Properties And Related Inequalities Of $\varphi$-frames In Normed Spaces

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Abstract

In this paper, we use the properties of sesquilinear forms to introduce a new class of frames, called $\varphi$-frames. The notion of continuous $\varphi$-frames, its various properties and characterizations in normed spaces are established. Also, some fundamental identities and certain inequalities related to $\varphi$-frames are obtained.

1 Notations and Preliminaries

The concept of frame in Hilbert spaces was introduced by Duffin and Schaeffer [14] to study some problems in non-harmonic Fourier series in 1952, reintroduced in 1986 by Daubechies, Grossmann, and Meyer [12] and popularized from then on. Now the theory of frames is widely studied by several authors and they have established a series of results (see [1, 4, 8, 9, 10]). A frame, which is redundant set of vectors in a Hilbert space $\mathcal{H}$ with the property that provides non unique representations of vectors in terms of the frame elements, has been applied in filter bank theory [6], sigma-delta quantization [5], signal and image processing [7] and many other fields. A frame for a complex Hilbert space $\mathcal{H}$ is a family of vectors $\{f_i\}_{i \in I}$ in $\mathcal{H}$ so that there are two positive constants $A$ and $B$ satisfying

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \ (f \in \mathcal{H}).$$

(1.1)

The constants $A$ and $B$ are called the lower and upper frame bounds, respectively. A frame is said to be tight whenever $A = B$ and if we can take $A = B = 1$ it is called a Parseval frame. If the right-hand inequality of (1.1) holds, then we say that $\{f_i\}_{i \in I}$

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is a Bessel sequence for $\mathcal{H}$ with bound $B$. The analytic operator associated to the frame $\{f_i\}_{i \in I}$ is defined as $T : L^2 \to \mathcal{H}$ by $T \{a_i\} = \sum_{i \in I} a_i f_i$. It is easy to see that $T^* : \mathcal{H} \to L^2$ such that $T^* (f) = \{\langle f, f_i \rangle \}_{i \in I}$. The frame operator for the frame is the positive, self-adjoint invertible operator $S = TT^* : \mathcal{H} \to \mathcal{H}$ satisfying

$$Sf = \sum_{i \in I} \langle f, f_i \rangle f_i, \quad (f \in \mathcal{H}).$$

This provides the frame decomposition

$$f = S^{-1}Sf = \sum_{i \in I} \langle f, f_i \rangle \tilde{f}_i = \sum_{i \in I} \langle f, \tilde{f}_i \rangle f_i,$$

where $\tilde{f}_i = S^{-1}f_i$. The family $\{\tilde{f}_i\}_{i \in I}$ is also a frame for $\mathcal{H}$, called the canonical dual frame of $\{f_i\}_{i \in I}$. If $\{f_i\}_{i \in I}$ is a Bessel sequence in $\mathcal{H}$, for every $J \subset I$ we define the operator $S_J$ by

$$S_Jf = \sum_{i \in J} \langle f, f_i \rangle f_i.$$

We refer to [9, 11, 18] for an introduction to the frame theory and its applications. In this section, we recall fundamental definitions, basic properties and notations of sesquilinear forms which are needed for a comprehensive reading of this paper. This background can be found in [13]. Let $\mathcal{E}$ be a vector space then $\varphi : \mathcal{E} \times \mathcal{E} \to \mathbb{C}$ is a sesquilinear form on $\mathcal{E}$ if the following two conditions hold:

(a) $\varphi (\alpha x_1 + \beta x_2, y) = \alpha \varphi (x_1, y) + \beta \varphi (x_2, y),$

(b) $\varphi (x, \alpha y_1 + \beta y_2) = \overline{\alpha} \varphi (x, y_1) + \overline{\beta} \varphi (x, y_2)$

for any scalars $\alpha$ and $\beta$ and any $x, x_1, x_2, y, y_1, y_2 \in \mathcal{E}$. Two typical examples of sesquilinear forms are as follows:

(I) Let $A$ and $B$ be operators on an inner product space $\mathcal{E}$. Then $\varphi_1 (x, y) = \langle Ax, y \rangle$, $\varphi_2 (x, y) = \langle x, By \rangle$, and $\varphi_3 (x, y) = \langle Ax, By \rangle$ are sesquilinear forms on $\mathcal{E}$.

(II) Let $f$ and $g$ be linear functionals on a vector space $\mathcal{E}$. Then $\varphi (x, y) = f (x) g (y)$ is a sesquilinear form on $\mathcal{E}$.

