A Refinement Of An Integral Inequality For The Polar Derivative Of A Polynomial

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Abstract

Certain refinements of a recently obtained integral inequality by Rather and Bhat for the polar derivative of a polynomial with restricted zeros are given.

1 Introduction

Let $\mathcal{P}_n$ be the set of all complex polynomials $P(z)$ of degree $n$. It was shown by Turan \cite{Turan} that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq 1$, then

$$n \max_{|z|=1} |P(z)| \leq 2 \max_{|z|=1} |P'(z)|. \quad (1)$$

Equality in (1) holds for $P(z) = az^n + \beta$, $|a| = |\beta|$.

Govil \cite{Govil} showed that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k, k \geq 1$, then

$$n \max_{|z|=1} |P(z)| \leq (1 + k^n) \max_{|z|=1} |P'(z)|. \quad (2)$$

The estimate is sharp and equality in (2) holds for $P(z) = (z^n + k^n)$.

Malik \cite{Malik} obtained an extension of (1) in the sense that the left hand side of (1) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$ by showing that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq 1$, then for each $q > 0$,

$$n \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} \left| 1 + e^{i\theta} \right|^q d\theta \right\}^{1/q} \max_{|z|=1} |P'(z)|. \quad (3)$$

Exremal polynomial is $P(z) = az^n + b$, $|a| = |b|$.

For the class of polynomials $P \in \mathcal{P}_n$ having all their zeros in $|z| \leq k, k \geq 1$, Aziz \cite{Aziz} proved for each $q > 0$,

$$n \left\{ \int_0^{2\pi} \left| P(e^{i\theta}) \right|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} \left| 1 + ke^{i\theta} \right|^q d\theta \right\}^{1/q} \max_{|z|=1} |P'(z)|. \quad (3)$$
Equality in (3) holds for $P(z) = z^n + k^n$. In the limiting case when $q \to \infty$, the inequality (3) reduces to inequality (2). In literature there exist other similar type of results on polynomial approximation theory (see [5, 9]).

For $\alpha \in \mathbb{C}$, the polar derivative $D_\alpha P(z)$ of a polynomial $P \in \mathcal{P}_n$ is defined by

$$D_\alpha P(z) := nP(z) + (\alpha - z)P'(z)$$

(see [6, 8]). The polynomial $D_\alpha P(z)$ is of degree at most $n - 1$ and it generalizes the ordinary $P'(z)$ of $P(z)$ in the sense that

$$\lim_{\alpha \to \infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$$

uniformly with respect $z$ for $|z| \leq R, R > 0$.

As an extension of inequality (2) to the polar derivative of a polynomial, Aziz and Rather [2] proved that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$ where $k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$,

$$n (|\alpha| - k) \max_{|z|=1} |P(z)| \leq (1 + k^n) \max_{|z|=1} |D_\alpha P(z)|.$$  \hfill (4)

More recently Rather and Bhat [11] extended inequality (3) to the polar derivative of polynomial and obtain a generalization of (4) in the sense that the left hand side of (4) is replaced by a factor involving the integral mean of $|P(z)|$ on $|z| = 1$ by showing that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$ where $k \geq 1$, then for $|\alpha| \geq k$ and $q > 0$,

$$n (|\alpha| - k) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^q d\theta \right\}^{1/q} \max_{|z|=1} |D_\alpha P(z)|$$  \hfill (5)

and under the same hypothesis, they [11] also proved that

$$n (|\alpha| - k) \left\{ \int_0^{2\pi} |P(e^{i\theta}) + \beta m|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^q d\theta \right\}^{1/q} \left\{ \max_{|z|=1} |D_\alpha P(z)| - nm/k^{n-1} \right\} \right.$$  \hfill (6)

where $|\beta| \leq 1$ and $m = \min_{|z|=k} |P(z)|$.

In this paper we first present the following refinement of inequality (5).

THEOREM 1. If $P(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree $n \geq 2$ having all its zeros in $|z| \leq k$ where $k \geq 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \geq k$ and for each $q > 0$,

$$n (|\alpha| - k) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^q d\theta \right\}^{1/q} \max_{|z|=1} |D_\alpha P(z)| - \phi(k) |na_0 + \alpha a_1|$$  \hfill (7)
where
\[ \phi(k) = (1 - 1/k^2) \] or \( (1 - 1/k) \) according as \( n > 2 \) or \( n = 2 \). \hfill (8)

Equality in (7) holds in the limiting case when \( \alpha \to \infty \) and the extremal polynomial is
\( P(z) = (z^n + k^n) \).

