Graph Convergence For $\eta$-Subdifferential Mapping With Application *

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Abstract

In this paper, we introduce the concept of graph convergence for $\eta$-subdifferential mapping of a nonconvex, proper, lower semi-continuous and subdifferential functional on Banach space and discuss its existence and Lipschitz continuity. Further, we prove equivalence between graph convergence and resolvent operator convergence. We propose a new iterative algorithm for solving the system of generalized implicit variational-like inclusions. Furthermore, we prove the existence of solution for the system of generalized implicit variational-like inclusions and discuss the convergence of iterative sequences generated by proposed algorithm.

1 Introduction

Variational inequality theory has become a very effective and powerful tool in pure and applied sciences and has been used in a large class of problems arising in differential equations, mechanics, optimization and control, contact problems in elasticity and general equilibrium problems, see, [1, 3, 5, 8, 9, 10, 14, 15, 16]. Variational inclusion is an important and useful generalization of the variational inequality. One of the most important and interesting problem in the theory of variational inequality is the development of an efficient and implementable iterative algorithm for solving the variational inequalities. Variational inclusions include variational, quasi-variational, variational-like inequalities as special cases. In 1994, Hassouni and Moudafi [17] introduced and studied a class of variational inclusions. Later, Adly [1], Huang [18], Ding [8, 11], Ding and Luo [9] and Ding and Feng [12] have obtained some important generalizations of the results in [17].

Recently, many authors have studied the perturbed algorithms for variational inequalities involving monotone mappings in Hilbert spaces. Using the concept of graph convergence for maximal monotone mappings, Attouch [2] showed the equivalence between graph convergence and resolvent operator convergence, they constructed some perturbed algorithm for variational inequality and proved the convergence of sequences generated by perturbed algorithm under some suitable conditions. Further Li and Huang [25] generalized the concept of graph convergence for $H(\cdot,\cdot)$-accretive mapping in Banach space.

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In recent past, Ding and Xia [13] introduced the concept of $P$-proximal mapping for a nonconvex, proper, lower semi-continuous and subdifferentiable functional on Banach space and prove the existence and Lipschitz continuity. Sun et al. [28], Kazmi and Bhat [21] and Kazmi et al. [22, 23] generalized the concept of $M$-proximal mappings.

Motivated and inspired by the research works going on in this direction, in this paper, we introduce a new concept of graph convergence for $\eta$-subdifferential mapping of a nonconvex, proper, lower semi-continuous and subdifferentiable functional on Banach space and shown its existence and Lipschitz continuity. Further, we prove equivalence between graph convergence and resolvent operator convergence. We propose a new iterative algorithm for solving the system of generalized implicit variational-like inclusions. Furthermore, we prove the existence of the solution for the system of generalized implicit variational-like inclusions and discuss the convergence of iterative sequences generated by proposed algorithm.

2 Preliminaries

Let $E$ be a real Banach space equipped with norm $\| \cdot \|$, $E^{*}$ be the topological dual of $E$ and $\langle \cdot, \cdot \rangle$, be the duality pairing between $E$ and $E^{*}$. Let $2^{E}$, (respectively, $CB(E)$) be the family of all nonempty (respectively, closed and bounded) subsets of $E$, let $D(\cdot, \cdot)$ be the Hausdorff metric on $CB(E)$ defined by

$$
D(A, B) = \max \left\{ \sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y) \right\},
$$

where

$$
A, B \in CB(E), \ d(x, B) = \inf_{y \in B} d(x, y) \quad \text{and} \quad d(A, y) = \inf_{x \in A} d(x, y).
$$

The normalized duality mapping $J : E \rightarrow 2^{E^{*}}$ is defined by

$$
J(x) = \left\{ f \in E^{*} : \langle x, f \rangle = \|x\|^{2}, \ \|f\|_{E^{*}} = \|x\| \right\}, \ \forall x \in E.
$$

It is well known that if $E$ is smooth, then $J$ is single-valued and if $E \equiv H$, a Hilbert space, then $J$ is the identity mapping.

DEFINITION 2.1 ([7]). A Banach space $E$ is called smooth, if for every $x \in E$ with $\|x\| = 1$, there exists a unique $f \in E^{*}$ such that $\|f\| = f(x) = 1$. The modulus of smoothness of $E$ is the function $\rho_{E} : [0, \infty) \rightarrow [0, \infty)$ defined by

$$
\rho_{E}(t) = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} - 1 : \|x\| \leq 1 \ \text{and} \ \|y\| \leq t \right\}.
$$

A Banach space $E$ is called uniformly smooth, if

$$
\lim_{t \rightarrow 0} \frac{\rho_{E}(t)}{t} = 0.
$$

LEMMA 2.1 ([4]). Let $E$ be a uniformly smooth Banach space and $J : E \rightarrow E^{*}$ be the normalized duality mapping. Then for all $x, y \in E$, we have
(i) \( \|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle \);

(ii) \( \langle x - y, J(x) - J(y) \rangle \leq 2d^2 \rho_E \left( \frac{2\|x-y\|}{d} \right) \), where \( d = \sqrt{\|x\|^2 + \|y\|^2}/2 \).

**LEMMA 2.2** ([26]). Let \( E \) be a complete metric space with metric \( d \), and let \( T : E \to CB(E) \) be a multi-valued mapping. Then for any \( \epsilon > 0 \) and for any \( x, y \in E \), \( u \in T(x) \), there exists \( v \in T(y) \) such that \( d(u, v) \leq D(Tx, Ty) \).

**LEMMA 2.3** ([27]). Let \( E \) be a real Banach space and \( J : E \to 2^{E^*} \) be the normalized duality mapping. Then for any \( x, y \in E \),

\[ \|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y) \rangle, \forall j(x + y) \in J(x + y). \]

**DEFINITION 2.3** ([31]). A functional \( f : E \times E \to \mathbb{R} \cup \{+\infty\} \) is said to be 0-diagonally quasi-concave (in short, 0-DQCV) in \( x \), if for any finite set \( \{x_1, x_2, \ldots, x_n\} \subset E \) and for any \( y = \sum_{i=1}^{n} \lambda_i x_i \) with \( \lambda_i \geq 0 \) and \( \sum_{i=1}^{n} \lambda_i = 1 \), \( \min_{1 \leq i \leq n} f(x_i, y) \leq 0 \) holds.