Let $\varphi$ be a sesquilinear form on vector space $\mathcal{E}$, then $\varphi$ is called symmetric if $\varphi (x, y) = \overline{\varphi (y, x)}$ for all $x, y \in \mathcal{E}$. A sesquilinear form $\varphi$ on vector space $\mathcal{E}$ is said to be positive if $\varphi (x, x) \geq 0$ for all $x \in \mathcal{E}$. Moreover, $\varphi$ is called Cauchy-Schwarz if $(\varphi (x, y))^2 \leq \varphi (x, x) \varphi (y, y)$ for each $x, y \in \mathcal{E}$. The corresponding quadratic form associated to $\varphi$ is defined as:

$$\Phi (x) = \varphi (x, x).$$

We remark that, if $\mathcal{E}$ be a normed space and $\varphi$ is a positive bounded sesquilinear form, then $\sqrt{\Phi (x)}$ defines a semi norm on $\mathcal{E}$ (see [16, p. 52]). Let $\mathcal{B} (\mathcal{E})$ denote the algebra
of all bounded linear operators on a complex vector space $\mathcal{E}$. For operator $A \in \mathcal{B}(\mathcal{E})$ there exist $B \in \mathcal{B}(\mathcal{E})$ such that for each $x$ and $y$ in $\mathcal{E}$

$$\varphi(Ax, y) = \varphi(x, By).$$

In this case, $B$ is $\varphi$-adjoint of $A$ and it is denoted by $A^*$. For more information on related ideas and concepts we refer [17, p. 88-90]. The operator $A$ in $\mathcal{B}(\mathcal{E})$ is called $\varphi$-positive if for all $x \in \mathcal{E}$, $\varphi(Ax, x) \geq 0$. We note that, $A \geq B$ if $A - B \geq 0$.

In this paper, we develop the existing notions of frames on Hilbert spaces by using the definition of sesquilinear form on a normed space $\mathcal{E}$. Section 2 is devoted to some elementary considerations concerning the $\varphi$-frames. Some properties and results of such frames are investigated. In Section 3, we derive some characterizations of continuous $\varphi$-frames. Finally, in the last section, we give new Parseval type identities and inequalities for $\varphi$-frames in normed spaces (see Corollary 4.1 and Proposition 4.1). Our results generalize the remarkable results obtained recently by Găvruţa.

## 2 $\varphi$-frames

The following basic results are essentially known as in [9], but our expression is a little bit different from those in [9]. In fact Hilbert space $\mathcal{H}$ and inner product $\langle \cdot, \cdot \rangle$ are replaced with vector space $\mathcal{E}$ and sesquilinear form $\varphi$ respectively. Recall that a sequence $\{e_k\}_{k=1}^m$ in a vector space $\mathcal{E}$ is a basis, if the following conditions are satisfied:

(a) $\mathcal{E} = \text{span} \{e_k\}_{k=1}^m$;
(b) $\{e_k\}_{k=1}^m$ is linearly independent.

As a consequence of above definition, every $f \in \mathcal{E}$ has a unique representation in terms of the elements in the basis, i.e., there exists unique scalar coefficients $\{c_k\}_{k=1}^m$ such that

$$f = \sum_{k=1}^m c_k e_k.$$ 

If $\{e_k\}_{k=1}^m$ is a $\varphi$-orthonormal basis, i.e., a basis for which

$$\varphi(e_k, e_j) = \delta_{k,j} = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j, \end{cases}$$

then the coefficients $\{c_k\}_{k=1}^m$ are easy to find

$$\varphi(f, e_j) = \varphi\left(\sum_{k=1}^m c_k e_k, e_j\right) = \sum_{k=1}^m c_k \varphi(e_k, e_j) = c_j.$$ 

