Oscillation of Nonlinear First Order Neutral Difference Equations *

Ying Gao and Guang Zhang††

Received 1 June 2000

Abstract

In this note, oscillation criteria are obtained for a class of nonlinear neutral difference equations.

Oscillatory behaviors of neutral difference equations of the form

$$\Delta (x_n - p_n x_{n-\tau}) + q_n x_{n-\sigma} = 0, \ n = 0, 1, \ldots$$

have been explored to some extent in a number of studies [1-3]. However, relatively few results are known for their nonlinear extensions of the form

$$\Delta (x_n - p_n x_{n-\tau}) + q_n \max_{s \in [n-\sigma,n]} x_s = 0, \ n = 0, 1, \ldots .$$

Nonlinear functional equations involving the maximum function are important since they appear naturally in automatic control theory, see e.g. Popov [4]. Some of the qualitative theory of these equations has been developed recently, see e.g. [5-8]. In this note, we will consider their oscillatory behaviors.

We will assume that \( \tau \) and \( \sigma \) are positive integers, that \( \{ p_n \} \) and \( \{ q_n \} \) are nonnegative real sequences, and \( \{ q_n \} \) has a positive subsequence. Let \( \mu = \max \{ \tau, \sigma \} \). By a solution of (1) or (2), we mean a real sequence \( \{ x_n \}_{n=0}^{\infty} \) which satisfies (1) or (2) respectively for \( n \geq 0 \). Such a solution is said to be oscillatory if it is neither eventually positive nor eventually negative.

It is easily seen that \( \{ x_n \} \) is an eventually positive solution of equation (1) if, and only if, \( \{ -x_n \} \) is its eventually negative solution. However, such a property is not valid for equation (2). Instead, \( \{ x_n \} \) is an eventually positive solution of (2) if, and only if, \( \{ -x_n \} \) is an eventually negative solution of the equation

$$\Delta (x_n - p_n x_{n-\tau}) + q_n \min_{s \in [n-\sigma,n]} x_s = 0, \ n = 0, 1, \ldots .$$

Thus, all solutions of (2) are oscillatory if, and only if, both (2) and (3) do not have any eventually positive solutions.

*Mathematics Subject Classifications: 39A10
†Department of Mathematics, Yanbei Normal College, Datong, Shanxi 037000, P. R. China
‡Supported by the Science Foundation of Shanxi Province of China
**Lemma 1.** Assume that there exists a nonnegative integer \( N \geq 0 \) such that \( p_{N+j} \leq 1 \) for \( j = 0, 1, 2, \ldots \). Let \( \{x_n\} \) be an eventually positive solution of (2) or (3). Set
\[
y_n = x_n - p_n x_{n-\tau}
\]
for all large \( n \). Then \( y_n > 0 \) eventually.

The proof is similar to the proof of Lemma 1 in [2].

**Theorem 1.** Assume that there exists a nonnegative integer \( N \geq 0 \) such that \( p_{N+j} \leq 1 \) for \( j = 0, 1, 2, \ldots \). Suppose further that either \( p_n > 0 \) or \( q_n \) does not vanish identically over any set of consecutive integers of the form \( \{a, a + 1, \ldots, a + \sigma\} \). Then equation (2) has an eventually positive solution if, and only if, the following functional inequality
\[
\Delta (x_n - p_n x_{n-\tau}) + q_n \max_{s \in [n-\sigma, n]} x_s \leq 0, \quad n = 0, 1, \ldots
\]
has an eventually positive solution; and equation (3) has an eventually positive solution if, and only if, the functional inequality
\[
\Delta (x_n - p_n x_{n-\tau}) + q_n \min_{s \in [n-\sigma, n]} x_s \leq 0, \quad n = 0, 1, \ldots
\]
has an eventually positive solution.

The proof of Theorem 1 is similar to that of Theorem 1 in [2], and is thus omitted.

**Theorem 2.** Assume that there exists a nonnegative integer \( N \geq 0 \) such that \( p_{N+j} \leq 1 \) for \( j = 0, 1, 2, \ldots \). Suppose further that there exists some positive integer \( T \) such that the functional inequality
\[
\Delta y_n + q_n \min_{s \in [n-\sigma, n]} \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{s-i\tau} y_{n-\tau} \leq 0
\]
does not have any eventually positive solutions. Then all solutions of (2) oscillate.

**Proof.** If \( \{x_n\} \) is an eventually positive solution of (2), then \( \Delta y_n \leq 0 \) and \( y_n = x_n - p_n x_{n-\tau} > 0 \) eventually. Thus,
\[
x_n = y_n + p_n x_{n-\tau} = y_n + p_n y_{n-\tau} + p_n p_{n-\tau} x_{n-2\tau}
\]
\[
= \cdots \geq y_n + p_n y_{n-\tau} + \cdots + \prod_{i=0}^{T-1} p_{n-i\tau} y_{n-(i+1)\tau}
\]
\[
\geq \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{n-i\tau} y_{n-\tau}.
\]
Hence,
\[
\max_{s \in [n-\sigma, n]} x_s \geq \max_{s \in [n-\sigma, n]} \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{s-i\tau} y_{s-\tau} \geq \max_{s \in [n-\sigma, n]} \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{s-i\tau} y_{n-\tau}.
\]
Substituting the last inequality into (2), we have
\[ \Delta y_n + q_n \max_{s \in [n-\sigma, n]} \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{s-ir} y_{n-r} \leq 0, \]
which is a contradiction. If \( \{z_n\} \) is an eventually negative solution of (2), then \( x_n = -z_n \) is an eventually positive solution of equation (3). Similarly, we have
\[ \Delta y_n + q_n \min_{s \in [n-\sigma, n]} \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{s-ir} y_{n-r} \leq \Delta y_n + q_n \max_{s \in [n-\sigma, n]} \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{s-ir} y_{n-r} \leq 0, \]
This is also a contradiction. The proof is complete.