To see this, we divide the two sides of inequality (7) by \( |\alpha| \), let \( \alpha \to \infty \) and use the
fact that \( \lim_{\alpha \to \infty} \frac{\Delta_{\alpha} \Phi_{\alpha}^*(z)}{\alpha} = \Phi_{\alpha}^*(z) \), we get

\[ n \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^q d\theta \right\}^{1/q} \max_{|z|=1} |\Phi_{\alpha}^*(z)| - \phi(k) |\alpha_1| \cdot \]

For the polynomial \( P(z) = (z^n + k^n) \), \( \max_{|z|=1} |\Phi_{\alpha}^*(z)| = n \) and \( \alpha_1 = 0 \). By using
property of definite of integrals, the left hand side of above inequality equals

\[ n \left\{ \int_0^{2\pi} |e^{i\theta} + k^n|^q d\theta \right\}^{1/q} = n \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^q d\theta \right\}^{1/q} \]

whereas the right hand side equals

\[ n \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^q d\theta \right\}^{1/q} \cdot \]

Thus the two sides of above inequality are equal. Therefore, the equality in Theorem 1
holds in limiting case when \( \alpha \to \infty \) and the extremal polynomial is \( P(z) = (z^n + k^n) \).

Further if we let \( q \to \infty \) in (7), we get a refinement of inequality (4). We next prove:

**THEOREM 2.** If \( P(z) = \sum_{j=0}^n a_j z^j \) is a polynomial of degree \( n \geq 2 \) having all its
zeros in \( |z| \leq k \) where \( k \geq 1 \) and \( m = \min_{|z|=k} |P(z)| \), then for every \( \alpha, \beta \in C \) with
\( |\alpha| \geq k, |\beta| \leq 1 \) and for each \( q > 0 \),

\[ n (|\alpha| - k) \left\{ \int_0^{2\pi} |P(e^{i\theta}) + \beta m|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^q d\theta \right\}^{1/q} \max_{|z|=1} |\Delta_{\alpha} P(z)| - \phi(k) |na_0 + \alpha a_1| \]

where \( \phi(k) \) is given by (8).

Equality in (9) holds in the limiting case when \( |\alpha| \to \infty \) and the extremal polynomial
is \( P(z) = (z^n + k^n) \) as can be verified as before since \( m = 0 \). For \( \beta = 0 \), Theorem 2
gives the following refinement of Theorem 1.

**COROLLARY 1.** If \( P(z) = \sum_{j=0}^n a_j z^j \) is a polynomial of degree \( n \geq 2 \) having all
its zeros in \( |z| \leq k \) where \( k \geq 1 \) and \( m = \min_{|z|=k} |P(z)| \), then for every \( \alpha, \beta \in C \) with
$|\alpha| \geq k$, $|\beta| \leq 1$ and for each $q > 0$,
\[
\left( \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right)^{1/q} \leq \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^q d\theta \right\}^{1/q} \left\{ \max_{|z|=1} |D_\alpha P(z)| - n m/k^{n-1} \right\}
- \phi(k) |n a_0 + \alpha a_1|
\]
where $\phi(k)$ is same as defined in Theorem 1.

Letting $q \to \infty$ in (9) and chosing the argument of $\beta$ with $|\beta| = 1$ suitably, we obtain the following refinement of inequality (4).

COROLLARY 2. If $P(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree $n \geq 2$ having all its zeros in $|z| \leq k$ where $k \geq 1$ and $m = \min_{|z|=k} |P(z)|$, then for every $\alpha \in C$ with $|\alpha| \geq k$,
\[
n (|\alpha| - k) \max_{|z|=1} |P(z)| + n \left( |\alpha| + 1/k^{n-1} \right) m + \phi(k) |n a_0 + \alpha a_1|
\leq (1 + k^n) \max_{|z|=1} |D_\alpha P(z)|
\]
where $\phi(k)$ is given by (8).

2 Lemmas

For the proofs of these theorems we need the following results. The first result is due to Frappier, Rahman and Ruscheweyh [3].

LEMMA 1. If $P(z) = \sum_{j=0}^n a_j z^j$ is a polynomial of degree $n \geq 1$, then for $R \geq 1$,
\[
\max_{|z|=R} |P(z)| \leq R^n \max_{|z|=1} |P(z)| - (R^n - R^{n-2})|P(0)|, \text{ if } n > 1
\]
and
\[
\max_{|z|=R} |P(z)| \leq R \max_{|z|=1} |P(z)| - (R - 1)|P(0)|, \text{ if } n = 1.
\]

Next result is due to Rahman and Schmeisser [10].

