Certain New Subclasses Of Analytic And m-Fold Symmetric Bi-Univalent Functions*

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Abstract

The purpose of the present paper is to introduce and investigate two new subclasses \( SS_m^{\Sigma_m} (\lambda, \gamma; \alpha) \) and \( S_m^{\Sigma_m} (\lambda, \gamma; \beta) \) of \( \Sigma_m \) consisting of analytic and \( m \)-fold symmetric bi-univalent functions defined in the open unit disk \( U \). We obtain upper bounds for the coefficients \( |a_{m+1}| \) and \( |a_{2m+1}| \) for functions belonging to these subclasses. Many of the well-known and new results are shown to follow as special cases of our results.

1 Introduction

Let \( \mathcal{A} \) denote the class of functions \( f \) that are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) and normalized by the conditions \( f(0) = f'(0) - 1 = 0 \) and having the form:

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k.
\] (1)

Let \( S \) be the subclass of \( \mathcal{A} \) consisting of the form (1) which are also univalent in \( U \). The Koebe one-quarter theorem (see [4]) states that the image of \( U \) under every function \( f \in S \) contains a disk of radius \( \frac{1}{4} \). Therefore, every function \( f \in S \) has an inverse \( f^{-1} \) which satisfies \( f^{-1}(f(z)) = z \), \( (z \in U) \) and \( f(f^{-1}(w)) = w \), \( (|w| < r_0(f), r_0(f) \geq \frac{1}{4}) \), where

\[
g(w) = f^{-1}(w) = w - a_2 w^2 + \left( 2a_2^2 - a_3 \right) w^3 - \left( 5a_2^3 - 5a_2 a_3 + a_4 \right) w^4 + \cdots.
\] (2)

A function \( f \in \mathcal{A} \) is said to be bi-univalent in \( U \) if both \( f \) and \( f^{-1} \) are univalent in \( U \). We denote by \( \Sigma \) the class of bi-univalent functions in \( U \) given by (1). For a brief history and interesting examples in the class \( \Sigma \) see [18], (see also [6, 7, 8, 10, 14, 15, 21, 22]).

For each function \( f \in S \), the function \( h(z) = (f(z^m))^{\frac{1}{m}} \), \( (z \in U, m \in \mathbb{N}) \) is univalent and maps the unit disk \( U \) into a region with \( m \)-fold symmetry. A function is said to be \( m \)-fold symmetric (see [9, 12]) if it has the following normalized form:

\[
f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \ (z \in U, m \in \mathbb{N}).
\] (3)

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We denote by $S_m$ the class of $m$-fold symmetric univalent functions in $U$, which are normalized by the series expansion (3). In fact, the functions in the class $S$ are one-fold symmetric.

In [19] Srivastava et al. defined $m$-fold symmetric bi-univalent functions analogues to the concept of $m$-fold symmetric univalent functions. They gave some important results, such as each function $f \in \Sigma$ generates an $m$-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of $f$ given by (3), they obtained the series expansion for $f^{-1}$ as follows:

$$g(w) = w - a_{m+1}w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right]w^{2m+1}$$

$$- \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right]w^{3m+1}$$

$$+ \cdots,$$

(4)

where $f^{-1} = g$. We denote by $\Sigma_m$ the class of $m$-fold symmetric bi-univalent functions in $U$. It is easily seen that for $m = 1$, the formula (4) coincides with the formula (2) of the class $\Sigma$. Some examples of $m$-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \left[\frac{1}{2} \log \left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}} \text{ and } [- \log (1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \left(\frac{e^{2w^m} - 1}{e^{2w^m} + 1}\right)^{\frac{1}{m}} \text{ and } \left(\frac{e^{w^m} - 1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

Recently, many authors investigated bounds for various subclasses of $m$-fold bi-univalent functions (see [1, 2, 5, 16, 17, 19, 20]).

The aim of the present paper is to introduce the new subclasses $SS_m^*(\lambda, \gamma; \alpha)$ and $S_{\Sigma_m}^*(\lambda, \gamma; \beta)$ of $\Sigma_m$ and find estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

In order to prove our main results, we require the following lemma.