For the equation
\[ \Delta (x_n - x_{n-\tau}) + q_n \max_{s \in [n-\sigma, n]} x_s = 0, \quad n = 0, 1, \ldots, \]  
we have the following result.

**THEOREM 3.** Equation (7) has nonoscillatory solutions if, and only if,
\[ \Delta^2 z_{n-1} + \frac{1}{\tau} q_n z_n = 0 \]  
also has nonoscillatory solutions.

**PROOF.** Assume that \( \{x_n\} \) is an eventually positive solution of (7). In view of Lemma 1, we see that there is an integer \( N_1 \) such that \( x_{n-\tau} > 0, y_n = x_n - x_{n-\tau} > 0, \) and \( \Delta y_n \leq 0 \) for \( n \geq N_1 \). Set \( m = \min \{x_{N_1-\tau}, x_{N_1-\tau+1}, \ldots, x_{N_1-1}\} \). For \( n \geq N_1 + \tau \), there exists a positive integer \( k \) such that
\[ N_1 + k\tau \leq n < N_1 + (k + 1)\tau \]
and
\[ x_n = x_{n-k\tau} + \sum_{j=0}^{k-1} y_{n-j\tau} \geq m + \sum_{j=0}^{k-1} y_{n-j\tau}. \]
Furthermore, since \( \Delta y_n \leq 0 \) for \( n \geq N_1 \), and since \( n-k\tau + \tau \leq N_1 + 2\tau = N_2 \),
\[ \sum_{j=0}^{k-1} y_{n-j\tau} \geq (y_{n-(k-1)\tau} + y_{n-(k-1)\tau+1} + \ldots + y_{n-(k-2)\tau-1}) + (y_{n-(k-2)\tau} + y_{n-(k-2)\tau+1} + \ldots + y_{n-(k-3)\tau-1}) + \ldots + (y_n + y_{n+1} + \ldots + y_{n+\tau-1}) \]
\[ \geq \sum_{i=N_2}^{n} y_i, \]
we have
\[ x_n \geq m + \frac{1}{\tau} \sum_{i=N_2}^{n} y_i. \]
Let
\[ z_n = m + \frac{1}{\tau} \sum_{i=N_2}^{n} y_i, \]
then \( z_n > 0 \) and
\[ \tau \Delta^2 z_{n-1} + q_n z_n = \Delta y_n + q_n \left( m + \frac{1}{\tau} \sum_{i=N_2}^{n} y_i \right) \]
\[ = \Delta y_n + q_n \max_{s \in [n-\sigma, n]} \left( m + \frac{1}{\tau} \sum_{i=N_2}^{s} y_i \right) \]
\[ \leq \Delta y_n + q_n \max_{s \in [n-\sigma, n]} x_s = 0. \]
In view of Lemma 1, we know that equation (8) has an eventually positive solution. If \( \{x_n\} \) is an eventually positive solution of the equation
\[ \Delta (x_n - x_{n-\tau}) + q_n \min_{s \in [n-\sigma, n]} x_s = 0, \quad n = 0, 1, \ldots \quad (9) \]
we can also prove that
\[ \tau \Delta^2 z_{n-1} + q_n z_n \leq 0 \]
has an eventually positive solution. In view of Theorem 2 in [3], we know that equation (8) has an eventually positive solution. We now show that the converse holds. Let \( \{z_n\} \) be an eventually positive solution of (8), then it is positive and concave for all large \( n \), so that \( \{\Delta z_n\} \) is eventually positive and nonincreasing. Thus it is easy to see that there exists a sufficiently large integer \( N \) such that \( 0 < \tau \Delta z_{n-1} \leq z_{n-\tau} \) for \( n \geq N \).
Let
\[ H_n = \begin{cases} \tau \Delta z_{n-1} & n \geq N \\ (n - N + \tau)\Delta z_{N-1} & N - \tau \leq n < N \\ 0 & n < N - \tau \end{cases} \]
and let
\[ x_n = z_{N-\tau} - \tau \Delta z_{N-1} + \sum_{i=0}^{\infty} H_{n-i}, \quad n \geq 0. \]
In view of the definition of \( H_n \), it is clear that \( 0 < x_n < \infty \) for all \( n \geq 0 \), that
\[ \max\{x_{N-\tau}, x_{N-\tau+1}, \ldots, x_{N-1}\} = z_{N-\tau} - \tau \Delta z_{N-1} + (\tau - 1)\Delta z_{N-1} \leq z_{N-\tau}, \]
and that
\[ x_n - x_{n-\tau} = H_n = \tau \Delta z_{n-1} \]
for \( n \geq N \). For any \( n \) which satisfies \( N \leq n \leq N + \tau - 1 \), we see that
\[ x_n = \tau \Delta z_{n-1} + x_{n-\tau} \leq \sum_{i=n-\tau}^{n-1} \Delta z_i + x_{n-\tau} \leq \sum_{i=N-\tau}^{n-1} \Delta z_i + z_{N-\tau} = z_n. \]
By induction, it is easy to prove that for any \( n \) which satisfies \( N + k \tau \leq n < N + (k + 1) \tau \) where \( k = 0, 1, 2, \ldots \), \( x_n \leq z_n \) is still valid. Thus
\[
\Delta(x_n - x_{n-}\tau) + q_n \max_{s \in [n-\sigma, n]} x_s \leq \tau \Delta^2 z_{n-1} + q_n z_n = 0
\]
as desired. Similarly, we have also
\[
x_n \leq \sum_{i=N-\tau}^{n-1} \Delta z_i + z_{N-\tau} \leq \sum_{i=N-\tau}^{n-1+\sigma} \Delta z_i + z_{N-\tau} = z_{n+\sigma}.
\]
Thus, we have
\[
\Delta(x_n - x_{n-\tau}) + q_n \min_{s \in [n-\sigma, n]} x_s \leq \tau \Delta^2 z_{n-1} + q_n \min_{s \in [n-\sigma, n]} z_{s+\sigma}
\]
\[
= \tau \Delta^2 z_{n-1} + q_n z_n = 0.
\]
The proof is complete.