So

$$f = \sum_{k=1}^m \varphi(f, e_k) e_k.$$
A sequence \( \{f_k\}_{k=1}^{\infty} \) in a vector space \( \mathcal{E} \) is called \( \varphi \)-frame if there exist \( A, B > 0 \) such that
\[
A \varphi(f, f) \leq \sum_{k=1}^{n} |\varphi(f, f_k)|^2 \leq B \varphi(f, f), \tag{2.1}
\]
for all \( f \in \mathcal{E} \). The constants \( A \) and \( B \) are called \( \varphi \)-frame bounds. If \( A = B \), this is a tight \( \varphi \)-frame and if \( A = B = 1 \) this is a Parseval \( \varphi \)-frame. Consider a vector space \( \mathcal{E} \) equipped with a frame \( \{f_k\}_{k=1}^{m} \) and define a linear mapping
\[
T : \mathbb{C}^m \to \mathcal{E}, \quad T \{c_k\}_{k=1}^{m} = \sum_{k=1}^{m} c_k f_k.
\]
\( T \) is called the \( \varphi \)-pre-frame operator. The adjoint operator is given by
\[
T^* : \mathcal{E} \to \mathbb{C}^m, \quad T^* f = \{\varphi(f, f_k)\}_{k=1}^{m}
\]
in fact by the usual inner product on \( \mathbb{C}^m \) as the sesquilinear form \( \varphi' \) we have
\[
\varphi(T x, y) = \varphi \left( \sum_{k=1}^{m} c_k f_k, y \right) = \sum_{k=1}^{m} c_k \varphi(f_k, y)
\]
and
\[
\varphi'(x, T^* y) = \varphi' \left( \{c_k\}_{k=1}^{m}, \{\varphi(y, f_k)\}_{k=1}^{m} \right) = \sum_{k=1}^{m} c_k \varphi(f_k, y).
\]
In this case, \( T^* \) is called the analytic operator and by composing \( T \) with its adjoint \( T^* \), we obtain the \( \varphi \)-frame operator
\[
S : \mathcal{E} \to \mathcal{E}, \quad S f = TT^* f = \sum_{k=1}^{m} \varphi(f, f_k) f_k.
\]
Note that in terms of the \( \varphi \)-frame operator,
\[
\varphi(T f, f) = \sum_{k=1}^{m} |\varphi(f, f_k)|^2, \quad f \in \mathcal{E}.
\]

**Remark 2.1.** Let \( \varphi \) be a Cauchy-Schwarz bounded sesquilinear form, then
\[
\sum_{k=1}^{m} |\varphi(f, f_k)|^2 \leq \sum_{k=1}^{m} \Phi(f_k) \Phi(f). \tag{2.2}
\]

**Proposition 2.1.** Let \( \{f_k\}_{k=1}^{m} \) be a sequence in \( \mathcal{E} \). Then \( \{f_k\}_{k=1}^{m} \) is a \( \varphi \)-frame for span \( \{f_k\}_{k=1}^{m} \).
PROOF. Assume that none of the $f_k$'s are zeros. From Remark 2.1, the upper $\varphi$-frame condition is satisfied with $B = \sum_{k=1}^{m} \Phi(f_k)$. Now let

$$W = \text{span} \{ f_k \}_{k=1}^{m}$$

and consider the continuous mapping

$$\psi : W \to \mathbb{R}, \quad \psi(f) = \sum_{k=1}^{m} |\varphi(f, f_k)|^2.$$ 

The unit ball in $W$ is compact since, $W$ is finite dimensional. So the function $\psi$ takes its infimum on the unit ball $W$. We can find $g \in W$ with $\sqrt{\Phi(g)} = 1$ such that

$$A = \sum_{k=1}^{m} |\varphi(g, f_k)|^2 = \inf \left\{ \sum_{k=1}^{m} |\varphi(f, f_k)|^2 : f \in W, \sqrt{\Phi(f)} = 1 \right\}.$$ 

It is clear that $A > 0$. Now for $f \in W, f \neq 0$, we have

$$\sum_{k=1}^{m} |\varphi(f, f_k)|^2 = \sum_{k=1}^{m} \varphi\left( \frac{f}{\sqrt{\Phi(f)}}, f_k \right)^2 |\Phi(f)| \geq A |\Phi(f)|.$$ 

COROLLARY 2.1. A family of elements $\{ f_k \}_{k=1}^{m}$ in $E$ is a $\varphi$-frame for $E$ if and only if $\text{span} \{ f_k \}_{k=1}^{m} = E$.

THEOREM 2.1. Let $\{ f_k \}_{k=1}^{m}$ be a $\varphi$-frame for $E$ with $\varphi$-frame operator $S$. Then

(a) $S$ is invertible and self adjoint.