**Lemma 1 ([4]).** If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions $h$ analytic in $U$ for which

$$\text{Re} \ (h(z)) > 0, \ (z \in U),$$

where

$$h(z) = 1 + c_1z + c_2z^2 + \cdots, \ (z \in U).$$
2 Coefficient Estimates for the Functions Class

\( SS^*_\Sigma_m(\lambda, \gamma; \alpha) \)

DEFINITION 1. A function \( f \in \Sigma_m \) given by (3) is said to be in the class \( SS^*_\Sigma_m(\lambda, \gamma; \alpha) \) if it satisfies the following conditions:

\[
\arg\left[ \frac{1}{2} \left( \frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}} + \left( \frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}} \right)^{1/2} \right) \right] < \frac{\alpha\pi}{2}, \quad (z \in U)
\]

and

\[
\arg\left[ \frac{1}{2} \left( \frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}} + \left( \frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}} \right)^{1/2} \right) \right] < \frac{\alpha\pi}{2}, \quad (w \in U),
\]

\((0 < \alpha \leq 1; 0 < \lambda \leq 1; \gamma \geq 0; m \in \mathbb{N})\),

where the function \( g = f^{-1} \) is given by (4).

REMARK 1. It should be remarked that the class \( SS^*_\Sigma_m(\lambda, \gamma; \alpha) \) is a generalization of well-known classes considered earlier. These classes are:

1. For \( \gamma = 0 \), the class \( SS^*_\Sigma_m(\lambda, \gamma; \alpha) \) reduce to the class \( S_{\Sigma_m}(\alpha, \lambda) \) which was introduced recently by Altinkaya and Yalcin [2];
2. For \( \lambda = 1 \) and \( \gamma = 0 \), the class \( SS^*_\Sigma_m(\lambda, \gamma; \alpha) \) reduce to the class \( S^*_\Sigma_m \) which was considered by Altinkaya and Yalcin [1];
3. For \( \lambda = \gamma = 1 \), the class \( SS^*_\Sigma_m(\lambda, \gamma; \alpha) \) reduce to the class \( H^0_{\Sigma_m} \) which was investigated by Srivastava et al. [19].

REMARK 2. For one-fold symmetric bi-univalent functions, we denote the class \( SS^*_\Sigma_1(\lambda, \gamma; \alpha) = SS^*_\Sigma(\lambda, \gamma; \alpha) \). Special cases of this class illustrated below:

1. For \( \lambda = 1 \), the class \( SS^*_\Sigma(\lambda, \gamma; \alpha) \) reduce to the class \( P_{\Sigma}(\alpha, \gamma) \) which was introduced by Prema and Keerthi [13];
2. For \( \lambda = 1 \) and \( \gamma = 0 \), the class \( SS^*_\Sigma(\lambda, \gamma; \alpha) \) reduce to the class \( S^*_\Sigma(\alpha) \) which was given by Brannan and Taha [3];
3. For \( \lambda = \gamma = 1 \), the class \( SS^*_\Sigma(\lambda, \gamma; \alpha) \) reduce to the class \( H^0_{\Sigma} \) which was investigated by Srivastava et al. [18].

THEOREM 1. Let \( f \in SS^*_\Sigma_m(\lambda, \gamma; \alpha) \) \((0 < \alpha \leq 1, 0 < \lambda \leq 1, \gamma \geq 0, m \in \mathbb{N})\) be given by (3). Then

\[
|a_{m+1}| \leq \frac{4\lambda\alpha}{(m+\gamma)\sqrt{(\lambda+1)\left(2\lambda\alpha\left(\frac{m}{m+\gamma}+1\right)+(1-\alpha)(\lambda+1)\right)+2\alpha(1-\lambda)}} \quad (7)
\]
\[ |a_{2m+1}| \leq \frac{8\lambda^2 \alpha^2 (m+1)}{(m+\gamma)^2 (\lambda+1)^2} + \frac{4\lambda \alpha}{(2m+\gamma)(\lambda+1)}. \quad (8) \]

