - b\rho_4^{-\sigma} > a - b = |a| + |b|\,.$$

The proof is complete.

## References

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