(b) Every $f \in E$ can be represented as

$$f = \sum_{k=1}^{m} \varphi(f, S^{-1}f_k) f_k = \sum_{k=1}^{m} \varphi(f, f_k) S^{-1}f_k. \quad (2.3)$$

PROOF. Since $S = TT^*$, it is clear that $S$ is a self adjoint. We have to prove that $S$ is injective. Let $f \in E$ and assume that $Sf = 0$. Then

$$0 = \varphi(Sf, f) = \sum_{k=1}^{m} |\varphi(f, f_k)|^2,$$

by the $\varphi$-frame condition $f = 0$. $S$ is injective implies that $S$ is surjective, but let us give direct proof. By Corollary 2.1, the $\varphi$-frame condition implies that $\text{span} \{ f_k \}_{k=1}^{m} = E$, so the $\varphi$-pre frame operator $T$ is surjective. For $f \in E$ we can find $g \in E$ such that
\( Tg = f \). We can choose \( g \in N_{\frac{T}{T^*}} = R_{T^*} \), so it follows that \( R_S = R_{TT^*} = \mathcal{E} \). Thus \( S \) is surjective. Each \( f \in \mathcal{E} \) has the representation

\[
\sum_{k=1}^{m} \varphi(S^{-1}f, f_k) f_k.
\]

Since \( S \) is self adjoint, we get

\[
\sum_{k=1}^{m} \varphi(f, S^{-1}f_k) f_k.
\]

The second representation in (2.3) is obtained in the same way, hence \( f = S^{-1}Sf \).

**THEOREM 2.2.** Let \( \{f_k\}_{k=1}^{m} \) be a \( \varphi \)-frame for \( \mathcal{E} \) with \( \varphi \)-frame operator \( T \). Then if \( f \in \mathcal{E} \) also has the representation \( f = \sum_{k=1}^{m} c_k f_k \) for some scalar coefficients \( \{c_k\}_{k=1}^{m} \), then

\[
\sum_{k=1}^{m} |c_k|^2 = \sum_{k=1}^{m} |\varphi(f, T^{-1}f_k)|^2 + \sum_{k=1}^{m} |c_k + \varphi(f, T^{-1}f_k)|^2. \tag{2.4}
\]

**PROOF.** Suppose that \( f = \sum_{k=1}^{m} c_k f_k \). We can write

\[
\{c_k\}_{k=1}^{m} = \{c_k\}_{k=1}^{m} - \{\varphi(f, T^{-1}f_k)\}_{k=1}^{m} + \{\varphi(f, T^{-1}f_k)\}_{k=1}^{m}.
\]

By the choice of \( \{c_k\}_{k=1}^{m} \) we have

\[
\sum_{k=1}^{m} (c_k - \varphi(f, T^{-1}f_k)) f_k = 0
\]

i.e.,

\[
\{c_k\}_{k=1}^{m} - \{\varphi(f, T^{-1}f_k)\}_{k=1}^{m} \in N_S = R_{\frac{T}{T^*}},
\]

since

\[
\{\varphi(f, T^{-1}f_k)\}_{k=1}^{m} = \{\varphi(T^{-1}f, f_k)\}_{k=1}^{m} \in R_S.
\]

we obtain (2.4).

**REMARK 2.2.** If \( \{f_k\}_{k=1}^{m} \) is a \( \varphi \)-frame but not a basis, there exist non zero sequences \( \{d_k\}_{k=1}^{m} \) such that \( \sum_{k=1}^{m} d_k f_k = 0 \). Therefore \( f \in \mathcal{E} \) can be written

\[
f = \sum_{k=1}^{m} \varphi(f, T^{-1}f_k) f_k + \sum_{k=1}^{m} d_k f_k
\]

and

\[
= \sum_{k=1}^{m} (\varphi(f, T^{-1}f_k) + d_k) f_k
\]
showing that \( f \) has many representations as superpositions of the \( \varphi \)-frame elements.

**Proposition 2.2.** Let \( \{f_k\}_{k=1}^m \) be a basis for \( \mathcal{E} \). Then there exists a unique family \( \{g_k\}_{k=1}^m \) in \( \mathcal{E} \) such that

\[
f = \sum_{k=1}^m \varphi(f, g_k) f_k, \quad (\forall f \in \mathcal{E}).
\] (2.5)

**Proof.** The existence of a family \( \{g_k\}_{k=1}^m \) satisfying (2.5) follows from Theorem 2.1, also the uniqueness part is direct.

**Remark 2.3.** Applying (2.5) on a fixed element \( f_j \) and since \( \{f_k\}_{k=1}^m \) is a basis, we get \( \varphi(f_j, g_k) = \delta_{j,k} \) for all \( k = 1, 2, \ldots, m \).

