Inequalities Of \((k, s), (k, h)\)-Type For Riemann-Liouville Fractional Integrals *

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Abstract

The main objective of this paper is to establish some new integral inequalities by using the \((k, h), (k, s)\)-Riemann-Liouville fractional integrals in the case of synchronous functions.

1 Introduction

During the last two decades, many authors have studied some well-known inequalities and their applications using Riemann-Liouville fractional derivative and integral. For more about these, see [1–10] and the related references therein.

DEFINITION 1 ([13]). Two integrable functions \(f\) and \(g\) are said to be synchronous on \([a, b]\) if

\[(f(x) - f(y))(g(x) - g(y)) \geq 0\]

for all \(x \in [a, b]\).

Recently, in [13] Dahmani gave the following fractional integral inequalities, using standard Riemann-Liouville fractional integral:

THEOREM 1 ([13, Theorem 2]). Let \(f, g\) be two synchronous functions on \([0, \infty)\) and let \(p, q, r : [0, \infty) \to [0, \infty)\), then for all \(t > a \geq 0, \alpha > 0\) the following \((k, h)\)-fractional integral inequality

\[
\begin{align*}
2J^\alpha r(t) [J^\alpha p(t) J^\alpha (qfg)(t) + J^\alpha q(t) J^\alpha (pf)(t)] &+ 2J^\alpha p(t) J^\alpha q(t) J^\alpha (rf)(t) \\
\geq J^\alpha r(t) [J^\alpha (pf)(t) J^\alpha (qg)(t) + J^\alpha (qf)(t) J^\alpha (pg)(t)] &+ J^\alpha p(t) [J^\alpha (rf)(t) J^\alpha (qg)(t) + J^\alpha (qf)(t) J^\alpha (rg)(t)] \\
&+ J^\alpha q(t) [J^\alpha (rf)(t) J^\alpha (pg)(t) + J^\alpha (pf)(t) J^\alpha (rg)(t)]
\end{align*}
\]

holds.

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THEOREM 2 ([13, Theorem 4]). Let \( f, g \) be two synchronous functions on \([0, \infty)\) and let \( p, q, r : [0, \infty) \to [0, \infty) \), then for all \( t > a \geq 0, \alpha > 0, \beta > 0 \) the following \((k, h)\)-fractional integral inequality

\[
J^\alpha r(t) \left[ J^\alpha q(t) J^\beta(pfg)(t) + 2J^\alpha p(t) J^\beta(qfg)(t) + J^\beta q(t) J^\alpha (pf)(t) \right] \\
+ \left[ J^\alpha p(t) J^\beta q(t) + J^\beta p(t) J^\alpha q(t) \right] J^\alpha (rfg)(t)
\]

\[
\geq J^\alpha r(t) \left[ J^\alpha (pf)(t) J^\beta(qg)(t) + J^\beta(qf)(t) J^\alpha (pg)(t) \right] \\
+ J^\alpha p(t) \left[ J^\alpha (rf)(t) J^\beta(qg)(t) + J^\beta(qf)(t) J^\alpha (rg)(t) \right] \\
+ J^\beta q(t) \left[ J^\alpha (rf)(t) J^\beta(pg)(t) + J^\beta(pf)(t) J^\alpha (rg)(t) \right]
\]

holds.

In literature few results have been obtained on some fractional integral inequalities for \( k \)-fractional integrals in [14, 15, 16]. Motivated by [16, 17, 18], our purpose in this work is to establish some inequalities for generalized \( k \)-fractional integrals that are called in the literature by \((k, s)\) and \((k, h)\)-Riemann-Liouville fractional integrals which are stated in Theorems 3 and 4 of the last section.

2 Preliminaries

Here, we will give the necessary notation and basic definitions. Due to page restrictions, only the basic definitions of the \((k, s)\), \((k, h)\)-Riemann-Liouville fractional integrals are given, and the reader is referred to [14–18] for more details.

DEFINITION 2. Let \( 0 < x < b, \alpha > 0 \) and \( f \in L_1(a, b) \), then the \( k \)-fractional integral of the Riemann-Liouville type is defined as follows:

\[
k J^\alpha f(x) = \frac{1}{k \Gamma_k(\alpha)} \int_0^x (x-t)^{\frac{\alpha}{k}-1} f(t) dt,
\]

where \( k \)-gamma function is defined by

\[
\Gamma_k(x) = \int_0^\infty t^{\frac{x}{k}-1} e^{-\frac{t}{k}} dt.
\]

Note that when \( k \to 1 \), then the \( k \)-fractional integral reduces to the classical Riemann-liouville fractional integral [11, 12].