LEMMA 2. If $P \in P_n$ and $P(z) \neq 0$ in $|z| < 1$, then for $R \geq 1$ and $q > 0$,
\[
\left\{ \int_0^{2\pi} |P(Re^{i\theta})|^q d\theta \right\}^{1/q} \leq C_q \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q d\theta \right\}^{1/q}
\]
where
\[
C_q = \left\{ \int_0^{2\pi} |1 + R^q e^{i\theta}|^q d\theta \right\}^{1/q} / \left\{ \int_0^{2\pi} |1 + e^{i\theta}|^q d\theta \right\}^{1/q}.
\]
3 Proofs of the Theorems

PROOF OF THEOREM 1. By hypothesis all the zeros of \( P(z) \) lie in \( |z| \leq k \), therefore, all the zeros of \( f(z) = P(kz) \) lie in \( |z| \leq 1 \). Applying inequality (5) with \( k = 1 \) to the polynomial \( f(z) \), we get for each \( q > 0 \) and \( |\beta| \geq 1 \),

\[
n(\beta - 1) \left\{ \int_0^{2\pi} |f(e^{i\theta})|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} |1 + e^{i\theta}|^q d\theta \right\}^{1/q} \max_{|z|=1} |D_\beta f(z)|.
\]

Setting \( \beta = \frac{\alpha}{k} \) in above inequality and noting that \( |\beta| = \frac{|\alpha|}{k} \geq 1 \), we have

\[
n \left\{ \frac{|\alpha|}{k} - 1 \right\} \left\{ \int_0^{2\pi} |f(e^{i\theta})|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^q d\theta \right\}^{1/q} \max_{|z|=1} |D_\beta f(z)| \quad (10)
\]

Let \( g(z) = z^n f(1/z) \). Then

\[
|g(z)| = |f(z)| \quad \text{for } |z| = 1
\]

and \( f(z) \neq 0 \) in \( |z| < 1 \). By Lemma 2 applied to the polynomial \( g(z) \) with \( R = k \geq 1 \), it follows that for each \( q > 0 \),

\[
\int_0^{2\pi} |g(k e^{i\theta})|^q d\theta \leq B_q \int_0^{2\pi} |f(e^{i\theta})|^q d\theta = B_q \int_0^{2\pi} |1 + k^n e^{i\theta}|^q d\theta, \quad (11)
\]

where

\[
B_q = \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^q d\theta \right\}^{1/q} \left\{ \int_0^{2\pi} |1 + e^{i\theta}|^q d\theta \right\}^{1/q}. \quad (12)
\]

Combining (10) and (11), we get for each \( q > 0 \),

\[
n (|\alpha| - k) \left\{ \int_0^{2\pi} |g(k e^{i\theta})|^q d\theta \right\}^{1/q} \leq k B_q \left\{ \int_0^{2\pi} |1 + e^{i\theta}|^q d\theta \right\}^{1/q} \max_{|z|=1} |D_\beta f(z)|
\]

\[
= k \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^q d\theta \right\}^{1/q} \max_{|z|=1} |D_\beta f(z)|. \quad (13)
\]

Also,

\[
g(z) = z^n f(1/z) = z^n P(k/z),
\]

gives for \( 0 \leq \theta < 2\pi \),

\[
|g(k e^{i\theta})| = |k^n e^{i\theta} P(e^{i\theta})| = k^n |P(e^{i\theta})|.
\]
Using this in (13), we get
\[
 nk^n (|\alpha| - k) \left\{ \int_0^{2\pi} |P(e^{i\theta})|^q \, d\theta \right\}^{1/q} \leq k \left\{ \int_0^{2\pi} \left| 1 + k^n e^{i\theta} \right|^q \, d\theta \right\}^{1/q} \max_{|z|=1} |D_{\varphi} f(z)| .
\] (14)

Again, noting that \( D_{\alpha} P(z) \) is a polynomial of degree at most \( n - 1 \) and
\[
 \max_{|z|=1} |D_{\varphi} f(z)| = \max_{|z|=k} |D_{\alpha} P(z)| ,
\]
by Lemma 1 for \( R = k \geq 1 \), we have
\[
 \max_{|z|=1} |D_{\varphi} f(z)| = \max_{|z|=k} |D_{\alpha} P(z)| \leq k^{n-1} \max_{|z|=1} |D_{\alpha} P(z)| - (k^{n-1} - k^n) |a_0 + \alpha a_1| ,
\] (15)
if \( n > 2 \) and
\[
 \max_{|z|=1} |D_{\varphi} f(z)| = \max_{|z|=k} |D_{\alpha} P(z)| \leq k \max_{|z|=1} |D_{\alpha} P(z)| - (k-1) |a_0 + \alpha a_1| ,
\] (16)
if \( n = 2 \). Combining (14), (15) and (16), we immediately get the desired result. This completes the proof of Theorem 1.

The proof of Theorem 2 follows on the lines of proof of Theorem 2 of [11]. However, for the sake of completeness we present a proof.

