Uniqueness And Value Distribution Of Differences Of Meromorphic Functions*

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Abstract

The purpose of the paper is to study the uniqueness problems of difference polynomials of meromorphic functions sharing a small function. The results of the paper improve and generalize the recent results due to Liu, et al. [11] and Liu, et al. [12].

1 Introduction, Definitions and Results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

We adopt the standard notations of value distribution theory (see [6]). For a non-constant meromorphic function $f$, we denote by $T(r, f)$ the Nevanlinna characteristic of $f$ and by $S(r, f)$ any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ possibly outside a set of finite logarithmic measure. We denote by $T(r)$ the maximum of $T(r, F)$ and $T(r, G)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \to \infty$, outside of a possible exceptional set of finite logarithmic measure.

A meromorphic function $a(z)$ is called a small function with respect to $f$, provided that $T(r, a) = S(r, f)$. The order of $f$ is defined by

$$
\sigma(f) = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}.
$$

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share $a(z)$ CM (counting multiplicities) if $f(z) - a(z)$ and $g(z) - a(z)$ have the same zeros with the same multiplicities and we say that $f(z)$, $g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities.

Recently, the topics of difference equations and difference products in complex plane $\mathbb{C}$ has attracted many mathematicians. Many papers have focused on value distribution of differences and differences operators analogues of Nevanlinna theory ([2, 4, 9, 10]) and many people dealt with the uniqueness problems related to meromorphic functions and their shifts or their difference operators and obtained some interesting results ([11, 12, 13, 16]).

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In 2011, K. Liu, X. L. Liu and T. B. Cao studied the uniqueness of the difference monomials and obtained the following results.

THEOREM A ([11]). Let \( f(z) \) and \( g(z) \) be two transcendental meromorphic functions with finite order. Suppose that \( c \in \mathbb{C} \setminus \{0\} \) and \( n \in \mathbb{N} \). If \( n \geq 14, f^n(z)f(z+c) \) and \( g^n(z)g(z+c) \) share 1 CM, then \( f(z) \equiv tg(z) \) or \( f(z)g(z) \equiv t \), where \( t^{n+1} = 1 \).

THEOREM B ([11]). Let \( f(z) \) and \( g(z) \) be two transcendental meromorphic functions with finite order. Suppose that \( c \in \mathbb{C} \setminus \{0\} \) and \( n \in \mathbb{N} \). If \( n \geq 26, f^n(z)f(z+c) \) and \( g^n(z)g(z+c) \) share 1 IM, then \( f(z) \equiv tg(z) \) or \( f(z)g(z) \equiv t \), where \( t^{n+1} = 1 \).

We now explain the notation of weighted sharing as introduced in [8].

DEFINITION 1 ([8]). Let \( k \in \mathbb{N} \cup \{0\} \cup \{\infty\} \). For \( a \in \mathbb{C} \cup \{\infty\} \) we denote by \( E_k(a; f) \) the set of all \( a \)-points of \( f \) where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k+1 \) times if \( m > k \). If \( E_k(a; f) = E_k(a; g) \), we say that \( f, g \) share the value \( a \) with weight \( k \).

We write \( f, g \) share \((a, k)\) to mean that \( f, g \) share the value \( a \) with weight \( k \). Clearly if \( f, g \) share \((a, k)\) then \( f, g \) share \((a, p)\) for any integer \( p, 0 \leq p < k \). Also we note that \( f, g \) share a value \( a \) IM or CM if and only if \( f, g \) share \((a, 0)\) or \((a, \infty)\) respectively.

In 2015, Y. Liu, J. P. Wang and F. H. Liu improved Theorems A, B and obtained the following results.

THEOREM C ([12]). Let \( c \in \mathbb{C} \setminus \{0\} \) and let \( f(z) \) and \( g(z) \) be two transcendental meromorphic functions with finite order, and \( n(\geq 14), k(\geq 3) \) be two positive integers. If \( E_k(1, f^n(z)f(z+c)) = E_k(1, g^n(z)g(z+c)) \), then \( f(z) \equiv t_1g(z) \) or \( f(z)g(z) \equiv t_2 \) for some constants \( t_1 \) and \( t_2 \) satisfying \( t_1^{n+1} = 1 \) and \( t_2^{n+1} = 1 \).