**DEFINITION 2.4** ([8]). Let \( \eta : E \times E \to \mathbb{R} \) is a single-valued mapping. A proper functional \( \phi : E \to \mathbb{R} \cup \{+\infty\} \) is said to be \( \eta \)-subdifferentiable at point \( x \in E \) if there exists a point \( f^* \in E^* \) such that

\[ \phi(y) - \phi(x) \geq \langle f^*, \eta(y, x) \rangle, \forall y \in E, \]

where \( f^* \) is called \( \eta \)-subgradient of \( \phi \) at \( x \). The set of all \( \eta \)-subgradients of \( \phi \) at \( x \) is denoted by \( \partial \phi(x) \). The mapping \( \partial \phi : E \to 2^{E^*} \) is defined by

\[ \partial \phi(x) = \{ f^* \in E^* : \phi(y) - \phi(x) \geq \langle f^*, \eta(y, x) \rangle, \forall y \in E \} \]

is said to be \( \eta \)-subdifferential of \( \phi \) at \( x \).

**DEFINITION 2.5.** Let \( \eta : E \times E \to \mathbb{R} \) and \( A, B : E \to E \) be single-valued mappings and let \( M : E \times E \to E^* \) be a nonlinear mapping. Then

(i) \( M(A, \cdot) \) is said to be \( \alpha \)-strongly \( \eta \)-monotone with respect to \( A \) if there exists a constant \( \alpha > 0 \) such that

\[ \langle M(Ax, u) - M(Ay, u), \eta(x, y) \rangle \geq \alpha \|x - y\|^2, \forall x, y, u \in E; \]

(ii) \( M(\cdot, B) \) is said to be \( \beta \)-relaxed \( \eta \)-monotone with respect to \( B \) if there exists a constant \( \beta > 0 \) such that

\[ \langle M(u, Bx) - M(u, By), \eta(x, y) \rangle \geq (-\beta) \|x - y\|^2, \forall x, y, u \in E; \]

(iii) \( M(A, B) \) is said to be \( \alpha \beta \)-symmetric \( \eta \)-monotone with respect to \( A \) and \( B \) if \( M(A, \cdot) \) is \( \alpha \)-strongly \( \eta \)-monotone with respect to \( A \) and \( M(\cdot, B) \) is \( \beta \)-relaxed \( \eta \)-monotone with respect to \( B \);
(iv) $M(\cdot,\cdot)$ is said to be $(\xi_1,\xi_2)$-mixed Lipschitz continuous if there exist constants $\xi_1, \xi_2 > 0$ satisfying

$$\|M(x,u) - M(y,v)\| \leq \xi_1\|x-y\| + \xi_2\|u-v\|, \forall x,y,u,v \in E.$$ 

DEFINITION 2.6. Let $\eta : E \times E \to E$ and $A, B : E \to E$ be single-valued mappings. Let $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and $\eta$-subdifferentiable (may not be convex) functional and $M : E \times E \to E^*$ be a nonlinear mapping. If for any given point $x^* \in E^*$ and $\rho > 0$, there exists a unique point $x \in E$ satisfying

$$\langle M(Ax,Bx) - x^*, \eta(y,x) \rangle + \rho \phi(y) - \rho \phi(x) \geq 0, \forall y \in E,$$

then the mapping $x^* \to x$, denoted by $R_{\rho,\eta}^{\phi}(x^*)$ is called resolvent operator of $\phi$. Then, we have $x^* - M(Ax,Bx) \in \partial \phi(x)$ and it follows that $R_{\rho,\eta}^{\phi}(x^*) = [M(A,B) + \rho \partial \phi]^{-1}(x^*)$.

LEMMA 2.4 ([23]). Let $E$ be a reflexive Banach space. Let $\eta : E \times E \to E$ be a continuous mapping such that $\eta(y,y') + \eta(y',y) = 0$ for all $y, y' \in E$; $M : E \times E \to E^*$ be $\alpha\beta$-symmetric $\eta$-monotone continuous with respect to $A$ and $B$; let for any $x^* \in E^*$, the function $h(y,x) = \langle x^* - M(Ax,Bx), \eta(y,x) \rangle$ be 0-DQC in $y$ and $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and $\eta$-subdifferentiable (may not be convex) functional. Then for any given constant $\rho > 0$ and $x^* \in E^*$, there exists a unique $x \in E$ such that

$$\langle M(Ax,Bx) - x^*, \eta(y,x) \rangle \geq \rho \phi(x) - \rho \phi(y), \forall y \in E,$$  

that is, $x = R_{\rho,\eta}^{\phi}(x^*)$.

LEMMA 2.5 ([23]). Let $\eta : E \times E \to E$ be $\tau$-Lipschitz continuous such that $\eta(y,y') + \eta(y',y) = 0$ for all $y, y' \in E$; $M : E \times E \to E^*$ be $\alpha\beta$-symmetric $\eta$-monotone continuous with respect to $A$ and $B$; let for any $x^* \in E^*$, the function $h(y,x) = \langle x^* - M(Ax,Bx), \eta(y,x) \rangle$ be 0-DQC in $y$ and $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and $\eta$-subdifferentiable functional and let $\rho > 0$ be any given constant. Then the resolvent operator $R_{\rho,M(\cdot,\cdot)}^{\phi}(\cdot)$ of $\phi$ is $\frac{\tau}{\alpha - \beta}$-Lipschitz continuous, that is, for any $x_1^*, x_2^* \in E^*$,

$$\|R_{\rho,M(\cdot,\cdot)}^{\phi}(x_1^*) - R_{\rho,M(\cdot,\cdot)}^{\phi}(x_2^*)\| \leq \frac{\tau}{\alpha - \beta}\|x_1^* - x_2^*\|.$$ 

3 Graph Convergence for $\eta$-Subdifferential Mapping

Let $\eta : E \times E \to E$ be a single-valued mapping. Let $\phi : E \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and $\eta$-subdifferentiable (may not be convex) functional and let $\partial \phi : E \to 2E^*$ be a $\eta$-subdifferential mapping of $\phi$. The graph of the $\eta$-subdifferential mapping $\partial \phi$ is defined by

$$\text{graph}(\partial \phi) = \{(x,y^*) \in E \times E^* : y^* \in \partial \phi(x)\}.$$
In this section, we introduce the notion of graph convergence for \( \nu \)-subdifferential mapping.