As an important corollary, the equation
\[
\Delta(x_n - x_{n-\tau}) + \frac{\alpha}{(n+1)^2} \min_{s \in [n-\sigma, n]} x_s = 0, \quad n = 0, 1, 2, \ldots
\]
in view of Theorem 3, is oscillatory if, and only if, \( \alpha/\tau > 1/4 \).

We make several additional remarks. Let us say that a solution of (1) or (2) is strongly oscillatory if for any given nonnegative integer \( N \), there is a corresponding integer \( m \geq N \) such that \( x_m x_{m+1} < 0 \). Assume that \( q_n > 0 \) for \( n \geq 0 \). In this case, let \( \{x_n\} \) be a nonnegative solution of (1) or (2), and let \( y_n \) be defined by (4). We have already seen that \( y_n > 0 \) and \( \Delta y_n \leq 0 \) eventually. Note that \( x_n \geq y_n \) for all large \( n \).

**THEOREM 4.** Assume that there exists a nonnegative integer \( N \geq 0 \) such that \( p_{N+j\tau} \leq 1 \) for \( j = 0, 1, 2, \ldots \). Suppose further that \( q_n > 0 \) for \( n \geq 0 \). Then every nontrivial solution of equation (1) is strongly oscillatory if, and only if, the inequality
\[
\Delta(x_n - p_n x_{n-\tau}) + q_n x_{n-\sigma} \leq 0, \quad n = 0, 1, \ldots
\]
does not have any eventually nonnegative solutions, and equation (2) has a nonnegative solution if, and only if, (5) has a nonnegative solution.

**THEOREM 5.** Assume that there exists a nonnegative integer \( N \geq 0 \) such that \( p_{N+j\tau} \leq 1 \) for \( j = 0, 1, 2, \ldots \). Assume further that \( q_n > 0 \) for \( n \geq 0 \), If there exists some integer \( T \) such that the functional inequality
\[
\Delta y_n + q_n \max_{s \in [n-\sigma, n]} \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{s-i\tau} y_{n-\tau} \leq 0
\]
does not have an eventually positive solution, then equation (2) does not have an eventually nonnegative solution. If there exists some integer \( T \) such that the functional inequality
\[
\Delta y_n + q_n \sum_{j=0}^{T-1} \prod_{i=0}^{j} p_{n-i\tau} y_{n-\tau} \leq 0
\]
does not have an eventually positive solution, then every solution of equation (1) is strongly oscillatory.

**THEOREM 6.** Assume that $q_n > 0$ for $n \geq 0$. Then all solutions of the equation

$$\Delta (x_n - x_{n-\tau}) + q_n x_{n-\sigma} = 0,$$

are strongly oscillatory if, and only if, equation (8) is oscillatory; and equation (7) has an eventually nonnegative solution if, and only if, (8) has an eventually positive solution.

The proofs of Theorem 4, Theorem 5 and Theorem 6 are similar to the proofs of Theorem 1, Theorem 2 and Theorem 3 respectively. They will be omitted. Results analogous to Theorem 4, Theorem 5 and Theorem 6 are not valid for equation (3). For example, the sequence $\{\sin(n\pi/2) - 1\}$ is a nonpositive solution of the equation

$$\Delta (x_n - x_{n-4}) + q_n \max_{s \in [n-6,n]} x_s = 0,$$

where $\{q_n\}$ is any real sequence.

**References**


