A Note On \( k \)-Derangements*

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Received 26 July 2017

Abstract

Let \( D_{k;n} \) denote the set of \( k \)-derangements in \( S_n \). In this paper, we determine the maximum of \( \Psi = \sum_{i=1}^{n} |\pi(i) - i| \), over all elements \( \pi \) of \( D_{k;n} \). Moreover, the structure of \( \pi \in D_{k;n} \) that maximizes \( \Psi \) is a particular bipartite graph.

1 Introduction

Suppose that \( S_n \) is the symmetric group on the set \([n] = \{1, 2, \ldots, n\}\). Let \([n]^k \) \((1 \leq k \leq n)\) denote the set of all subsets containing \( k \) distinct elements of \([n]\). The group \( S_n \) acts in the natural way on \([n]^k\). In other words, for each \( \pi \in S_n \),

\[
\{i_1, \ldots, i_k\}^\pi = \{\pi(i_1), \ldots, \pi(i_k)\}.
\]

A \textit{k-derangement} in \( S_n \) is a permutation \( \pi \) on \([n]\) that leaves no \( k \)-subset of elements fixed. In other words, \( x^\pi \neq x \) for all \( x \in [n]^k \). Let \( D_{k,n} \) denote the set of \( k \)-derangements of \( S_n \). Specifically, if \( k = 1 \), then \( D_n = D_{1,n} \) is the set of derangements in \( S_n \), that is, the set of permutations in \( S_n \) without fixed points. Suppose that \( \pi \in S_n \). Construct a bipartite graph \( \Gamma_\pi = (X \cup Y, E) \), corresponding to \( \pi \), where \( X = \{x_i : i \in [n]\} \), \( Y = \{y_i : i \in [n]\} \) and \( E = \{(x_i, y_j) : i, j \in [n], \pi(i) = j\} \).

In the current paper, we measure how much derangement is actually disordered. For this, the following term is defined:

\[
\Psi = \sum_{i=1}^{n} |\pi(i) - i|.
\]

Then, let \( \Psi \) denote the maximum of \( \Psi \) over all elements \( \pi \) of \( D_{k,n} \). \( \Psi \) is determined and it is shown that \( \Psi \) is independent of \( k \).

The following proposition determines all permutations that belong to \( D_{k,n} \).

PROPOSITION 1 ([2]). A permutation \( \sigma \in S_n \) is a \( k \)-derangement if and only if the cycle decomposition of \( \sigma \) does not contain a set of cycles whose lengths partition \( k \).

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*Mathematics Subject Classifications: 05A05, 05C99.

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2 Computing $\Psi$ for $k$-Derangements

The main result of this paper is the following theorem.

THEOREM 1. Suppose $k$ and $n$ are integers and $1 \leq k < n$. Then $\Psi$ is independent of $k$ and we have

$$\Psi = \begin{cases} \frac{1}{2}n^2 & \text{if } n \text{ is even,} \\ \frac{1}{2}(n^2 - 1) & \text{if } n \text{ is odd.} \end{cases}$$

PROOF. Let us first prove a reduced form of the theorem for $k = 1$. Consider the case that $n$ is even, i.e., $n = 2m$ for an integer $m \geq 1$. Let $\pi \in D_{2m}$ be a derangement such that $\pi(i) > m$ if and only if $i \leq m$. We claim that $\Psi_\pi$ is maximized over all elements $\pi$ of $D_{2m}$ and its value is equal to $\Psi$. Suppose by the contrary that $\Psi_\pi$ is maximized for some $\sigma \in D_{2m}$ such that $\Gamma_\sigma$ contains the edge $(x_i, y_{\pi(i)})$ where $\sigma(i) \leq m$ for some $i \leq m$. Let $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ be a partition of vertices of $\Gamma_\sigma$ such that $X_1 = \{x_1, \ldots, x_m\}$, $X_2 = \{x_{m+1}, \ldots, x_{2m}\}$, $Y_1 = \{y_1, \ldots, y_m\}$ and $Y_2 = \{y_{m+1}, \ldots, y_{2m}\}$. So $x_i \in X_1$ and there exists $j \in [m]$ such that $y_j \in Y_1$ and $\sigma(i) = j$. Since each vertex of $\Gamma_\sigma$ has degree 1 and $|X_2| = |Y_1\setminus\{y_j\}| + 1$, by the pigeonhole principle, there exists an edge from a vertex $x_r$ in $X_2$ to a vertex $y_s$ in $Y_2$. Now consider a new permutation $\pi'$ such that $\pi'(i) = s, \pi'(r) = j$ and $\pi'(t) = \sigma(t)$ for other $t \neq i, r$. Since $1 \leq i, j \leq m$ and $m + 1 \leq r, s \leq 2m$, we have

$$|i - j| + |r - s| \leq |i - s| + |r - j|.$$ 

From the above inequality, it is easy to deduce that $\Psi_\sigma < \Psi_{\pi'}$, which is a contradiction. Now we show that the value of $\Psi_\pi$ is constant for each $\pi \in D_{2m}$ such that $\pi(i) > m$ if and only if $i \leq m$. This condition implies that $\pi(i) - i$ is positive if $i \in [m]$, otherwise it is negative. Compute $\Psi_\pi$ as