**DEFINITION 3.1.** Let \( \eta : E \times E \to E; A, B : E \to E \) be single-valued mappings. Let \( \phi : E \to \mathbb{R} \cup \{+\infty\} \) be a proper, lower semicontinuous and \( \eta \)-subdifferentiable (may not be convex) functional; let \( M : E \times E \to E^* \) be a nonlinear mapping. Let \( \partial \phi_n, \partial \phi : E \to 2^{E^*} \) be the \( \eta \)-subdifferential mappings of \( \phi \) for \( n = 0, 1, 2, \ldots \). The sequence \( \{\partial \phi_n\} \) is said to be graph convergence to \( \partial \phi \), denoted by \( \partial \phi_n \rightarrow G \partial \phi \), if for every \( (x, y^*) \in \text{graph}(\partial \phi) \) there exists a sequence \( (x_n, y_n^*) \in \text{graph}(\partial \phi_n) \) such that

\[
x_n \to x, \quad y_n^* \to y^* \quad \text{as} \quad n \to \infty.
\]

**THEOREM 3.1.** Let \( \eta : E \times E \to E \) be \( \tau \)-Lipschitz continuous such that \( \eta(y, y') + \eta(y', y) = 0 \) for all \( y, y' \in E \); let \( M : E \times E \to E^* \) be \( \alpha \beta \)-symmetric \( \eta \)-monotone continuous with respect to \( A \) and \( B \) such that \( M \) is \( \gamma_1 \)-Lipschitz continuous with respect to \( A \) and \( \gamma_2 \)-Lipschitz continuous with respect to \( B \). Let for any \( x^* \in E^* \), the function \( h(y, x) = (x^* - M(AX, BX), \eta(y, x)) \) be 0-DQC\( \nu \) in \( y \) and let \( \phi : E \to \mathbb{R} \cup \{+\infty\} \) be a proper, lower semicontinuous and \( \eta \)-subdifferentiable functional and let \( \rho > 0 \) be any given constant. Then \( \partial \phi_n \rightarrow G \partial \phi \) if and only if

\[
R_{\rho, M(\cdot, \cdot)}^{\partial \phi_n}(x^*) \to R_{\rho, M(\cdot, \cdot)}^{\partial \phi}(x^*), \quad \forall x^* \in E^*.
\]

**PROOF.** Suppose that \( \partial \phi_n \rightarrow G \partial \phi \). For any \( x^* \in E^* \), let

\[
z_n = R_{\rho, M(\cdot, \cdot)}^{\partial \phi_n}(x^*), \quad z = R_{\rho, M(\cdot, \cdot)}^{\partial \phi}(x^*).
\]

It follows that \( z = [M(A, B) + \rho \partial \phi]^{-1}(x^*) \),

then,

\[
\frac{1}{\rho}[x^* - M(Az, Bz)] \in \partial \phi(z),
\]

that is, \((z, \frac{1}{\rho}[x^* - M(Az, Bz)]) \in \text{graph}(\partial \phi)\). It follows from the definition of the graph convergence that there exists a sequence \((z'_n, y'_n) \in \text{graph}(\partial \phi_n)\) such that

\[
z'_n \to z \quad \text{and} \quad y'_n \to \frac{1}{\rho}[x^* - M(Az, Bz)] \quad \text{as} \quad n \to \infty. \tag{2}
\]

Since \( y'_n \in \partial \phi_n(z'_n) \), we have

\[M(Az'_n, Bz'_n) + \rho y'_n \in [M(A, B) + \rho \partial \phi_n](z'_n)\]

that is, \( z'_n = R_{\rho, M(\cdot, \cdot)}^{\partial \phi_n}[M(Az'_n, Bz'_n) + \rho y'_n] \). Now,

\[
\|z_n - z\| \leq \|z_n - z'_n\| + \|z'_n - z\|
\]

\[
= \|R_{\rho, M(\cdot, \cdot)}^{\partial \phi_n}(x^*) - R_{\rho, M(\cdot, \cdot)}^{\partial \phi_n}[M(Az'_n, Bz'_n) + \rho y'_n]\|
\]

\[
+ \|z'_n - z\|,
\]

\[
\|z_n - z\| \leq \|z_n - z'_n\| + \|z'_n - z\|
\]

\[= \|R_{\rho, M(\cdot, \cdot)}^{\partial \phi_n}(x^*) - R_{\rho, M(\cdot, \cdot)}^{\partial \phi_n}[M(Az'_n, Bz'_n) + \rho y'_n]\|
\]

\[+ \|z'_n - z\|,
\]

\[
= \|z_n - z'_n\| + \|z'_n - z\|
\]

\[\leq \|z_n - z'_n\| + \|z'_n - z\|,
\]