**Theorem 2.3.** Let \( \{f_k\}_{k=1}^m \) be a \( \varphi \)-frame for subspace \( F \) of the vector space \( \mathcal{E} \). Then the \( \varphi \)-orthogonal projection of \( \mathcal{E} \) onto \( F \) is given by

\[
P_f = \sum_{k=1}^m \varphi(f, T^{-1} f_k) f_k.
\] (2.6)

**Proof.** It is enough to prove that if we define \( P \) by (2.6), then

\[
P_f = f \quad \text{for} \quad f \in F \quad \text{and} \quad P_f = 0 \quad \text{for} \quad f \in F^\perp.
\]

The first equation follows by Theorem 2.1, and the second by the fact that the range of \( T^{-1} \) equals \( F \) because \( T \) is a bijection on \( F \).

### 3 Continuous \( \varphi \)-Frames

In this section, we introduce the concept of continuous \( \varphi \)-frames, which is a partial extension of continuous frames. To prove our main result related to continuous \( \varphi \)-frames, we need the following essential definitions. Let \( I \) be a locally compact group, and \( \mathcal{E} \) be a vector space, and \( \varphi \) be a sesquilinear form on \( \mathcal{E} \). A function

\[
f : I \to \mathcal{E}
\]

is called a continuous \( \varphi \)-frame in \( \mathcal{E} \), if there are positive numbers \( A, B \), such that for all \( x \) in \( \mathcal{E} \)

\[
A \varphi(x, x) \leq \int_I |\varphi(x, f_i)|^2 \, di \leq B \varphi(x, x),
\] (3.1)

where \( di \) is a Haar measure on \( I \). The constants \( A \) and \( B \) are called the frame bounds. In this case, we define the corresponding frame operator as \( S : I \to I \) such that

\[
S(x) = \int_I \varphi(x, f_i) \, di.
\] (3.2)
Moreover, we can define the analysis operator as this $T : \mathcal{E} \to L^2(I)$ such that

$$x \to (\varphi(x, f_i))_{i \in I}. \tag{3.3}$$

The notation $(\varphi(x, f_i))_{i \in I}$ in (3.3) denotes the function in $L^2(I)$

$$i \to (\varphi(x, f_i))_{i \in I}.$$

It easy to prove that $T^* : L^2(I) \to \mathcal{E}$ which

$$g \to \int_I f_i g_i \, di,$$

and it implies that

$$S = T^* T.$$

**THEOREM 3.1.** Let $I$ be a locally compact group, $\varphi$ be a symmetric sesquilinear form on a vector space $\mathcal{E}$, and let $f : I \to \mathcal{E}$ be a $\varphi$-frame in $\mathcal{E}$, with frame bounds $A$ and $B$. Then the operator $S$ is a positive, self adjoint, invertible operator on $\mathcal{E}$, moreover

$$AI_{\mathcal{E}} \leq S \leq BI_{\mathcal{E}}.$$ 

**PROOF.** By definition, we can write

$$\varphi(Sx, x) = \varphi \left( \int_I \varphi(x, f_i) f_i \, di, x \right) = \int_I \varphi(\varphi(x, f_i) f_i, x) \, di$$

$$= \int_I \varphi(x, f_i) \varphi(f_i, x) \, di$$

$$= \int_I \varphi(x, f_i) \overline{\varphi(x, f_i)} \, di$$

$$= \int_I |\varphi(x, f_i)|^2 \, di.$$

Therefore from definition of frame bounds, we conclude that

$$A \varphi(x, x) \leq \varphi(Sx, x) \leq B \varphi(x, x)$$

which is equivalent to

$$AI_{\mathcal{E}} \leq S \leq BI_{\mathcal{E}}.$$ 

**EXAMPLE 3.1.** Let $I$ be the positive real number, and $\mathcal{E}$ be $L^2(R)$. Define $f : R^+ \to L^2(R)$ which

$$\alpha \to f_\alpha$$
where

\[ f_\alpha(x) = e^{2\pi i \alpha x}. \]

Then it easy to show that the frame operator corresponding to the inner product of \( L^2(\mathbb{R}) \) is the identity on \( \mathcal{E} \). In other words, for any function \( f \)

\[ f = \int_0^{+\infty} \varphi(f, f_\alpha) f_\alpha d\alpha \]

or equivalently

\[ f(x) = \int_0^{+\infty} \left( \int_{-\infty}^{+\infty} f(x) \overline{f_\alpha(x)} dx \right) f_\alpha(x) d\alpha \]

or

\[ f(x) = \int_0^{+\infty} \left( \int_{-\infty}^{+\infty} f(x) e^{-2\pi i \alpha x} dx \right) e^{2\pi i \alpha x} d\alpha. \]

This is the Fourier integral for the function \( f \).