**PROOF.** It follows from conditions (5) and (6) that
\[
\frac{1}{2} \left( \frac{z^{1-\gamma} f'(z)}{(f(z))^{1-\gamma}} + \left( \frac{z^{1-\gamma} f'(z)}{(f(z))^{1-\gamma}} \right)^\frac{1}{\gamma} \right) = \alpha p(z),
\quad (9)
\]
and
\[
\frac{1}{2} \left( \frac{w^{1-\gamma} g'(w)}{(g(w))^{1-\gamma}} + \left( \frac{w^{1-\gamma} g'(w)}{(g(w))^{1-\gamma}} \right)^\frac{1}{\gamma} \right) = \alpha q(w),
\quad (10)
\]
where \( g = f^{-1} \) and \( p, q \) in \( \mathcal{P} \) have the following series representations:
\[
p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots
\quad (11)
\]
and
\[
q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots
\quad (12)
\]
Comparing the corresponding coefficients of (9) and (10) yields
\[
\frac{(m+\gamma)(\lambda+1)}{2\lambda} a_{m+1} = \alpha p_m,
\quad (13)
\]
\[
\frac{(2m+\gamma)(\lambda+1)}{4\lambda} \left( 2a_{2m+1} + (\gamma-1)a_{m+1}^2 \right) + \frac{(m+\gamma)^2 (1-\lambda)}{4\lambda^2} a_{m+1}^2
\]
\[
= \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2} p_m^2,
\quad (14)
\]
\[
-\frac{(m+\gamma)(\lambda+1)}{2\lambda} a_{m+1} = \alpha q_m
\quad (15)
\]
and
\[
\frac{(2m+\gamma)(\lambda+1)}{4\lambda} \left[ (2m+\gamma+1)a_{m+1}^2 - 2a_{2m+1} \right] + \frac{(m+\gamma)^2 (1-\lambda)}{4\lambda^2} a_{m+1}^2
\]
\[
= \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2} q_m^2.
\quad (16)
\]
Making use of (13) and (15), we obtain
\[
p_m = -q_m
\quad (17)
\]
and
\[
\frac{(m+\gamma)^2 (\lambda+1)^2}{2\lambda^2} a_{m+1}^2 = \alpha^2 (p_m^2 + q_m^2).
\quad (18)\]
Also, from (14), (16) and (18), we find that

\[
\frac{(2m + \gamma)(m + \gamma)(\lambda + 1)}{2\lambda} + \frac{(m + \gamma)^2(1 - \lambda)}{2\lambda^2} a_{m+1}^2
\]

\[
= \alpha(p_{2m} + q_{2m}) + \frac{\alpha(\alpha - 1)}{2} p_m^2 + q_m^2
\]

\[
= \alpha(p_{2m} + q_{2m}) + \frac{(\alpha - 1)(m + \gamma)^2(\lambda + 1)^2}{4\lambda^2 \alpha} a_{m+1}^2.
\]

Therefore, we have

\[
a_{m+1}^2 = \frac{4\lambda^2 \alpha(p_{2m} + q_{2m})}{(m + \gamma)^2 \left[ (\lambda + 1) \left( 2\lambda \alpha \left( \frac{m}{m + \gamma} + 1 \right) + (1 - \alpha)(\lambda + 1) \right) + 2\alpha(1 - \lambda) \right]}. \tag{19}
\]

Now, taking the absolute value of (19) and applying Lemma 1 for the coefficients \(p_{2m}\) and \(q_{2m}\), we obtain

\[
|a_{m+1}| \leq \frac{4\lambda \alpha}{(m + \gamma) \left( (\lambda + 1) \left( 2\lambda \alpha \left( \frac{m}{m + \gamma} + 1 \right) + (1 - \alpha)(\lambda + 1) \right) + 2\alpha(1 - \lambda) \right)}.
\]

This gives the desired estimate for \(|a_{m+1}|\) as asserted in (7). In order to find the bound on \(|a_{2m+1}|\), by subtracting (16) from (14), we get

\[
\frac{(2m + \gamma)(\lambda + 1)}{\lambda} a_{2m+1} = \frac{(2m + \gamma)(m + 1)(\lambda + 1)}{2\lambda} a_{m+1}^2
\]

\[
= \alpha(p_{2m} - q_{2m}) + \frac{\alpha(\alpha - 1)}{2} (p_m^2 - q_m^2). \tag{20}
\]