\[
= \|z'_n - z\|,
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\[
\leq \|z_n - z'_n\| + \|z'_n - z\|
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\[\leq \|z_n - z'_n\| + \|z'_n - z\|
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= \|z'_n - z\|,
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\leq \|z_n - z'_n\| + \|z'_n - z\|
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\[\leq \|z_n - z'_n\| + \|z'_n - z\|
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\[\leq \|z_n - z'_n\| + \|z'_n - z\|
\]
By using the Lipschitz continuity of the resolvent operator $R_{\rho,M}(\cdot,\cdot)$, we have
\[
\|z_n - z\| \leq \frac{\tau}{\alpha - \beta} \|x^* - [M(Az_n, Bz_n) + \rho y_n^*]\| + \|z_n' - z\|
\leq \frac{\tau}{\alpha - \beta} \|x^* - [M(Az, Bz) + \rho y_n^*]\|
+ \frac{\tau}{\alpha - \beta} \|M(Az, Bz) - M(Az_n, Bz_n)\| + \|z_n' - z\|.
\]
Since $M$ is $\gamma_1$-Lipschitz continuous with respect to $A$ and $\gamma_2$-Lipschitz continuous with respect to $B$, we have
\[
\|z_n - z\| \leq \frac{\tau}{\alpha - \beta} \|x^* - [M(Az, Bz) + \rho y_n^*]\| + \frac{\tau(\gamma_1 + \gamma_2)}{\alpha - \beta} \|z - z_n\| + \|z_n' - z\|
= \frac{\tau}{\alpha - \beta} \|x^* - [M(Az, Bz) + \rho y_n^*]\| + \left[1 + \frac{\tau(\gamma_1 + \gamma_2)}{\alpha - \beta}\right] \|z_n' - z\|.
\]
By (2), we have
\[
\|z_n' - z\| \to 0, \quad \frac{1}{\rho} \|x^* - [M(Az, Bz) + \rho y_n^*]\| \to 0, \quad \text{as } n \to \infty,
\]
hence $\|z_n - z\| \to 0$ as $n \to \infty$, that is,
\[
R_{\rho,M}(\cdot,\cdot)(x^*) \to R_{\rho,M}(\cdot,\cdot)(x^*), \quad \forall x^* \in E^*.
\]
Conversely, suppose that $R_{\rho,M}(\cdot,\cdot)(x^*) \to R_{\rho,M}(\cdot,\cdot)(x^*), \quad \forall x^* \in E^*, \rho > 0$. For any $(x,y^*) \in \text{graph}(\partial \phi)$, we have, $y^* \in \partial \phi(x)$, that is,
\[
M(Ax, Bx) + \rho y^* \in [M(A, B) + \rho \partial \phi](x),
\]
and so $x = R_{\rho,M}(\cdot,\cdot)[M(Ax, Bx) + \rho y^*]$. Let $x_n = R_{\rho,M}(\cdot,\cdot)[M(Ax, Bx) + \rho y^*]$, then
\[
\frac{1}{\rho} [M(Ax, Bx) - M(Ax_n, Bx_n) + \rho y^*] \in \partial \phi_n(x_n).
\]
Suppose that $y_n^* = \frac{1}{\rho} [M(Ax, Bx) - M(Ax_n, Bx_n) + \rho y^*]$. Now,
\[
\|y_n^* - y^*\| = \frac{1}{\rho} \|M(Ax, Bx) - M(Ax_n, Bx_n) + \rho y^* - y^*\|
\leq \frac{1}{\rho} \|M(Ax, Bx) - M(Ax_n, Bx_n)\|
\leq \frac{(\gamma_1 + \gamma_2)}{\rho} \|x_n - x\|.
\] (3)
Since $R_{\rho,M}(\cdot,\cdot)(x^*) \to R_{\rho,M}(\cdot,\cdot)(x^*)$ for any $x^* \in E^*$, we have $\|x_n - x\| \to 0$ as $n \to \infty$. It follows from (3) that $\|y_n^* - y^*\| \to 0$ as $n \to \infty$. Hence $\partial \phi_n G \partial \phi$. 

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4 System of Generalized Implicit Variational-like Inclusions

Let for each \(i \in \{1, 2\}\), \(E_i\) be a real Banach space with norm \(\| \cdot \|_i\) and \(E_i^*\) be its dual space with norm \(\| \cdot \|_{i*}\). Let \(\langle \cdot, \cdot \rangle_i\) denotes the duality pairing between \(E_i\) and \(E_i^*\); let \(\eta_i : E_i \times E_i \rightarrow E_i, N_i : E_i^* \times E_i^* \rightarrow E_i^*\) and \(S_i : E_i \rightarrow E_i^*\) be single-valued mappings; let \(q_1 : E_2 \rightarrow CB(E_2)\) and \(g_2 : E_1 \rightarrow CB(E_1^*)\) be multi-valued mappings. Let \(\phi_i : E_i \rightarrow \mathbb{R} \cup \{+\infty\}\) be a proper, lower semicontinuous and \(\eta_i\)-subdifferentiable functional. We consider the following system of generalized implicit variational-like inclusions (in short, SGIVLI).

Find \((x, y, u, v)\) such that \(x \in E_1, y \in E_2, u \in g_1(y), v \in g_2(x)\) and

\[
\begin{align*}
\langle N_1(S_1(x), u), \eta_1(a, x) \rangle &\geq \rho_1[\phi_1(x) - \phi_1(a)], \forall a \in E_1, \\
\langle N_2(v, S_2(y)), \eta_2(b, y) \rangle &\geq \rho_2[\phi_2(y) - \phi_2(b)], \forall b \in E_2,
\end{align*}
\]

(4)

where \(\rho_1, \rho_2 > 0\) are some constants.

REMARK 4.1. For suitable choices of mappings \(A_i, B_i, N_i, g_i, S_i, M_i, \eta_i, \phi_i\) and underlying spaces \(E_i\), SGIVLI (4) reduces to various known classes of systems of variational inclusions and variational inequalities, see for examples, \([6, 19, 20, 24, 29, 30]\).