**EXAMPLE 3.2.** In the previous, let \( I \) be the set of all positive integers, then we have

\[ f = \sum_{0}^{\infty} \varphi(f, f_n) f_n \]

or

\[ f(x) = \sum_{0}^{\infty} \left( \int_{-\infty}^{+\infty} f(x) e^{-2\pi i \alpha x} dx \right) e^{2\pi i \alpha x} d\alpha. \]

which is the Fourier series for the function \( f \).

Example 3.2 shows that the Fourier system is a continuous \( \varphi \)-frame, which has a discrete sub frame, but not in a same measure.

**REMARK 3.1.** In general, it is not necessary for \( I \) to be a group, it is enough that \( I \) is a subset of a locally compact group with a suitable measure. As we see in the examples, it is important to define an integral or summation on \( I \).

4 Applications

As an application of previous sections, we prove the following inequalities and by using the model technique of Balan et al. [2, 3] and Gavruta [15], we obtain an analogue, called Parseval’s identity of \( \varphi \)-frames in normed spaces.

**THEOREM 4.1.** Let \( \{f_i\}_{i \in I} \) be a \( \varphi \)-frame for a vector space \( \mathcal{E} \) with frame bounds \( A, B \). Let \( J \subset I \), so that \( \{f_i\}_{i \in J} \) has Bessel bound \( B(J) < A \). Then \( \{f_i\}_{i \in J} \) is a \( \varphi \)-frame for \( \mathcal{E} \).
Properties and Related Inequalities Of $\varphi$-frames in Normed Spaces

PROOF. Since $\{f_i\}_{i \in J^c}$ has $B$ as a Bessel bound, we only need to check its lower frame bound. For this just compute for any $f \in \mathcal{E}$

$$\sum_{i \in J^c} |\varphi(f, f_i)|^2 = \sum_{i \in I} |\varphi(f, f_i)|^2 - \sum_{i \in J} |\varphi(f, f_i)|^2$$

$$\geq A\Phi(f) - B(J)\Phi(f) = (A - B(J))\Phi(f).$$

Since $A - B(J) > 0$, we deduce the desired result.

COROLLARY 4.1. Let $\{f_i\}_{i \in I}$ be a Parseval $\varphi$-frame for $\mathcal{E}$ and $J \subset I$. In order for $\{f_i\}_{i \in J}$ to be a $\varphi$-frame for $\mathcal{E}$ is necessary and sufficient that $B(J^c) < 1$. In this case, the optimal lower frame bound for $\{f_i\}_{i \in J}$ is $1 - B(J^c)$.

PROOF. For any $f \in \mathcal{E}$ we have

$$\sum_{i \in J} |\varphi(f, f_i)|^2 = \sum_{i \in I} |\varphi(f, f_i)|^2 - \sum_{i \in J^c} |\varphi(f, f_i)|^2$$

$$\geq \Phi(f) - B(J^c)\Phi(f) = (1 - B(J^c))\Phi(f).$$

It is easy to see that the inequality above is optimal, hence the proof.

The following result can be stated as well.

THEOREM 4.2. Assume that $\varphi$ is a bounded positive sesquilinear form. If $U, V \in \mathcal{L}(\mathcal{E})$ are $\varphi$-self adjoint operators satisfying $U + V = 1_\mathcal{E}$, then for all $f \in \mathcal{E}$ we have

$$\varphi(Uf, f) + \Phi(Vf) = \varphi(Vf, f) + \Phi(Vf) \geq \frac{3}{4}\Phi(f).$$

PROOF. We have

$$\varphi(Uf, f) + \Phi(Vf) = \varphi(Uf, f) + \varphi(Vf, Vf)$$

$$= \varphi((I_\mathcal{E} - V)f, f) + \varphi(V^2f, f)$$

$$= \varphi((V^2 - V + I_\mathcal{E})f, f)$$

$$= \varphi(Vf, f) + \varphi(Uf, Uf) + \varphi((I_\mathcal{E} - V)^2f, f)$$

$$= \varphi((V^2f - V + I_\mathcal{E})f, f)$$

$$= \varphi \left( \left( V - \frac{1}{2}I_\mathcal{E} \right)^2 + \frac{3}{4}I_\mathcal{E} \right) f, f \right)$$

$$\geq \frac{3}{4}\Phi(f).$$

This completes the proof of Theorem 4.2.