It follows from (17), (18) and (20) that

\[
a_{2m+1} = \frac{\lambda^2 \alpha^2(m + 1)(p_m^2 + q_m^2)}{(m + \gamma)^2 (\lambda + 1)^2} + \frac{\lambda \alpha(p_{2m} - q_{2m})}{(2m + \gamma)(\lambda + 1)}.
\]

Taking the absolute value of (21) and applying Lemma 1 once again for the coefficients \(p_m, p_{2m}, q_m\) and \(q_{2m}\), we obtain

\[
|a_{2m+1}| \leq \frac{8\lambda^2 \alpha^2(m + 1)}{(m + \gamma)^2 (\lambda + 1)^2} + \frac{4\lambda \alpha}{(2m + \gamma)(\lambda + 1)}.
\]

which completes the proof of Theorem 1.

**REMARK 3.** In Theorem 1, if we choose

1. \(\gamma = 0\), then we obtain the results which was proven by Altinkaya and Yalcin [[2], Theorem 1];
2. \( \lambda = \gamma = 1 \), then we obtain the results which was proven by Srivastava et al. \([19], \text{Theorem 2}\).

For one-fold symmetric bi-univalent functions, Theorem 1 reduces to the following corollary:

**COROLLARY 1.** Let \( f \in S\Sigma^*_m(\lambda, \gamma; \alpha) \) \((0 < \alpha \leq 1, 0 < \lambda \leq 1, \gamma \geq 0)\) be given by (1). Then
\[
|a_2| \leq \frac{4\lambda\alpha}{(1 + \gamma)\sqrt{(\lambda + 1)\left(\frac{2\lambda\alpha(1+\gamma)}{1+\gamma} + (1 - \alpha)(\lambda + 1)\right) + 2\alpha(1 - \lambda)}},
\]
and
\[
|a_3| \leq \frac{16\lambda^2\alpha^2}{(1 + \gamma)^2(\lambda + 1)^2} + \frac{4\lambda\alpha}{(2 + \gamma)(\lambda + 1)}.
\]

**REMARK 4.** In Corollary 1, if we choose
1. \( \lambda = 1 \), then we have the results which was given by Prema and Keerthi \([13], \text{Theorem 2.2}\);  
2. \( \lambda = 1 \) and \( \gamma = 0 \), then we have the results obtained by Murugusundaramoorthy et al. \([11], \text{Corollary 6}\);  
3. \( \lambda = \gamma = 1 \), then we obtain the results obtained by Srivastava et al. \([18], \text{Theorem 1}\).

### 3 Coefficient Estimates for the Functions Class \( S^{*}_{\Sigma_m}(\lambda, \gamma; \beta) \)

**DEFINITION 2.** A function \( f \in \Sigma_m \) given by (3) is said to be in the class \( S^{*}_{\Sigma_m}(\lambda, \gamma; \beta) \) if it satisfies the following conditions:
\[
Re \left\{ \frac{1}{2} \left( \frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}} + \left( \frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}} \right)^{\frac{1}{\gamma}} \right) \right\} > \beta, \quad (z \in U)
\]
and
\[
Re \left\{ \frac{1}{2} \left( \frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}} + \left( \frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}} \right)^{\frac{1}{\gamma}} \right) \right\} > \beta, \quad (w \in U),
\]
\((0 \leq \beta < 1, : 0 < \lambda \leq 1, : \gamma \geq 0, : m \in \mathbb{N})\),

where the function \( g = f^{-1} \) is given by (4).

**REMARK 5.** It should be remarked that the class \( S^{*}_{\Sigma_m}(\lambda, \gamma; \beta) \) is a generalization of well-known classes consider earlier. These classes are:
1. For $\gamma = 0$, the class $S_{\Sigma m}^2(\lambda; \gamma; \beta)$ reduce to the class $S_{\Sigma m}(\beta, \lambda)$ which was introduced recently by Altinkaya and Yalcin [2];

2. For $\lambda = 1$ and $\gamma = 0$, the class $S_{\Sigma m}^2(\lambda; \gamma; \beta)$ reduce to the class $S_{\Sigma m}^2$ which was considered by Altinkaya and Yalcin [1];

3. For $\lambda = \gamma = 1$, the class $S_{\Sigma m}^2(\lambda; \gamma; \beta)$ reduce to the class $H_{\Sigma m}(\beta)$ which was investigated by Srivastava et al. [19].

