On Generalized Absolute Cesàro Summability Of Factored Infinite Series

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Abstract

In this paper, we generalize a known result dealing with the absolute Cesàro summability factors of infinite series. Some new and known results are also obtained.

1 Introduction

Let \( \sum a_n \) be a given infinite series with partial sums \( (s_n) \). We denote by \( t_n^{\alpha,\beta} \) the \( n \)th Cesàro mean of order \( (\alpha, \beta) \), with \( \alpha + \beta > -1 \), of the sequence \( (na_n) \), that is (see [5])

\[
t_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^\beta a_v,
\]

where

\[ A_n^{\alpha+\beta} = O(n^{\alpha+\beta}), \quad A_0^{\alpha+\beta} = 1, \quad \text{and} \quad A_n^{\alpha+\beta} = 0 \quad \text{for} \quad n > 0. \]

A series \( \sum a_n \) is said to be summable \( |C, \alpha, \beta, k| \), if (see [2])

\[
\sum_{n=1}^{\infty} n^{k+k-1} \frac{|t_n^{\alpha,\beta}|^k}{n^k} < \infty.
\]

If we take \( \sigma = 1 \), then \( |C, \alpha, \beta, k| \) summability reduces to \( |C, \alpha, \beta|_k \) summability (see [3]). If we set \( \sigma = 1 \) and \( \delta = 0 \), then we obtain the \( |C, \alpha, \beta|_k \) summability (see [6]). Also, if we take \( \beta = 0 \), then we have \( |C, \alpha, \sigma|_k \) summability (see [10]). Furthermore, if we take \( \sigma = 1 \), \( \beta = 0 \), and \( \delta = 0 \), then we get \( |C, \alpha|_k \) summability (see [7]). Finally, if we set \( \sigma = 1 \) and \( \beta = 0 \), then we get \( |C, \alpha|_k \) (see [8]). For any sequence \( (\lambda_n) \) we write that \( \Delta^2 \lambda_n = \Delta \lambda_n - \Delta \lambda_{n+1} \) and \( \Delta \lambda_n = \lambda_n - \lambda_{n+1} \). Let \( (\theta_n^{\alpha,\beta}) \) be a sequence defined by (see [1])

\[
\theta_n^{\alpha,\beta} = \begin{cases} 
|t_n^{\alpha,\beta}|, & \text{for } \alpha = 1, \beta > -1, \\
\max_{1 \leq v \leq n} |t_v^{\alpha,\beta}|, & \text{for } 0 < \alpha < 1, \beta > -1.
\end{cases}
\]

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2 Known Result

The following theorem is known dealing with the generalized absolute Cesàro summability factors of infinite series.

THEOREM 1 ([4]). Let \( (\theta_n^{\alpha,\beta}) \) be a sequence defined as in (2). If \( (\lambda_n) \) is a non-negative and non-increasing sequence such that the series \( \sum \frac{\lambda_n}{n} \) is convergent, 

\[
\Delta n \lambda_n \to 0 \quad \text{as} \quad n \to \infty, \quad (3)
\]

\[
\sum_{n=1}^{\infty} (n+1)^2 \lambda_n \quad (4)
\]

is convergent and the condition

\[
\sum_{n=1}^{m} (n^\delta \theta_n^{\alpha,\beta})^k = O(m) \quad \text{as} \quad m \to \infty \quad (5)
\]

holds, then the series \( \sum a_n \lambda_n \) is summable \( |C, \alpha, \beta; \delta| \), \( 0 < \alpha \leq 1, \beta > -1, k \geq 1, \delta \geq 0, \) and \( (\alpha + \beta - \delta) > 0 \).

3 Main Result

The aim of this paper is to generalize Theorem 1 for the \( |C, \alpha, \beta, \sigma; \delta| \) summability method. Now, we shall prove the following theorem.