THEOREM D ([12]). Let \( c \in \mathbb{C} \setminus \{0\} \) and let \( f(z) \) and \( g(z) \) be two transcendental meromorphic functions with finite order, and \( n(\geq 16) \) be a positive integer. If \( E_2(1, f^n(z)f(z+c)) = E_2(1, g^n(z)g(z+c)) \), then \( f(z) \equiv t_1g(z) \) or \( f(z)g(z) \equiv t_2 \), for some constants \( t_1 \) and \( t_2 \) satisfying \( t_1^{n+1} = 1 \) and \( t_2^{n+1} = 1 \).

THEOREM E ([12]). Let \( c \in \mathbb{C} \setminus \{0\} \) and let \( f(z) \) and \( g(z) \) be two transcendental meromorphic functions with finite order, and \( n(\geq 22) \) be a positive integer. If \( E_1(1, f^n(z)f(z+c)) = E_1(1, g^n(z)g(z+c)) \), then \( f(z) \equiv t_1g(z) \) or \( f(z)g(z) \equiv t_2 \), for some constants \( t_1 \) and \( t_2 \) satisfying \( t_1^{n+1} = 1 \) and \( t_2^{n+1} = 1 \).

Now it is quite natural to ask the following question.

QUESTION 1. What can be said if the sharing value 1 in Theorems C, D and E is replaced by a nonzero polynomial?

Now taking the possible answer of the above question into background we obtain the following results.
THEOREM 1. Let \( f(z) \) and \( g(z) \) be two transcendental meromorphic functions of finite order, \( c \in \mathbb{C} \setminus \{0\} \) and \( n \in \mathbb{N} \) be such that \( n \geq 14 \). Let \( p(z)(\neq 0) \) be a polynomial such that \( \deg(p) < (n-1)/2 \). If \( f^n(z)f(z+c) - p(z) \) and \( g^n(z)g(z+c) - p(z) \) share \((0,2)\), then one of the following two cases holds:

1. \( f(z) \equiv tg(z) \) for some constant \( t \) such that \( t^{n+1} = 1 \),
2. \( f(z)g(z) \equiv t \), where \( p(z) \) reduces to a nonzero constant \( c \) and \( t \) is a constant such that \( t^{n+1} = c^2 \).

THEOREM 2. Let \( f(z) \) and \( g(z) \) be two transcendental meromorphic functions of finite order, \( c \in \mathbb{C} \setminus \{0\} \) and \( n \in \mathbb{N} \) be such that \( n \geq 16 \). Let \( p(z)(\neq 0) \) be a polynomial such that \( \deg(p) < (n-1)/2 \). Suppose \( f^n(z)f(z+c) - p(z) \) and \( g^n(z)g(z+c) - p(z) \) share \((0,1)\). Then conclusion of Theorem 1 holds.

THEOREM 3. Let \( f(z) \) and \( g(z) \) be two transcendental meromorphic functions of finite order, \( c \in \mathbb{C} \setminus \{0\} \) and \( n \in \mathbb{N} \) be such that \( n \geq 26 \). Let \( p(z)(\neq 0) \) be a polynomial such that \( \deg(p) < (n-1)/2 \). Suppose \( f^n(z)f(z+c) - p(z) \) and \( g^n(z)g(z+c) - p(z) \) share \((0,0)\). Then conclusion of Theorem 1 holds.

We now make the following definitions and notations which are used in the paper.

DEFINITION 2 ([7]). Let \( a \in \mathbb{C} \cup \{\infty\} \). For \( p \in \mathbb{N} \) we denote by \( N(r,a;f | \leq p) \) the counting function of those \( a \)-points of \( f \) (counted with multiplicities) whose multiplicities are not greater than \( p \). By \( \bar{N}(r,a;f | \leq p) \) we denote the corresponding reduced counting function.