LEMMA 4.1. For each \(i \in \{1, 2\}\), let \(E_i\) be a reflexive Banach space; let \(\eta_i : E_i \times E_i \rightarrow E_i^*\) be a continuous mapping such that \(\eta_i(y_i, y_i) + \eta_i(y_i, y_i) = 0\), for all \(y_i, y_i \in E_i\). Let \(A_i, B_i : E_i \rightarrow E_i\) be single-valued mappings; let the mappings \(M_i : E_i \times E_i \rightarrow E_i^*\) be \(\alpha_\beta\)-symmetric \(\eta_i\)-monotone continuous with respect to \(A_i\) and \(B_i\); let for any \(x_i^* \in E_i^*\), the function \(h_i(y_i, x_i) = \langle x_i^* - M_i(A_i(x_i, B_i(x_i), \eta_i(x_i, x_i))\rangle\) be 0-DQC\(V\) in \(y_i\) and let \(\phi_i : E_i \rightarrow \mathbb{R} \cup \{+\infty\}\) be a proper, lower semicontinuous and \(\eta_i\)-subdifferentiable functional. Then for \((x, y, u, v)\), where \(x \in E_1, y \in E_2, u \in g_1(y), v \in g_2(x)\) is a solution of SGIVLI (4), if and only if \((x, y, u, v)\) satisfies the relation

\[
\begin{align*}
x &= R_{\rho_1, M_1(\cdot)}^{\phi_1}[M_1(A_1 x, B_1 x) - N_1(S_1 x, u)], \\
y &= R_{\rho_2, M_2(\cdot)}^{\phi_2}[M_2(A_2 y, B_2 y) - N_2(v, S_2 y)],
\end{align*}
\]

where \(\rho_1, \rho_2\) are some constants, \(R_{\rho_1, M_1(\cdot)}^{\phi_1}(x^*) = [M_1(A_1, B_1) + \rho_1 \partial \phi_1]^{-1}(x^*)\) and \(R_{\rho_2, M_2(\cdot)}^{\phi_2}(y^*) = [M_2(A_2, B_2) + \rho_2 \partial \phi_2]^{-1}(y^*)\).

PROOF. The conclusion follows directly from the definition of resolvent operators \(R_{\rho_1, M_1(\cdot)}^{\phi_1}\) and \(R_{\rho_2, M_2(\cdot)}^{\phi_2}\).

We note that \((E_1 \times E_2, \| \cdot \|_*)\) is a Banach space with norm \(\| \cdot \|_*\) defined as

\[
\|(x, y)\|_* = \|x\|_1 + \|y\|_2, \forall (x, y) \in E_1 \times E_2.
\]

Next, we prove existence and uniqueness for SGIVLI (4).
THEOREM 4.1. For each $i \in \{1, 2\}$, let $E_i$ be a uniformly smooth Banach space with $\rho_{E_i}(t) \leq C_i t^2$ for some $C_i > 0$; let $\eta_i : E_i \times E_i \to E_i$ be a continuous mapping such that $\eta_i(y_i, y_i') + \eta_i(y_i, y_i) = 0$, for all $y_i, y_i' \in E_i$; let $A_i, B_i : E_i \to E_i$ be nonlinear mappings; let $M_i : E_i \times E_i \to E_i$ be $\alpha_i \beta_i$-symmetric $\eta_i$-monotone continuous with respect to $A_i, B_i$; let for any given $x_i \in E_i$, the function $h_i(y_i, x_i) = (x_i^* - M_i(A_i x_i, B_i x_i), \eta_i(y_i, x_i))$ be 0-DQCV in $y_i$. Let $\phi_i : E_i \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semicontinuous and $\eta_i$-subdifferentiable functional. Let $N_i : E_i \times E_i^* \to E_i^*$ be $\delta_i$, $\eta_i$-mixed Lipschitz continuous; let $g_1 : E_2 \to CB(E_2^*)$ and $g_2 : E_1 \to CB(E_1^*)$ be $\lambda_{D_{\eta_i}}$ and $\lambda_{D_{\eta_i}}$-Lipschitz continuous with respect to second and first argument, respectively; let $N_i(S_i(\cdot), u_i)$ be $\epsilon_i$-strongly accretive with respect to $M_i(A_i, B_i)$ and $N_i(v_1, S_2(\cdot))$ is $\epsilon_2$-strongly accretive with respect to $M_i(A_2, B_2)$; let $M_i(A_i, B_i)$ is $\lambda_{M_i}$-Lipschitz continuous with respect to $A_i$ and $B_i$. Suppose that there exist constants $\rho_1, \rho_2 > 0$ such that

\[
\begin{align*}
G_1 &= u_1 + L_2 \delta_2 D g_2 < 1, \\
G_2 &= u_2 + L_1 r_1 \lambda_{D_{\eta_1}} < 1,
\end{align*}
\]

where

\[
\begin{align*}
u_1 &= L_1 \sqrt{\frac{\lambda^2_{M_1} - 2 \epsilon_1 + 64 C_1 \delta^2_1}{\alpha_1 - \beta_1}}, \\
u_2 &= L_2 \sqrt{\frac{\lambda^2_{M_2} - 2 \epsilon_2 + 64 C_2 \delta^2_2}{\alpha_2 - \beta_2}}, \\
L_1 &= \frac{\tau_1}{\alpha_1 - \beta_1}, \\
L_2 &= \frac{\tau_2}{\alpha_2 - \beta_2}.
\end{align*}
\]

Then SGIVLI (4) has a unique solution.

PROOF. It follows that for $(x, y) \in E_1 \times E_2$, the resolvent operators $R^{\delta_{\phi_1}}_{\rho_1, M_1(\cdot, \cdot)}$ and $R^{\delta_{\phi_2}}_{\rho_2, M_2(\cdot, \cdot)}$ are $L_1$ and $L_2$-Lipschitz continuous, respectively.