DEFINITION 3. Let \( a \leq x \leq b \) and \( f \in L_1(a, b) \), then the \((k, s)\)-Riemann-Liouville fractional integral of \( f \) of order \( \alpha > 0 \) is defined by

\[
s_k J^\alpha_a f(x) = \frac{(s+1)^{1-\frac{s}{k}}}{k \Gamma_k(\alpha)} \int_a^x (x^{s+1} - t^{s+1})^{\frac{s}{k}-1} t^s f(t) dt,
\]

where \( k > 0 \) and \( s \in \mathbb{R} \setminus \{-1\} \).
DEFINITION 4. Let \(a \leq x \leq b\) and \(f \in L_1(a, b)\), then the \((k, h)\)-Riemann-Liouville fractional integral of \(f\) of order \(\alpha > 0\) is defined by

\[
(k^{\alpha}_{a^+, h}) f(x) = \frac{1}{k \Gamma(k)} \int_a^x (h(x) - h(t))^{\frac{\alpha}{k} - 1} h'(t) f(t) dt,
\]

where \(k > 0\).

3 Main Results

To obtain the first main theorem, we prove the following Lemma 1.

LEMMA 1. Let \(f, g\) be two synchronous functions on \([0, \infty)\) and let \(y, z \geq 0\), then for all \(t > a \geq 0\) and \(\alpha > 0\), the following inequality for \((k, h)\)-fractional integrals

\[
\begin{align*}
(k^{\alpha}_{a^+, h}) y(t) & \left( k^{\alpha}_{a^+, h} \right) (zf)g(t) + \left( k^{\alpha}_{a^+, h} \right) z(t) \left( k^{\alpha}_{a^+, h} \right) (yf)g(t) \\
& \geq \left( k^{\alpha}_{a^+, h} \right) (yf)g(t) \left( k^{\alpha}_{a^+, h} \right) (zf) + \left( k^{\alpha}_{a^+, h} \right) (zf)g(t) \left( k^{\alpha}_{a^+, h} \right) (yg)g(t)
\end{align*}
\]

holds.

PROOF. Since \(f\) and \(g\) are two synchronous functions on \([0, \infty)\) then for all \(\tau, \xi \geq 0\), we have

\[(f(\tau) - f(\rho))(g(\xi) - g(\rho)) \geq 0.\]

This leads to

\[f(\xi)g(\xi) + f(\rho)g(\rho) \geq f(\xi)g(\rho) + f(\rho)g(\xi).\]  \hspace{1cm} (2)

Multiplying both sides of (2) by \(\frac{h(t) - h(\xi)}{k \Gamma(k)} h'(\xi) y(\xi)\) for \(\xi \in (a, t)\), then integrating the resulting inequalities with respect to \(\xi\) over \((a, t)\), respectively, we obtain

\[
\begin{align*}
\left( k^{\alpha}_{a^+, h} \right) (yf)g(t) + f(\rho)g(\rho) \left( k^{\alpha}_{a^+, h} \right) y(t) & \geq g(\rho) \left( k^{\alpha}_{a^+, h} \right) (yf)g(t) + f(\rho) \left( k^{\alpha}_{a^+, h} \right) (yg)g(t).
\end{align*}
\]

Multiplying both sides of (3) by \(\frac{h(t) - h(\rho)}{k \Gamma(k)} h'(\rho) z(\rho)\) for \(\rho \in (a, t)\), then integrating the resulting inequalities with respect to \(\rho\) over \((a, t)\), we have

\[
\begin{align*}
\left( k^{\alpha}_{a^+, h} \right) y(t) \left( k^{\alpha}_{a^+, h} \right) (zf)g(t) + \left( k^{\alpha}_{a^+, h} \right) z(t) \left( k^{\alpha}_{a^+, h} \right) (yf)g(t) & \geq \left( k^{\alpha}_{a^+, h} \right) (yf)g(t) \left( k^{\alpha}_{a^+, h} \right) (zf) + \left( k^{\alpha}_{a^+, h} \right) (zf)g(t) \left( k^{\alpha}_{a^+, h} \right) (yg)g(t),
\end{align*}
\]

and so the proof is completed.
THEOREM 3. Let \( f, g \) be two synchronous functions on \([0, \infty)\) and let \( y, z \geq 0 \), then for all \( t > a \geq 0, \alpha > 0 \) the following \((k,h)\)-fractional integral inequality