In an analogous manner we can define \( N(r,a;f | \geq p) \) and \( \bar{N}(r,a;f | \geq p) \).

DEFINITION 3 ([8]). Let \( k \in \mathbb{N} \cup \{\infty\} \). We denote by \( N_k(r,a;f) \) the counting function of \( a \)-points of \( f \), where an \( a \)-point of multiplicity \( m \) is counted \( m \) times if \( m \leq k \) and \( k \) times if \( m > k \). Then

\[
N_k(r,a;f) = \bar{N}(r,a;f) + \bar{N}(r,a;f | \geq 2) + \ldots + \bar{N}(r,a;f | \geq k).
\]

Clearly \( N_1(r,a;f) = \bar{N}(r,a;f) \).

2 Lemma

In this section we present the lemma which will be needed in the sequel.

Let \( F, G \) be two non-constant meromorphic functions. Henceforth we shall denote by \( H \) the following function

\[
H = \left( \frac{F''}{F'} - \frac{2F'}{F - 1} \right) - \left( \frac{G''}{G'} - \frac{2G'}{G - 1} \right).
\] (1)
LEMMA 1 ([14]). Let \( f \) be a non-constant meromorphic function and let \( a_n(z) \neq 0, a_{n-1}(z), \ldots, a_0(z) \) be meromorphic functions such that \( T(r,a_i(z)) = S(r,f) \) for \( i = 0,1,2,\ldots,n \). Then

\[
T(r,a_n f^n + a_{n-1} f^{n-1} + \ldots + a_1 f + a_0) = nT(r,f) + S(r,f).
\]

LEMMA 2 ([2]). Let \( f(z) \) be a meromorphic function of finite order \( \sigma \), and let \( c \in \mathbb{C} \setminus \{0\} \) be fixed. Then for each \( \varepsilon > 0 \), we have

\[
m(r, \frac{f(z+c)}{f(z)}) + m(r, \frac{f(z)}{f(z+c)}) = O(r^{\sigma-1+\varepsilon}).
\]

The following lemma is a slight modifications of the original version (Theorem 2.1 of [2])

LEMMA 3. Let \( f(z) \) be a transcendental meromorphic function of finite order, \( c \in \mathbb{C} \setminus \{0\} \) be fixed. Then

\[
T(r,f(z+c)) = T(r,f) + S(r,f).
\]

LEMMA 4 ([3]). Let \( f \) be a non-constant meromorphic function of finite order and \( c \in \mathbb{C} \). Then

\[
N(r,0; f(z+c)) \leq N(r,0; f(z)) + S(r,f), \quad N(r,\infty; f(z+c)) \leq N(r,\infty; f) + S(r,f),
\]

\[
\bar{N}(r,0; f(z+c)) \leq \bar{N}(r,0; f(z)) + S(r,f), \quad \bar{N}(r,\infty; f(z+c)) \leq \bar{N}(r,\infty; f) + S(r,f).
\]

Taking \( m = 0 \) in Lemma 2.4 [11], we obtain the following lemma.

LEMMA 5. Let \( f(z) \) be a transcendental meromorphic function of finite order, \( c \in \mathbb{C} \setminus \{0\} \) be fixed and let \( \Phi(z) = f^n(z)f(z+c) \), where \( n \in \mathbb{N} \). Then we have

\[
(n-1) T(r,f) \leq T(r,\Phi) + S(r,f).
\]

LEMMA 6. Let \( f(z), g(z) \) be two transcendental meromorphic functions of finite order, \( c \in \mathbb{C} \setminus \{0\} \) and \( n \in \mathbb{N} \) such that \( n \geq 2 \). Let \( p(z) \) be a nonzero polynomial such that \( \deg(p) < (n-1)/2 \). Then

1. if \( \deg(p) \geq 1 \), then \( f^n(z)f(z+c)g^n(z)g(z+c) \neq p^2(z) \);