THEOREM 2. Let \( (\theta_n^{\alpha,\beta}) \) be a sequence defined as in (2). If \( (\lambda_n) \) is a non-negative and non-increasing sequence such that the series \( \sum \frac{\lambda_n}{n} \) is convergent, the conditions (3), (4), and

\[
\sum_{n=1}^{m} \frac{\lambda_n}{n^{\delta + k + 1}} (\theta_n^{\alpha,\beta})^k = O(m) \quad \text{as} \quad m \to \infty \quad (6)
\]

hold, then the series \( \sum a_n \lambda_n \) is summable \( |C, \alpha, \beta, \sigma; \delta|, k \geq 1, 0 \leq \delta < \alpha \leq 1, \sigma \in R, \) and \( (\alpha + \beta + 1)k - \sigma(\delta k + k - 1) > 1 \).

We need the following lemma for the proof of our theorem.

LEMMA 1 ([1]). If \( 0 < \alpha \leq 1, \beta > -1, \) and \( 1 \leq v \leq n \), then

\[
\left| \sum_{p=0}^{v} A_{n-p}^{\alpha,\beta} a_p \right| \leq \max_{1 \leq m \leq v} \left| \sum_{p=0}^{m} A_{m-p}^{\alpha,\beta} a_p \right|.
\]
4 Proof of Theorem 2

Let \((T_n^{\alpha,\beta})\) be the \(n\)th \((C,\alpha,\beta)\) mean of the sequence \((a_n,\lambda_n)\). Then, by (1), we have that
\[
T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v \lambda_v.
\]

First applying Abel’s transformation and then using Lemma 1, we have that
\[
T_n^{\alpha,\beta} = \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^{v} A_{n-v}^{\alpha-1} A_p^{\beta} p a_p + \frac{\lambda_n}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v,
\]

\[
\left| T_n^{\alpha,\beta} \right| \leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} \left| \Delta \lambda_v \right| \left| \sum_{p=1}^{v} A_{n-v}^{\alpha-1} A_p^{\beta} p a_p \right| + \frac{\left| \lambda_n \right|}{A_n^{\alpha+\beta}} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} A_v^{\beta} v a_v
\]

\[
\leq \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} |\Delta \lambda_v| + |\lambda_n| \theta_n^{\alpha,\beta}
\]

\[
= T_{n,1}^{\alpha,\beta} + T_{n,2}^{\alpha,\beta}.
\]

To complete the proof, by Minkowski’s inequality, it is sufficient to show that
\[
\sum_{n=1}^{\infty} n^{\sigma(\delta k+k-1)-k} \left| T_{n,r}^{\alpha,\beta} \right|^k < \infty, \quad \text{for} \quad r = 1, 2.
\]

Whenever \(k > 1\), we can apply H"older’s inequality with indices \(k\) and \(k'\) where
\[
\frac{1}{k} + \frac{1}{k'} = 1,
\]

we get that
\[
\sum_{n=2}^{m+1} n^{\sigma(\delta k+k-1)-k} \left| T_{n,1}^{\alpha,\beta} \right|^k \leq \sum_{n=2}^{m+1} n^{\sigma(\delta k+k-1)-k} \left| \frac{1}{A_n^{\alpha+\beta}} \sum_{v=1}^{n-1} A_v^{\alpha} A_v^{\beta} \theta_v^{\alpha,\beta} \Delta \lambda_v \right|^k
\]

\[
= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{(\alpha+\beta+1)k-\sigma(\delta k+k-1)}} \left\{ \sum_{v=1}^{n-1} n^{\sigma(\delta k+k-1)-k} \theta_v^{\alpha,\beta} \Delta \lambda_v \right\} \left\{ \sum_{v=1}^{n-1} \Delta \lambda_v \right\}^{k-1}
\]

\[
= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \Delta \lambda_v \theta_v^{\alpha,\beta} \left( \int_{x}^{\infty} \frac{dx}{x^{(\alpha+\beta+1)k-\sigma(\delta k+k-1)}(\theta_v^{\alpha,\beta})^k} \right)
\]