REMARK 4.1. We consider now $\{f_i\}_{i \in I}$, a $\varphi$-frame for $\mathcal{E}$ with $S$ its frame operator and $\{\tilde{f}_i\}_{i \in I}$ its canonical dual frame and $J \subset I$. We have

$$S_J + S_{J^c} = S,$$
hence
\[ S^{-\frac{1}{2}} S J S^{-\frac{1}{2}} + S^{-\frac{1}{2}} S J f S^{-\frac{1}{2}} = 1_{\mathcal{F}}. \]

PROOF. If in the Theorem 4.2 we take \( U = S^{-\frac{1}{2}} S J S^{-\frac{1}{2}} \), \( V = S^{-\frac{1}{2}} S J f S^{-\frac{1}{2}} \) and \( S f \) instead of \( f \), we get
\[ \varphi \left( S^{-\frac{1}{2}} S J f, S f \right) + \Phi \left( S^{-\frac{1}{2}} S J f, S f \right) = \varphi \left( S^{-\frac{1}{2}} S J f, S f \right) \]
\[ \geq \frac{3}{4} \Phi \left( S f \right), \]
or
\[ \varphi \left( S J f, f \right) + \varphi \left( S^{-\frac{1}{2}} S J f, S^{-\frac{1}{2}} S J f \right) = \varphi \left( S J f, f \right) + \varphi \left( S^{-1} S J f, S J f \right) \]
\[ \geq \frac{3}{4} \varphi \left( S f, f \right). \]

The following result also holds (see [15, Theorem 3.2] for the case of Hilbert space).

**Theorem 4.3.** Let \( \{f_i\}_{i \in I} \) be a \( \varphi \)-frame for \( \mathcal{F} \) and \( \{g_i\}_{i \in I} \) be an alternative dual of \( \{f_i\}_{i \in I} \). Then for all \( J \subset I \) and all \( f \in \mathcal{F} \), we have
\[ \Re \sum_{i \in J} \varphi(f, g_i) \overline{\varphi(f, f_i)} + \Phi \left( \sum_{i \in J} \varphi(f, g_i) f_i \right) \]
\[ = \Re \sum_{i \in J} \varphi(f, g_i) \overline{\varphi(f, f_i)} + \Phi \left( \sum_{i \in J} \varphi(f, g_i) f_i \right) \]
\[ \geq \frac{3}{4} \Phi (f). \]

PROOF. For every \( J \subset I \) we define the operator \( L_J \) by
\[ L_J f = \sum_{i \in J} \varphi(f, g_i) f_i. \]
By the Cauchy-Schwarz inequality it follows that this series converges unconditionally and \( L_J \in L(\mathcal{F}) \). Since \( L_J + L_{J^c} = I_\mathcal{F} \),
\[ \varphi ((L_J L_J) f, f) + \frac{1}{2} \varphi ((L_J, L_{J^c}) f, f) = \varphi ((L_J L_{J^c}) f, f) + \frac{1}{2} \varphi ((L_J + L_{J^c}) f, f) \]
\[ \geq \frac{3}{4} \Phi (f), \]
or
\[ \Phi \left( \sum_{i \in J} \varphi(f, g_i) f_i \right) + \frac{1}{2} \left( \varphi(L_{J^c} f, f) + \varphi(L_J f, f) \right) \]
\[ = \Phi \left( \sum_{i \in J^c} \varphi(f, g_i) f_i \right) + \frac{1}{2} \left( \varphi(L_J f, f) + \varphi(L_J f, f) \right) \]
\[ \geq \frac{3}{4} \Phi (f). \]
To prove Theorem 4.4, we need the following lemma.