Now, we define a mapping $Q : E_1 \times E_2 \to E_1 \times E_2$ by

\[
Q(x, y) = (T(x, y), P(x, y)), \quad \forall (x, y) \in E_1 \times E_2,
\]

where $T : E_1 \times E_2 \to E_1$ and $P : E_1 \times E_2 \to E_2$ are defined by

\[
\begin{align*}
T(x, y) &= R^{\delta_{\phi_1}}_{\rho_1, M_1(\cdot, \cdot)}[M_1(A_1 x_1, B_1 x_1) - N_1(S_1 x_1, u_1)], \\
P(x, y) &= R^{\delta_{\phi_2}}_{\rho_2, M_2(\cdot, \cdot)}[M_2(A_2 y_2, B_2 y_2) - N_2(S_2 y_2)].
\end{align*}
\]

For any $(x_1, y_1), (x_2, y_2) \in E_1 \times E_2$, using (7), (8) and Lipschitz continuity of $R^{\delta_{\phi_1}}_{\rho_1, M_1(\cdot, \cdot)}$ and $R^{\delta_{\phi_2}}_{\rho_2, M_2(\cdot, \cdot)}$, we have

\[
\begin{align*}
\|T(x_1, y_1) - T(x_2, y_2)\|_1 &= \|R^{\delta_{\phi_1}}_{\rho_1, M_1(\cdot, \cdot)}[M_1(A_1 x_1, B_1 x_1) - N_1(S_1 x_1, u_1)] \\
&\quad - R^{\delta_{\phi_1}}_{\rho_1, M_1(\cdot, \cdot)}[M_1(A_1 x_2, B_1 x_2) - N_1(S_1 x_2, u_2)]\|_1 \\
&\leq L_1 \| M_1(A_1 x_1, B_1 x_1) - M_1(A_1 x_2, B_1 x_2) \\
&\quad - (N_1(S_1 x_1, u_1) - N_1(S_1 x_2, u_2))\|_1, \\
&\leq L_1 \| M_1(A_1 x_1, B_1 x_1) - M_1(A_1 x_2, B_1 x_2) \\
&\quad - (N_1(S_1 x_1, u_1) - N_1(S_1 x_2, u_2))\|_1 + L_2 \| N_1(S_1 x_2, u_1) - N_1(S_1 x_2, u_2)\|_1,
\end{align*}
\]
\[ \| M_1(A_1x_1, B_1x_1) - M_1(A_1x_2, B_1x_2) - (N_1(S_1x_1, u_1) - N_1(S_1x_2, u_1)) \|_{\lambda_{S_i}}^2 \]
\[ \leq \| M_1(A_1x_1, B_1x_1) - M_1(A_1x_2, B_1x_2) \|_{\lambda_{S_i}}^2 \\
- 2(N_1(S_1x_1, u_1) - N_1(S_1x_2, u_1), J_1^*(M_1(A_1x_1, B_1x_1) - M_1(A_1x_2, B_1x_2)))_1 \\
+ 2(N_1(S_1x_1, u_1) - N_1(S_1x_2, u_1), J_1^*(M_1(A_1x_1, B_1x_1) - M_1(A_1x_2, B_1x_2)))_1 \\
- J_1^*(M_1(A_1x_1, B_1x_1) - M_1(A_1x_2, B_1x_2)) - (N_1(S_1x_1, u_1) - N_1(S_1x_2, u_1))_1. \]

Since \( M_1 \) is \( \lambda_{S_i} \)-Lipschitz continuous with respect to \( A_1 \) and \( B_1 \), \( N_1(S_1(\cdot), u_1) \) is \( \epsilon_1 \)-strongly accretive with respect to \( M_1(A_1, B_1) \), \( N_1 \) is \( (\delta_1, r_1) \)-mixed Lipschitz continuous and \( g_1 \) is \( \lambda_{D_{x_1}} \)-Lipschitz continuous in the second argument, we have
\[ \| M_1(A_1x_1, B_1x_1) - M_1(A_1x_2, B_1x_2) - (N_1(S_1x_1, u_1) - N_1(S_1x_2, u_1)) \|_{\lambda_{S_i}}^2 \]
\[ \leq \lambda_{S_i}^2 \| x_1 - x_2 \|_1^2 - 2\epsilon_1 \| x_1 - x_2 \|_1^2 + 64C_1\delta_1^2 \| x_1 - x_2 \|_1^2, \quad (10) \]
where \( J_1^*: E_1^* \rightarrow E_1 \) is normalized duality mapping and
\[ \| N_1(S_1x_1, u_1) - N_1(S_1x_2, u_2) \|_{\lambda_{S_i}^2} \]
\[ \leq r_1 \| u_1 - u_2 \|_{\lambda_{S_i}^2} \]
\[ \leq r_1 D(g_1(y_1), g_1(y_2)) \]
\[ \leq r_1 \lambda_{D_{y_1}} \| y_1 - y_2 \|_2. \quad (11) \]

From (9)–(11), we have
\[ \| T(x_1, y_1) - T(x_2, y_2) \|_1 \]
\[ \leq L_1 \sqrt{\lambda_{S_i}^2 - 2\epsilon_1 + 64C_1\delta_1^2 \| x_1 - x_2 \|_1 + L_1 r_1 \lambda_{D_{y_1}} \| y_1 - y_2 \|_2. \quad (12) \]

\[ \| P(x_1, y_1) - P(x_2, y_2) \|_2 \]
\[ \leq \| R_{r_2, M_2(\cdot, \cdot)}^{\lambda_{S_i}^2} [M_2(A_2y_1, B_2y_1) - N_2(v_1, S_2y_1)] \\
- R_{r_2, M_2(\cdot, \cdot)}^{\lambda_{S_i}^2} [M_2(A_2y_2, B_2y_2) - N_2(v_2, S_2y_2)] \|_2 \]
\[ \leq L_2 \| [M_2(A_2y_1, B_2y_1) - M_2(A_2y_2, B_2y_2)] \\
- (N_2(v_1, S_2y_1) - N_2(v_2, S_2y_2)) \|_{\lambda_{S_i}^2} \]
\[ \leq L_2 \| [M_2(A_2y_1, B_2y_1) - M_2(A_2y_2, B_2y_2)] \\
- (N_2(v_1, S_2y_1) - N_2(v_2, S_2y_2)) \|_{\lambda_{S_i}^2} \]
\[ + L_2 \| N_2(v_1, S_2y_2) - N_2(v_2, S_2y_2) \|_{\lambda_{S_i}^2}. \quad (13) \]

\[ \| M_2(A_2y_1, B_2y_1) - M_2(A_2y_2, B_2y_2) - (N_2(v_1, S_2y_1) - N_2(v_2, S_2y_2)) \|_{\lambda_{S_i}^2}^2 \]
\[ \leq \| M_2(A_2y_1, B_2y_1) - M_2(A_2y_2, B_2y_2) \|_{\lambda_{S_i}^2}^2 \\
- 2(N_2(v_1, S_2y_1) - N_2(v_1, S_2y_2), J_2^*(M_2(A_2y_1, B_2y_1) - M_2(A_2y_2, B_2y_2)))_2 \\
+ 2(N_2(v_1, S_2y_1) - N_2(v_1, S_2y_2), J_2^*(M_2(A_2y_1, B_2y_1) - M_2(A_2y_2, B_2y_2)))_2 \\
- J_2^*(M_2(A_2y_1, B_2y_1) - M_2(A_2y_2, B_2y_2)) - (N_2(v_1, S_2y_1) - N_2(v_1, S_2y_2))_2. \]