**PROOF OF THEOREM 2.** Since \( f(z) = P(kz) \) has all its zeros in \( |z| \leq 1 \), therefore, applying the inequality (6) to the polynomial \( f(z) \) (with \( k = 1 \) and \( \alpha \) replaced by \( \alpha/k \)), we get for each \( q > 0, |\beta| \leq 1 \) and \( |\alpha| \geq k \),
\[
 n \left( \frac{|\alpha|}{k} - 1 \right) \left\{ \int_0^{2\pi} \left| f(e^{i\theta}) + \beta \min_{|z|=1} |f(z)| \right|^q \, d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} \left| 1 + e^{i\theta} \right|^q \, d\theta \right\}^{1/q} \left\{ \max_{|z|=1} |D_{\varphi} f(z)| - n \min_{|z|=1} |f(z)| \right\} .
\] (17)

Also since
\[
 m = \min_{|z|=k} |P(z)| = \min_{|z|=1} |P(kz)| = \min_{|z|=1} |f(z)| ,
\]
therefore, from (17), we obtain for each \( q > 0, |\beta| \leq 1 \) and \( |\alpha| \geq k \),
\[
 n (|\alpha| - k) \left\{ \int_0^{2\pi} \left| f(e^{i\theta}) + \beta m \right|^q \, d\theta \right\}^{1/q} \leq k \left\{ \int_0^{2\pi} \left| 1 + e^{i\theta} \right|^q \, d\theta \right\}^{1/q} \left\{ \max_{|z|=1} |D_{\varphi} f(z)| - nm \right\} .
\] (18)
Moreover, \( f(z) = 0 \) in \( |z| \leq 1 \) and
\[
 m \leq |f(z)| \quad \text{for} \quad |z| = 1 ,
\]
it follows by the maximum modulus theorem,

$$m|z|^n < |f(z)| \text{ for } |z| > 1. \quad (19)$$

We show all the zeros of polynomial $g(z) = f(z) + \beta m$ lie in $|z| \leq 1$ for every $\beta$ with $|\beta| \leq 1$. This is obvious if $m = 0$, that is, if $f(z)$ has a zero on $|z| = 1$. Assume that $f(z)$ has no zero on $|z| = 1$ so that $m \neq 0$. If there is a point $z = z_0$ with $|z_0| > 1$ such that $g(z_0) = f(z_0) + \beta m = 0$, then we have

$$|f(z_0)| = |\beta| m < m|z_0|^n, \quad |z_0| > 1,$$

a contradiction to (19). Hence, the polynomial $g(z)$ has all its zeros in $|z| \leq 1$ and therefore, the polynomial $h(z) = z^n g(1/z) \neq 0$ in $|z| < 1$. Applying Lemma 2 to the polynomial $h(z)$ with $R = k \geq 1$, it follows that for each $q > 0$,

$$\int_0^{2\pi} |h(ke^{i\theta})|^q d\theta \leq B_q^q \int_0^{2\pi} |h(e^{i\theta})|^q d\theta = B_q^q \int_0^{2\pi} |g(e^{i\theta})|^q d\theta = B_q^q \int_0^{2\pi} |f(e^{i\theta}) + \beta m|^q d\theta \quad (20)$$

where $B_q$ is the same as given by (12). Using (18) in (20), we obtain for each $q > 0$,

$$n(|\alpha| - k) \left\{ \int_0^{2\pi} |h(ke^{i\theta})|^q d\theta \right\}^{1/q} \leq k \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^q d\theta \right\}^{1/q} \left\{ \max_{|z|=1} |D_{\frac{q}{k}} f(z)| - nm \right\}. \quad (21)$$

But

$$h(z) = z^n g(1/z) = z^n f(1/z) + \bar{\beta} z^n m,$$

therefore, for $|z| = 1$, we get

$$|h(kz)| = |k^n z^n f(1/kz) + \bar{\beta} z^n mk^n| = k^n |f(z/k) + \bar{\beta} m| = k^n |P(z) + \beta m| \quad (22)$$

From (15), (16), (21) and (22), we deduce after short simplification for each $q > 0, |\beta| \leq 1$ and $|\alpha| \geq k$,

$$n(|\alpha| - k) \left\{ \int_0^{2\pi} |P(e^{i\theta}) + \beta m|^q d\theta \right\}^{1/q} \leq \left\{ \int_0^{2\pi} |1 + k^n e^{i\theta}|^q d\theta \right\}^{1/q} \left\{ \max_{|z|=1} |D_{\frac{q}{k}} P(z)| - nm/k^{n-1} \right\} - \phi(k) na_0 + c\alpha_1.$$

This proves Theorem 2.

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References