\[
2 \left( kJ^{\alpha}_{a^+, h} \right) r(t) \left[ kJ^{\alpha}_{a^+, h} p(t) \left( kJ^{\alpha}_{a^+, h} \right) (qfg)(t) \right] + \left( kJ^{\alpha}_{a^+, h} \right) q(t) \left( kJ^{\alpha}_{a^+, h} \right) (pf)(t) \\
+ 2 \left( kJ^{\alpha}_{a^+, h} \right) p(t) \left( kJ^{\alpha}_{a^+, h} \right) q(t) \left( kJ^{\alpha}_{a^+, h} \right) (rfg)(t) \\
\geq \left( kJ^{\alpha}_{a^+, h} \right) r(t) \left[ \left( kJ^{\alpha}_{a^+, h} \right) p(t) \left( kJ^{\alpha}_{a^+, h} \right) (qg)(t) \right] + \left( kJ^{\alpha}_{a^+, h} \right) (qf)(t) \left( kJ^{\alpha}_{a^+, h} \right) (pg)(t) \\
+ \left( kJ^{\alpha}_{a^+, h} \right) (pf)(t) \left( kJ^{\alpha}_{a^+, h} \right) (rg)(t) \tag{4}
\]

holds.

PROOF. Put \( v = p \) and \( w = q \) into inequality (1), and then multiplying the resulting inequality by \( \left( kJ^{\alpha}_{a^+, h} \right) r(t) \), we find

\[
\left( kJ^{\alpha}_{a^+, h} \right) r(t) \left[ \left( kJ^{\alpha}_{a^+, h} \right) p(t) \left( kJ^{\alpha}_{a^+, h} \right) (qfg)(t) + \left( kJ^{\alpha}_{a^+, h} \right) q(t) \left( kJ^{\alpha}_{a^+, h} \right) (pf)(t) \right] \\
\geq \left( kJ^{\alpha}_{a^+, h} \right) r(t) \left[ \left( kJ^{\alpha}_{a^+, h} \right) (qf)(t) \left( kJ^{\alpha}_{a^+, h} \right) (pg)(t) \right] \tag{5}
\]

Again, put \( v = r \) and \( w = q \) into inequality (1) and multiplying the result by \( \left( kJ^{\alpha}_{a^+, h} \right) p(t) \), we get

\[
\left( kJ^{\alpha}_{a^+, h} \right) p(t) \left[ \left( kJ^{\alpha}_{a^+, h} \right) r(t) \left( kJ^{\alpha}_{a^+, h} \right) (qfg)(t) + \left( kJ^{\alpha}_{a^+, h} \right) q(t) \left( kJ^{\alpha}_{a^+, h} \right) (rfg)(t) \right] \\
\geq \left( kJ^{\alpha}_{a^+, h} \right) p(t) \left[ \left( kJ^{\alpha}_{a^+, h} \right) (rf)(t) \left( kJ^{\alpha}_{a^+, h} \right) (rg)(t) \right] \tag{6}
\]

Similarly, we can obtain

\[
\left( kJ^{\alpha}_{a^+, h} \right) q(t) \left[ \left( kJ^{\alpha}_{a^+, h} \right) r(t) \left( kJ^{\alpha}_{a^+, h} \right) (pdf)(t) + \left( kJ^{\alpha}_{a^+, h} \right) (pf)(t) \right] \\
+ \left( kJ^{\alpha}_{a^+, h} \right) (qf)(t) \left( kJ^{\alpha}_{a^+, h} \right) (pg)(t) \\
\geq \left( kJ^{\alpha}_{a^+, h} \right) q(t) \left[ \left( kJ^{\alpha}_{a^+, h} \right) (rf)(t) \left( kJ^{\alpha}_{a^+, h} \right) (pg)(t) \right] \tag{7}
\]
Adding the inequalities (5)–(7), we get the required inequality (4).

REMARK 1. If we choose \( h(x) = \frac{s^{\alpha + 1}}{x^{\alpha + 1}} \), \( s \neq -1 \), then the inequality (4) reduces to the following \((k, s)\)-fractional integral inequality

\[
2 \frac{\alpha}{h_s} \int_a^t f(t)^{\frac{\alpha}{h_s}} (qfg)(t) dt + 2 \frac{\alpha}{h_s} \int_a^t g(t)^{\frac{\alpha}{h_s}} (pfhg)(t) dt \\
+ 2 \frac{\alpha}{h_s} \int_a^t p(t) \frac{\alpha}{h_s} \int_a^t q(t)^{\frac{\alpha}{h_s}} (pf)(t) dt \\
\geq \frac{\alpha}{h_s} \int_a^t f(t)^{\frac{\alpha}{h_s}} (qf)(t)^{\frac{\alpha}{h_s}} (pq)(t) dt + \frac{\alpha}{h_s} \int_a^t g(t)^{\frac{\alpha}{h_s}} (gf)(t)^{\frac{\alpha}{h_s}} (rg)(t) dt \\
+ \frac{\alpha}{h_s} \int_a^t p(t)^{\frac{\alpha}{h_s}} (pfg)(t) dt \\
+ \frac{\alpha}{h_s} \int_a^t q(t)^{\frac{\alpha}{h_s}} (pfg)(t) dt \\
+ \frac{\alpha}{h_s} \int_a^t r(t)^{\frac{\alpha}{h_s}} (pfg)(t) dt.
\]

To obtain the second theorem, we need the following Lemma 2.