**Lemma 4.1.** If $S, T$ are operators on $\mathcal{E}$ satisfying $S + T = I$, then $S - T = S^2 - T^2$.

**Proof.** Easy computation and simplification yield

$$S - T = S - (I - S) = 2S - I = S^2 - (I - 2S + S^2) = S^2 - (I - S)^2 = S^2 - T^2.$$  

**Theorem 4.4.** Let $\{f_i\}_{i \in I}$ be a $\varphi$-frame for $\mathcal{E}$ with canonical frame $\{\tilde{f}_i\}_{i \in I}$. Then for all $J \subset I$ and for all $f \in \mathcal{E}$ we have

$$\sum_{i \in J} |\varphi(f, f_i)|^2 - \sum_{i \in I} |\varphi(S_J f, \tilde{f}_i)|^2 = \sum_{i \in J^c} |\varphi(f, f_i)|^2 - \sum_{i \in I} |\varphi(S_{J^c} f, \tilde{f}_i)|^2.$$  

**Proof.** Let $S$ denote the frame operator for $\{f_i\}_{i \in I}$. Since $S = S_J + S_{J^c}$, it follows that $I = S^{-1}S_J + S^{-1}S_{J^c}$. Applying Lemma 4.1 to the two operators $S^{-1}S_J$ and $S^{-1}S_{J^c}$ yields

$$S^{-1}S_J - S^{-1}S_JS^{-1}S_J = S^{-1}S_{J^c}S^{-1}S_{J^c}. \quad (4.1)$$

Further, for every $f, g \in \mathcal{E}$ we obtain

$$\varphi(S^{-1}S_J f, g) - \varphi(S^{-1}S_J S^{-1}S_J f, g) = \varphi(S_J f, S^{-1}g) - \varphi(S^{-1}S_J f, S_J S^{-1}g). \quad (4.2)$$

Now, we choose $g$ to be $g = Sf$. Then we can continue the equality (4.2) in the following as

$$\varphi(S_J f, f) - \varphi(S^{-1}S_J f, S_J f) = \sum_{i \in I} |\varphi(f, f_i)|^2 - \sum_{i \in I} |\varphi(S_J f, \tilde{f}_i)|^2.$$  

Setting equality (4.2) equal to the corresponding equality for $J^c$ and using (4.1), we finally get

$$\sum_{i \in J} |\varphi(f, f_i)|^2 - \sum_{i \in I} |\varphi(S_J f, \tilde{f}_i)|^2 = \sum_{i \in J^c} |\varphi(f, f_i)|^2 - \sum_{i \in I} |\varphi(S_{J^c} f, \tilde{f}_i)|^2.$$  

**Proposition 4.1.** Let $\{f_i\}_{i \in I}$ be a Parseval $\varphi$-frame for $\mathcal{E}$. For every subset $J \subset I$ and every $f \in \mathcal{E}$, we have

$$\sum_{i \in J} |\varphi(f, f_i)|^2 - \Phi(\varphi(f, f_i) f_i) = \sum_{i \in J^c} |\varphi(f, f_i)|^2 - \Phi\left(\sum_{i \in J^c} \varphi(f, f_i) f_i\right).$$
PROOF. Let \( \{ \tilde{f}_i \}_{i \in I} \) denote the dual frame of \( \{ f_i \}_{i \in I} \). Since \( \{ f_i \}_{i \in I} \) is a Parseval \( \phi \)-frame, its frame operator equal identity operator and hence \( \tilde{f}_i = f_i \) for all \( i \in I \).

Employing Theorem 4.4 and the fact that \( \{ f_i \}_{i \in I} \) is a Parseval \( \phi \)-frame yields

\[
\sum_{i \in J} |\varphi (f, f_i)|^2 - \Phi \left( \sum_{i \in J} \varphi (f, f_i) f_i \right) = \sum_{i \in J} |\varphi (f, f_i)|^2 - \Phi (S_J f) \\
= \sum_{i \in J} |\varphi (f, f_i)|^2 - \sum_{i \in I} |\varphi (S_J f, f_i)|^2 \\
= \sum_{i \in J} |\varphi (f, f_i)|^2 - \sum_{i \in I} |\varphi (S_J f, \tilde{f}_i)|^2 \\
= \sum_{i \in J^c} |\varphi (f, f_i)|^2 - \sum_{i \in I} |\varphi (S_J f, \tilde{f}_i)|^2 \\
= \sum_{i \in J^c} |\varphi (f, f_i)|^2 - \Phi (S_J f) \\
= \sum_{i \in J^c} |\varphi (f, f_i)|^2 - \Phi \left( \sum_{i \in J^c} \varphi (f, f_i) f_i \right).
\]

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References