$$\Psi_\pi = \sum_{i=1}^{m} |\pi(i) - i| + \sum_{i=m+1}^{2m} |\pi(i) - i| = \sum_{i=1}^{m} (\pi(i) - i) + \sum_{i=m+1}^{2m} (i - \pi(i))$$

$$= \sum_{i=1}^{m} \pi(i) + \sum_{i=m+1}^{2m} i - \pi(i) = \sum_{i=1}^{m} \pi(i) - \sum_{i=m+1}^{2m} \pi(i).$$

By the structure of $\pi$, it is easy to see that

$$\sum_{i=1}^{m} \pi(i) = \sum_{i=m+1}^{2m} i \quad \text{and} \quad \sum_{i=m+1}^{2m} \pi(i) = \sum_{i=1}^{m} i.$$

Hence

$$\Psi_\pi = 2 \sum_{i=m+1}^{2m} i - 2 \sum_{i=1}^{m} i = 2m^2 = \frac{n^2}{2}.$$ 

This completes the proof in the case that $n$ is even.
Consider the case that \( n \) is odd, i.e., \( n = 2m + 1 \) for an integer \( m \geq 1 \). Let \( \pi \in D_{2m+1} \) be a derangement such that \( \Gamma_\pi \) has an edge \((x_i, y_{2m+1})\). Let \( X = X_1 \cup X_2 \) and \( Y = Y_1 \cup Y_2 \) be a partition of vertices of \( \Gamma_\pi \) such that \( X_1 = \{x_1, \ldots, x_{m+1}\} \), \( X_2 = \{x_{m+2}, \ldots, x_{2m+1}\} \), \( Y_1 = \{y_1, \ldots, y_m\} \) and \( Y_2 = \{y_{m+1}, \ldots, y_{2m+1}\} \). Then vertex \( x_i \) must belong to \( X_1 \). Otherwise, by the pigeonhole principle, there exists an edge from a vertex \( x_r \) in \( X_1 \) to a vertex \( y_s \) in \( Y_1 \). For the same reason as above, we can also get a contradiction. Let \( \Gamma'_{\pi'} \) be the graph obtained from \( \Gamma_\pi \) by deleting the two vertices \( x_i \) and \( y_{2m+1} \). Then \( \Gamma'_{\pi'} \) is a bipartite graph such that it has an even number of vertices on both sides. Clearly, \( \Psi_\pi \) is maximized if and only if \( \Psi_{\pi'} \) is also.

As in the even case, \( \Psi_{\pi'} \) is maximized exactly when \( \pi'(i) \) is a derangement such that \( \pi'(i) > m \) if and only if \( i \leq m + 1 \). Obviously, \( \Psi_\pi = \Psi_{\pi'} + 2m + 1 - i \). Moreover, the same of argument as in the even case we can show that \( \Psi_{\pi'} = (n^2 - 1)/2 - n + i \). So \( \Psi_\pi = (n^2 - 1)/2 \) and its value is equal to \( \Psi \). This completes the proof in the case that \( n \) is odd.

Now let \( k \) be an arbitrary positive integer. Let

\[ \pi = (1 \ 2n \ 2n - 1 \ \ldots \ n - 1 \ n + 2 \ n \ n + 1) \]

be a cyclic permutation of length \( 2n \). In other words, the permutation \( \pi \in S_{2n} \) is representing the following mapping:

\[ \pi(i) = \begin{cases} 
2n + 1 - i & \text{for } 1 \leq i \leq n, \\
1 & \text{for } i = n + 1, \\
2n + 2 - i & \text{for } n + 1 < i \leq 2n.
\end{cases} \]

Since the cycle structure of the permutation \( \pi \) is one cycle of length \( 2n \), so by Proposition 1, \( \pi \) is a \( k \)-derangement for \( k \in [2n - 1] \). It is easy to see that \( \pi(i) > n \) if and only if \( i \leq n \). Similarly, if

\[ \pi(i) = \begin{cases} 
n + i & \text{for } 1 \leq i \leq n + 1, \\
i - n - 1 & \text{for } n + 1 < i \leq 2n + 1,
\end{cases} \]

then \( \pi = (1 \ n + 1 \ 2n + 1 \ n \ \ldots \ 3 \ n + 3 \ 2 \ n + 2) \) is a cyclic permutation of length \( 2n + 1 \) in \( S_{2n+1} \). So by Proposition 1, \( \pi \) is a \( k \)-derangement permutation for \( k \in [2n] \). Also, \( \Psi_\pi \) satisfies the maximum condition in the odd case. Hence, this completes the proof of the theorem.

Acknowledgment. I thank Professor M. Hassani for fruitful discussions, helpful suggestions and encouragement. Also, I deeply acknowledge the anonymous referees for their helpful comments and suggestions.

References