LEMMA 2. Let \( f, g \) be two synchronous functions on \([0, \infty)\) and let \( y, z \geq 0 \). Then for all \( t > a \geq 0, \alpha > 0, \beta > 0 \), we have

\[
\left( k^{\alpha/h_s} \right) y(t) \left( k^{\beta/h_s} \right) (zf)(t) + \left( k^{\beta/h_s} \right) z(t) \left( k^{\alpha/h_s} \right) (yfg)(t) \\
\geq \left( k^{\alpha/h_s} \right) (yf)(t) \left( k^{\beta/h_s} \right) (zf)(t) + \left( k^{\beta/h_s} \right) (zf)(t) \left( k^{\alpha/h_s} \right) (yfg)(t).
\]

PROOF. Multiplying both sides of (3) by \( \frac{(h(t) - h(s))^\frac{\alpha}{\beta} - 1}{h'(s)} \), \( \rho \in (a, t) \), then integrating the resulting inequalities with respect to \( \rho \) over \((a, t)\), we have

\[
\left( k^{\alpha/h_s} \right) y(t) \left( k^{\beta/h_s} \right) (zf)(t) + \left( k^{\beta/h_s} \right) z(t) \left( k^{\alpha/h_s} \right) (yfg)(t) \\
\geq \left( k^{\alpha/h_s} \right) (yf)(t) \left( k^{\beta/h_s} \right) (zf)(t) + \left( k^{\beta/h_s} \right) (zf)(t) \left( k^{\alpha/h_s} \right) (yfg)(t).
\]

This completes the proof of inequality (9).

THEOREM 4. Let \( f, g \) be two synchronous functions on \([0, \infty)\) and let \( y, z \geq 0 \). Then for all \( t > a \geq 0, \alpha > 0, \beta > 0 \) the following \((k, h)\)-fractional integral inequality

\[
\left( k^{\alpha/h_s} \right) r(t) \left( k^{\alpha/h_s} \right) q(t) \left( k^{\beta/h_s} \right) (pf)(t) \\
+ \left( k^{\beta/h_s} \right) p(t) \left( k^{\alpha/h_s} \right) (zf)(t) + \left( k^{\alpha/h_s} \right) q(t) \left( k^{\beta/h_s} \right) (pf)(t) \\
+ \left( k^{\alpha/h_s} \right) r(t) \left( k^{\beta/h_s} \right) (qg)(t) + \left( k^{\beta/h_s} \right) p(t) \left( k^{\alpha/h_s} \right) (qg)(t) \\
+ \left( k^{\alpha/h_s} \right) q(t) \left( k^{\beta/h_s} \right) (rf)(t) + \left( k^{\beta/h_s} \right) p(t) \left( k^{\alpha/h_s} \right) (rf)(t) \\
\geq \left( k^{\alpha/h_s} \right) (rf)(t) \left( k^{\beta/h_s} \right) (qg)(t) + \left( k^{\beta/h_s} \right) (qg)(t) \left( k^{\alpha/h_s} \right) (rf)(t).
\]

(10)
Again, using inequality (9) with $y = p$ and $z = q$, and then multiplying the resulting inequality by $(k J^\alpha_{a+}, h) r(t)$, we find

$$
\left(k J^\alpha_{a+}, h\right) r(t) \left[ \left(k J^\alpha_{a+}, h\right) p(t) \left(k J^\beta_{a+}, h\right) (q f g)(t) + \left(k J^\beta_{a+}, h\right) q(t) \left(k J^\alpha_{a+}, h\right) (p f g)(t) \right]
$$

$$
\geq \left(k J^\alpha_{a+}, h\right) r(t) \left[ \left(k J^\alpha_{a+}, h\right) (p f)(t) \left(k J^\beta_{a+}, h\right) (q f g)(t) + \left(k J^\beta_{a+}, h\right) q(t) \left(k J^\alpha_{a+}, h\right) (p f g)(t) \right]
$$

$$
+ \left(k J^\beta_{a+}, h\right) (q f)(t) \left(k J^\alpha_{a+}, h\right) (p g)(t).
$$