REMARK 6. For one-fold symmetric bi-univalent functions, we denote the class $S_{\Sigma 1}(\lambda; \gamma; \beta) = S_{\Sigma 1}^2(\lambda; \gamma; \beta)$. Special cases of this class illustrated below:

1. For $\lambda = 1$, the class $S_{\Sigma 1}^2(\lambda; \gamma; \beta)$ reduce to the class $P_{\Sigma}(\beta, \gamma)$ which was introduced by Prema and Keerthi [13];

2. For $\lambda = 1$ and $\gamma = 0$, the class $S_{\Sigma 1}^2(\lambda; \gamma; \beta)$ reduce to the class $S_{\Sigma 2}^2(\beta)$ which was given by Brannan and Taha [3];

3. For $\lambda = \gamma = 1$, the class $S_{\Sigma 1}^2(\lambda; \gamma; \beta)$ reduce to the class $H_{\Sigma 1}(\beta)$ which was investigated by Srivastava et al. [18].

THEOREM 2. Let $f \in S_{\Sigma m}^2(\lambda; \gamma; \beta)$ (0 $\leq \beta < 1$, 0 $< \lambda \leq 1$, $\gamma \geq 0$, $m \in \mathbb{N}$) be given by (3). Then

$$|a_{m+1}| \leq \frac{2\lambda}{m+\gamma} \sqrt{\frac{2(1-\beta)}{\left(\frac{m}{m+\gamma} + 1\right)\lambda^2 + \frac{m}{m+\gamma} \lambda + 1}}$$

and

$$|a_{2m+1}| \leq \frac{8\lambda^2(m+1)(1-\beta)^2}{(m+\gamma)^2(\lambda+1)^2} + \frac{4\lambda(1-\beta)}{(2m+\gamma)(\lambda+1)}.$$

PROOF. It follows from conditions (22) and (23) that there exist $p, q \in \mathcal{P}$ such that

$$\frac{1}{2} \left( \frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}} + \left( \frac{z^{1-\gamma}f'(z)}{(f(z))^{1-\gamma}} \right)^{\frac{1}{\lambda}} \right) = \beta + (1-\beta)p(z)$$

(26)

and

$$\frac{1}{2} \left( \frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}} + \left( \frac{w^{1-\gamma}g'(w)}{(g(w))^{1-\gamma}} \right)^{\frac{1}{\lambda}} \right) = \beta + (1-\beta)q(w),$$

(27)

where $p(z)$ and $q(w)$ have the forms (11) and (12), respectively. Equating coefficients (26) and (27) yields

$$\frac{(m+\gamma)(\lambda+1)}{2\lambda} a_{m+1} = (1-\beta)p_m,$$

(28)

$$\frac{(2m+\gamma)(\lambda+1)}{4\lambda} \left( 2a_{2m+1} + (\gamma-1)a_{2m+1}^2 \right) + \frac{(m+\gamma)^2(1-\lambda)}{4\lambda^2} a_{2m+1}^2 = (1-\beta)p_{2m},$$

(29)
\[-\frac{(m + \gamma)(\lambda + 1)}{2\lambda} a_{m+1} = (1 - \beta)q_m \quad (30)\]

and
\[
\frac{(2m + \gamma)(\lambda + 1)}{4\lambda} ((2m + \gamma + 1)a_{m+1}^2 - 2a_{2m+1}) + \frac{(m + \gamma)^2 (1 - \lambda)}{4\lambda^2} a_{m+1}^2 = (1 - \beta)q_{2m}. \quad (31)
\]