Since \( M_2 \) is \( \lambda_{S_i} \)-Lipschitz continuous with respect to \( A_2 \) and \( B_2 \), \( N_2(v_1, S_2(\cdot)) \) is \( \epsilon_2 \)-strongly accretive with respect to \( M_2(A_2, B_2) \), \( N_2 \) is \( (\delta_2, r_2) \)-mixed Lipschitz continuous.
and $g_2$ is $\lambda_{D_{g_2}}$-Lipschitz continuous in the first argument, we have

$$\|M_2(A_2y_1, B_2y_1) - M_2(A_2y_2, B_2y_2) - (N_2(v_1, S_2y_1) - N_2(v_1, S_2y_2))\|_2^2 \leq \lambda_{M_2}^2 \|y_1 - y_2\|_2^2 - 2\epsilon_2 \|y_1 - y_2\|_2^2 + 64C_2r_2^2 \|y_1 - y_2\|_2^2,$$

(14)

where $J_*^2 : E_*^2 \to E_2$ is normalized duality mapping and

$$\|N_2(v_1, S_2x_2) - N_2(v_2, S_2x_2)\| \leq \delta_2 \|v_1 - v_2\|_\ast, \quad \delta_2 \|v_1 - v_2\|_\ast \leq \delta_2 D(g_2(x_1), g_2(x_2)) \leq \delta_2 \lambda_{D_{g_2}} \|x_1 - x_2\|_1.$$

(15)

From (13)–(15), we have

$$\|P(x_1, y_1) - P(x_2, y_2)\|_2 \leq L_2 \sqrt{\lambda_{M_1}^2 - 2\epsilon_1 + 64C_1\delta_1^2} \|y_1 - y_2\|_2 + L_2 \delta_2 \lambda_{D_{g_2}} \|x_1 - x_2\|_1.$$

(16)

From (12) and (16), we have

$$\|T(x_1, y_1) - T(x_2, y_2)\|_1 + \|S(x_1, y_1) - S(x_2, y_2)\|_2 \leq G_1 \|x_1 - x_2\|_1 + G_2 \|y_1 - y_2\|_2 \leq \max\{G_1, G_2\} \|x_1 - x_2\|_1 + \|y_1 - y_2\|_2,$$

(17)

where

$$\left\{ \begin{array}{l} G_1 = u_1 + L_2 \delta_2 \lambda_{D_{g_2}}, \\ G_2 = u_2 + L_1 r_1 \lambda_{D_{g_1}} \end{array} \right.$$

(18)

and

$$u_1 = L_1 \sqrt{\lambda_{M_1}^2 - 2\epsilon_1 + 64C_1\delta_1^2}, \quad u_2 = L_2 \sqrt{\lambda_{M_2}^2 - 2\epsilon_2 + 64C_2r_2^2}.$$ 

Now, we define the norm $\|\cdot\|_\ast$ on $E_1 \times E_2$ by

$$\|(x, y)\|_\ast = \|x\|_1 + \|y\|_2, \quad \forall (x, y) \in E_1 \times E_2.$$

(19)

Since $(E_1 \times E_2, \|\cdot\|_\ast)$ is a Banach space and hence from (6), (17) and (19), we have

$$\|Q(x_1, y_1) - Q(x_2, y_2)\|_\ast = \|T(x_1, y_1) - T(x_2, y_2)\|_1 + \|P(x_1, y_1) - P(x_2, y_2)\|_2 \leq \max\{G_1, G_2\} \|(x_1, y_1) - (x_2, y_2)\|_\ast.$$

(20)

By condition (5), $\max\{G_1, G_2\} < 1$, hence $Q$ is a contraction mapping. It follows from Banach contraction principle, there exists a point $(x, y) \in E_1 \times E_2$ such that

$$Q(x, y) = (x, y),$$

which implies that

$$x = R_{\rho_1, M_1(\cdot, \cdot)}^{\partial \phi_1}[M_1(A_1x, B_1x) - N_1(S_1x, u)],$$

$$y = R_{\rho_2, M_2(\cdot, \cdot)}^{\partial \phi_2}[M_2(A_2y, B_2y) - N_2(v, S_2y)],$$

where $\rho_1, \rho_2$ are appropriate constants.
Then by Lemma 4.1, \((x, y, u, v)\) is a unique solution of SGIVLI (4).

**Algorithm 4.1.** For any \((x_0, y_0) \in E_1 \times E_2\), compute the sequence \((x_n, y_n) \in E_1 \times E_2, u_0 \in g_1(y_0), v_0 \in g_2(x_0)\) by the following iterative scheme:

\[
x_{n+1} = R_{\rho_1, M_1}^\phi [M_1(A_1 x_n, B_1 x_n) - N_1(S_1 x_n, u_n)],
\]

\[
y_{n+1} = R_{\rho_2, M_2}^\phi [M_2(A_2 y_n, B_2 y_n) - N_2(v_n, S_2 y_n)],
\]

where \(n = 0, 1, 2, \ldots; \rho_1 > 0, \rho_2 > 0\) are some constants.

**Theorem 4.2.** For each \(i \in \{1, 2\}\), let \(A_i, B_i, S_i, g_i, N_i, M_i, \phi_i\) and \(\eta_i\) be same as in Theorem 4.1. Suppose that \(\partial \phi_i \subset G_{\partial \phi_i}\) and the condition (5) holds. Then approximate solution \((x_n, y_n)\) generated by Algorithm 4.1 converges strongly to unique solution \((x, y)\) of SGIVLI (4).