(11)

Again, using inequality (9) with $y = r$ and $z = q$, we obtain

$$
\left(k J^\alpha_{a+}, h\right) r(t) \left( k J^\beta_{a+}, h \right) (q f g)(t) + \left(k J^\beta_{a+}, h\right) q(t) \left(k J^\alpha_{a+}, h\right) (r f g)(t)
$$

$$
\geq \left(k J^\alpha_{a+}, h\right) r(t) \left(k J^\beta_{a+}, h\right) (q g)(t) + \left(k J^\beta_{a+}, h\right) q(t) \left(k J^\alpha_{a+}, h\right) (r g)(t)
$$

(12)

Multiplying both sides of (12) by $(k J^\alpha_{a+}, h) p(t)$, we get

$$
\left(k J^\alpha_{a+}, h\right) p(t) \left(k J^\alpha_{a+}, h\right) r(t) \left(k J^\beta_{a+}, h\right) (q f g)(t) + \left(k J^\beta_{a+}, h\right) q(t) \left(k J^\alpha_{a+}, h\right) (r f g)(t)
$$

$$
\geq \left(k J^\alpha_{a+}, h\right) p(t) \left(k J^\alpha_{a+}, h\right) (r f)(t) \left(k J^\beta_{a+}, h\right) (q g)(t) + \left(k J^\beta_{a+}, h\right) q(t) \left(k J^\alpha_{a+}, h\right) (r g)(t)
$$

$$
+ \left(k J^\beta_{a+}, h\right) (q f)(t) \left(k J^\alpha_{a+}, h\right) (r g)(t).
$$

(13)

Similarly, we can obtain

$$
\left(k J^\alpha_{a+}, h\right) q(t) \left(k J^\beta_{a+}, h\right) r(t) \left(k J^\beta_{a+}, h\right) (p f g)(t) + \left(k J^\beta_{a+}, h\right) q(t) \left(k J^\alpha_{a+}, h\right) (r f g)(t)
$$

$$
\geq \left(k J^\alpha_{a+}, h\right) q(t) \left(k J^\beta_{a+}, h\right) (q f)(t) \left(k J^\beta_{a+}, h\right) (p g)(t) + \left(k J^\beta_{a+}, h\right) q(t) \left(k J^\alpha_{a+}, h\right) (r g)(t)
$$

$$
+ \left(k J^\beta_{a+}, h\right) (p f)(t) \left(k J^\alpha_{a+}, h\right) (r g)(t).
$$

(14)

Adding the inequalities (11)–(14), we get the inequality (9).

**REMARK 2.** If we choose $h(x) = \frac{x + 1}{s + 1}$, $s \neq -1$, then the equality (10) reduces to the following $(k, s)$-fractional integral

$$
\begin{align*}
\left[ k J^\alpha_{a+} \right] r(t) \left[ k J^\alpha_{a+} q(t) k J^\alpha_{a+} (p f g)(t) + 2 k J^\alpha_{a+} p(t) k J^\alpha_{a+} (q f g)(t) + 2 k J^\alpha_{a+} q(t) k J^\alpha_{a+} (p f g)(t) \right] \\
+ \left[ k J^\alpha_{a+} p(t) k J^\alpha_{a+} q(t) + k J^\alpha_{a+} p(t) k J^\alpha_{a+} q(t) \right] k J^\alpha_{a+} (r f g)(t) \\
\geq \left[ k J^\alpha_{a+} r(t) \right] k J^\alpha_{a+} (p f g)(t) k J^\alpha_{a+} (q f g)(t) + k J^\alpha_{a+} (q f)(t) k J^\alpha_{a+} (p f g)(t) \\
+ \left[ k J^\alpha_{a+} p(t) \right] k J^\alpha_{a+} (r f)(t) k J^\alpha_{a+} (q g)(t) + k J^\alpha_{a+} (q f)(t) k J^\alpha_{a+} (r g)(t) \\
+ \left[ k J^\alpha_{a+} q(t) \right] k J^\alpha_{a+} (r f)(t) k J^\alpha_{a+} (p g)(t) + k J^\alpha_{a+} (p f)(t) k J^\alpha_{a+} (r g)(t).
\end{align*}
$$

**REMARK 3.** If $f, g, r, p$ and $q$ satisfy the following conditions,
(i) The function $f$ and $g$ are asynchronous on $[0, \infty)$.

(ii) The function $r, p, q$ are negative on $[0, \infty)$.

(iii) Two of the function $r, p, q$ are positive and the third is negative on $[0, \infty)$.

then the inequality (4) and (10) are reversed.

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References