\[
= O(1) \sum_{v=1}^{m} v^{(\alpha+\beta)k} \Delta \lambda_v \theta_v^{\alpha,\beta} \int_{v}^{\infty} dx \int_{x}^{\infty} \frac{dx}{x^{(\alpha+\beta+1)k-\sigma(\delta k+k-1)}(\theta_v^{\alpha,\beta})^k}
\]
\[ \begin{align*}
&= O(1) \sum_{v=1}^{m} \Delta \lambda_v v^{\sigma(\delta k+k-1)} \left( \frac{\theta_{v}^{\alpha,\beta}}{v^{k-1}} \right)^k \\
&= O(1) \sum_{v=1}^{m} \Delta(\Delta \lambda_v) \sum_{p=1}^{v} p^{\sigma(\delta k+k-1)} \left( \frac{\theta_{p}^{\alpha,\beta}}{p^{k-1}} \right) + O(1) \Delta \lambda_m \sum_{v=1}^{m} v^{\sigma(\delta k+k-1)} \left( \frac{\theta_{v}^{\alpha,\beta}}{v^{k-1}} \right) \\
&= O(1) \sum_{v=1}^{m} v \Delta^2 \lambda_v + O(1)m \Delta \lambda_m \\
&= O(1) \text{ as } m \to \infty,
\end{align*} \]

in view of hypotheses of Theorem 2.

Similarly, we have that

\[ \sum_{n=1}^{m} n^{\sigma(\delta k+k-1)-k} | \lambda_n \theta_{n}^{\alpha,\beta} |^k = O(1) \sum_{n=1}^{m} \frac{\lambda_n}{n} n^{\sigma(\delta k+k-1)} \left( \frac{\theta_{n}^{\alpha,\beta}}{n^{k-1}} \right) \]
\[ = O(1) \sum_{n=1}^{m} \Delta(\Delta \lambda_n) \sum_{v=1}^{n} v^{\sigma(\delta k+k-1)} \left( \frac{\theta_{v}^{\alpha,\beta}}{v^{k-1}} \right) + O(1) \lambda_m \sum_{n=1}^{m} n^{\sigma(\delta k+k-1)} \left( \frac{\theta_{n}^{\alpha,\beta}}{n^{k-1}} \right) \]
\[ = O(1) \sum_{n=1}^{m} \Delta \lambda_n + O(1) \sum_{n=1}^{m} \frac{\lambda_{n+1}}{n+1} + O(1) \lambda_m \]
\[ = O(1) \sum_{n=1}^{m} \Delta \lambda_n + O(1) \sum_{n=2}^{m-1} \frac{\lambda_{n}}{n} + O(1) \lambda_m \]
\[ = O(1)(\lambda_1 - \lambda_m) + O(1) \sum_{n=1}^{m-1} \frac{\lambda_{n}}{n} + O(1) \lambda_m \]
\[ = O(1) \text{ as } m \to \infty, \]

by virtue of hypotheses of Theorem 2. This completes the proof of Theorem 2.

### 5 Conclusions

If we take \( \beta = 0 \) and \( \sigma = 1 \), then we get a new result for \(|C, \alpha; \delta|_k\) summability factors of infinite series. If we set \( \sigma = 1 \), then we get Theorem 1. Because in this case condition (6) reduces to condition (5). Also, if we take \( \beta = 0 \) and \( \delta=0 \), then we get a result concerning the \(|C, \alpha|_k\) summability. Furthermore, if we take \( \sigma = 1 \), \( \beta = 0 \), \( \alpha = 1 \), and \( \delta = 0 \), then we obtain a new result for the \(|C, 1|_k\) summability factors. Finally, if we take \( \delta = 0 \), \( \beta = 0 \), \( \sigma = 1 \), and \( k = 1 \), then we get the known result of Pati dealing with \(|C, \alpha|\) summability factors of infinite series (see [9]).
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References