From (28) and (30), we get
\[
p_m = -q_m \quad (32)
\]
and
\[
\frac{(m + \gamma)^2 (\lambda + 1)^2}{2\lambda^2} a_{m+1}^2 = (1 - \beta)^2 (p_m^2 + q_m^2). \quad (33)
\]
Adding (29) and (31), we obtain
\[
\left(\frac{(2m + \gamma)(m + \gamma)(\lambda + 1)}{2\lambda} + \frac{(m + \gamma)^2 (1 - \lambda)}{2\lambda^2}\right) a_{m+1}^2 = (1 - \beta)(p_{2m} + q_{2m}). \quad (34)
\]
Therefore, we have
\[
a_{m+1}^2 = \frac{2\lambda^2(1 - \beta)(p_{2m} + q_{2m})}{(m + \gamma)^2 \left[\left(\frac{m}{m+\gamma} + 1\right)\lambda^2 + \frac{m}{m+\gamma}\lambda + 1\right]}.
\]
Applying Lemma 1 for the coefficients \(p_{2m}\) and \(q_{2m}\), we obtain
\[
|a_{m+1}| \leq \frac{2\lambda}{m + \gamma} \sqrt{\frac{2(1 - \beta)}{\left(\frac{m}{m+\gamma} + 1\right)\lambda^2 + \frac{m}{m+\gamma}\lambda + 1}}.
\]
This gives the desired estimate for \(|a_{m+1}|\) as asserted in (24). In order to find the bound on \(|a_{2m+1}|\), by subtracting (31) from (29), we get
\[
\frac{(2m + \gamma)(\lambda + 1)}{\lambda} a_{2m+1} - \frac{(2m + \gamma)(m + 1)(\lambda + 1)}{2\lambda} a_{m+1}^2 = (1 - \beta) (p_{2m} - q_{2m}),
\]
or equivalently
\[
a_{2m+1} = \frac{(m + 1)}{2} a_{m+1}^2 + \frac{\lambda(1 - \beta) (p_{2m} - q_{2m})}{(2m + \gamma)(\lambda + 1)}.
\]
Upon substituting the value of \(a_{m+1}^2\) from (33), it follows that
\[
a_{2m+1} = \frac{\lambda^2(m + 1)(1 - \beta)^2 (p_{2m}^2 + q_{2m}^2)}{(m + \gamma)^2 (\lambda + 1)^2} + \frac{\lambda(1 - \beta) (p_{2m} - q_{2m})}{(2m + \gamma)(\lambda + 1)}.
\]
Applying Lemma 1 once again for the coefficients \(p_m, p_{2m}, q_m\) and \(q_{2m}\), we obtain
\[
|a_{2m+1}| \leq \frac{8\lambda^2(m + 1)(1 - \beta)^2}{(m + \gamma)^2 (\lambda + 1)^2} + \frac{4\lambda(1 - \beta)}{(2m + \gamma)(\lambda + 1)}.
\]
which completes the proof of Theorem 2.

REMARK 7. In Theorem 2, if we choose

1. \( \gamma = 0 \), then we obtain the results which was proven by Altinkaya and Yalçin \([2], \text{Theorem 2}\);  

2. \( \lambda = \gamma = 1 \), then we obtain the results which was proven by Srivastava et al. \([19], \text{Theorem 3}\).

For one-fold symmetric bi-univalent functions, Theorem 2 reduces to the following corollary:

COROLLARY 2. Let \( f \in S_2^c(\lambda, \gamma; \beta) \) \((0 \leq \beta < 1, \ 0 < \lambda \leq 1, \ \gamma \geq 0)\) be given by (1). Then

\[
|a_2| \leq \frac{2\lambda}{1+\gamma} \sqrt{\frac{2(1-\beta)}{1+\gamma} + \frac{1}{1+\gamma} \lambda + 1}
\]

and

\[
|a_3| \leq \frac{16\lambda^2 (1-\beta)^2}{(1+\gamma)^2 (\lambda+1)^2} + \frac{4\lambda (1-\beta)}{(2+\gamma)(\lambda+1)}.
\]

REMARK 8. In Corollary 2, if we choose

1. \( \lambda = 1 \), then we have the results which was given by Prema and Keerthi \([13], \text{Theorem 3.2}\);  

2. \( \lambda = 1 \) and \( \gamma = 0 \), then we have the results obtained by Murugusundaramoorthy et al. \([11], \text{Corollary 7}\);  

3. \( \lambda = \gamma = 1 \), then we obtain the results obtained by Srivastava et al. \([18], \text{Theorem 2}\).

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