**Proof.** It follows from Theorem 4.1 that there exists a unique solution \((x, y, u, v)\) of SGIVLI (4). By Algorithm 4.1 and Lipschitz continuity of the resolvent operators, we have

\[
\|x_{n+1} - x\|_1 = \|R_{\rho_1, M_1}^\phi [M_1(A_1 x_n, B_1 x_n) - N_1(S_1 x_n, u_n)]
- \|R_{\rho_1}^\phi [M_1(A_1 x_n, B_1 x_n) - N_1(S_1 x_n, u_n)]\|_1 
\leq \|R_{\rho_1, M_1}^\phi [M_1(A_1 x_n, B_1 x_n) - N_1(S_1 x_n, u_n)]
- \|R_{\rho_1}^\phi [M_1(A_1 x_n, B_1 x_n) - N_1(S_1 x_n, u_n)]\|_1 + \|R_{\rho_1}^\phi [M_1(A_1 x_n, B_1 x_n) - N_1(S_1 x_n, u_n)]
- \|R_{\rho_1}^\phi [M_1(A_1 x_n, B_1 x_n) - N_1(S_1 x_n, u_n)]\|_1 (23)
\]

and

\[
\|y_{n+1} - y\|_2 = \|R_{\rho_2, M_2}^\phi [M_2(A_2 y_n, B_2 y_n) - N_2(v_n, S_2 y_n)]
- \|R_{\rho_2}^\phi [M_2(A_2 y_n, B_2 y_n) - N_2(v_n, S_2 y_n)]\|_2 
\leq \|R_{\rho_2, M_2}^\phi [M_2(A_2 y_n, B_2 y_n) - N_2(v_n, S_2 y_n)]
- \|R_{\rho_2}^\phi [M_2(A_2 y_n, B_2 y_n) - N_2(v_n, S_2 y_n)]\|_2 + \|R_{\rho_2}^\phi [M_2(A_2 y_n, B_2 y_n) - N_2(v_n, S_2 y_n)]
- \|R_{\rho_2}^\phi [M_2(A_2 y_n, B_2 y_n) - N_2(v_n, S_2 y_n)]\|_2 . (24)
\]
Now, using the same arguments as from (9)–(12), we have
\[
\left\| R_{\rho_1, M_{1n}(\cdot, \cdot)}^{\partial \phi_1} \left[ M_1(A_1 x_n, B_1 x_n) - N_1(S_1 x_n, u_n) \right] \right\| \\
\leq u_1 \| x_n - x \|_1 + L_1 r_1 \lambda_{D_{\eta_1}} \| y_n - y \|_2,
\]
and following the same arguments as from (13)–(16), we have
\[
\left\| R_{\rho_2, M_{2n}(\cdot, \cdot)}^{\partial \phi_2} \left[ M_2(A_2 y_n, B_2 y_n) - N_2(v_n, S_2 y_n) \right] \right\|_2 \\
\leq u_2 \| y_n - y \|_2 + L_2 \delta_2 \lambda_{D_{\eta_2}} \| x_n - x \|_1.
\]
By Theorem 3.1, we have
\[
R_{\rho_1, M_{1n}(\cdot, \cdot)}^{\partial \phi_1} \left[ M_1(A_1 x, B_1 x) - N_1(S_1 x, u) \right] \\
\rightarrow R_{\rho_1, M_{1}(\cdot, \cdot)}^{\partial \phi_1} \left[ M_1(A_1 x, B_1 x) - N_1(S_1 x, u) \right],
\]
\[
R_{\rho_2, M_{2n}(\cdot, \cdot)}^{\partial \phi_2} \left[ M_2(A_2 y, B_2 y) - N_2(v, S_2 y) \right] \\
\rightarrow R_{\rho_2, M_{2}(\cdot, \cdot)}^{\partial \phi_2} \left[ M_2(A_2 y, B_2 y) - N_2(v, S_2 y) \right].
\]
Let
\[
a_n = R_{\rho_1, M_{1n}(\cdot, \cdot)}^{\partial \phi_1} \left[ M_1(A_1 x, B_1 x) - N_1(S_1 x, u) \right] \\
- R_{\rho_1, M_{1}(\cdot, \cdot)}^{\partial \phi_1} \left[ M_1(A_1 x, B_1 x) - N_1(S_1 x, u) \right]
\]
and
\[
b_n = R_{\rho_2, M_{2n}(\cdot, \cdot)}^{\partial \phi_2} \left[ M_2(A_2 y, B_2 y) - N_2(v, S_2 y) \right] \\
- R_{\rho_2, M_{2}(\cdot, \cdot)}^{\partial \phi_2} \left[ M_2(A_2 y, B_2 y) - N_2(v, S_2 y) \right].
\]
Then
\[
a_n, b_n \rightarrow 0 \quad n \rightarrow \infty.
\]
From (23)–(26), (27) and (28), we have
\[
\| x_{n+1} - x \|_1 + \| y_{n+1} - y \|_2 \leq G_1 \| x_n - x \|_1 + G_2 \| y_n - y \|_2 + a_n + b_n
\]
\[
\leq \max\{G_1, G_2\} (\| x_n - x \|_1 + \| y_n - y \|_2) + a_n + b_n.
\]
It follows from (19) that \((E_1 \times E_2, \| \cdot \|_*)\) is a Banach space, we have
\[
\| (x_{n+1}, y_{n+1}) - (x, y) \|_* = \max\{G_1, G_2\} (\| (x_n, y_n) - (x - y) \|_* + a_n + b_n).
\]
From condition (5) and (29), (30), we have
\[
\| (x_{n+1}, y_{n+1}) - (x, y) \|_* \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]
Thus \(\{(x_n, y_n)\}\) converges strongly to the unique solution \((x, y)\) of SGIVLI (4). This completes the proof.

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**References**


Graph Convergence for $\eta$-Subdifferential Mappings