2. if \( p(z) = c \in \mathbb{C} \setminus \{0\} \), then the relation \( f^n(z)f(z+c)g^n(z)g(z+c) = p^2(z) \) always implies that \( fg = t \), where \( t \) is a constant such that \( t^{n+1} = c^2 \).
PROOF. Suppose
\[ f^n(z)f(z + c)g^n(z)g(z + c) \equiv p^2(z). \] (2)
Let \( h_1 = fg \). Then by (2), we have
\[ h_1^n(z) \equiv \frac{p^2(z)}{h_1(z + c)}. \] (3)
We now consider following two cases.

Case 1. Suppose \( h_1 \) is a transcendental meromorphic function. Now by Lemmas 1, 2 and 4, we get
\[ n T(r, h_1) = T(r, h_1^n) + \log r + O(1), \] (5)
which is a contradiction.

Case 2. Suppose \( h_1 \) is a rational function. Let
\[ h_1 = \frac{h_2}{h_3}, \] (4)
where \( h_2 \) and \( h_3 \) are two nonzero relatively prime polynomials. By (4), we have
\[ T(r, h_1) = \max\{\deg(h_2), \deg(h_3)\} \log r + O(1). \] (5)
Now by (3)–(5), we have
\[ n \max\{\deg(h_2), \deg(h_3)\} \log r \]
\[ = T(r, h_1^n) + O(1) \]
\[ \leq T(r, h_1(z + c)) + 2 T(r, p) + O(1) \]
\[ = \max\{\deg(h_2), \deg(h_3)\} \log r + 2 \deg(p) \log r + O(1). \]
We see that \( \max\{\deg(h_2), \deg(h_3)\} \geq 1 \). Now by (6), we deduce that \( (n-1)/2 \leq \deg(p) \), which contradicts our assumption that \( \deg(p) < (n-1)/2 \). Hence \( h_1 \) must be a nonzero constant. Let
\[ h_1 = t \in \mathbb{C} \setminus \{0\}. \] (7)
Now when \( \deg(p) \geq 1 \), by (3) and (7), we arrive at a contradiction. Therefore in this case we have \( f^n(z)f(z + c)g^n(z)g(z + c) \neq p^2(z) \). Suppose \( p(z) = c \in \mathbb{C} \setminus \{0\} \). So by (3) we see that \( h_1^n+1 \equiv c^2 \). By (7) we get \( t^{n+1} \equiv c^2 \). This completes the proof.

LEMMA 7 ([8]). Let \( f \) and \( g \) be two non-constant meromorphic functions sharing \( (1, 2) \). Then one of the following holds:
Lemma 8 ([1]). Let $F$ and $G$ be two non-constant meromorphic functions sharing \((1, 1)\) and $H \neq 0$. Then

$$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \frac{1}{2} \bar{N}(r, 0; F) + \frac{1}{2} \bar{N}(r, \infty; F) + S(r, F) + S(r, G).$$

Lemma 9 ([1]). Let $F$ and $G$ be two non-constant meromorphic functions sharing \((1, 0)\) and $H \neq 0$. Then

$$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + 2 \bar{N}(r, 0; F) + \bar{N}(r, 0; G) + 2 \bar{N}(r, \infty; F) + \bar{N}(r, \infty; G) + S(r, F) + S(r, G).$$

Lemma 10 ([15]). Let $H$ be defined as in (1). If $H \equiv 0$ and

$$\limsup_{r \to \infty} \frac{\bar{N}(r, 0; F) + \bar{N}(r, 0; G) + \bar{N}(r, \infty; F) + \bar{N}(r, \infty; G)}{T(r)} < 1, \quad r \in I,$$

where $I$ is a set of infinite linear measure, then $F \equiv G$ or $F \cdot G \equiv 1$.

3 Proofs of the Theorems

Proof of Theorem 1. Let

$$F(z) = \frac{f^n(z)f(z + c)}{p(z)} \text{ and } G(z) = \frac{g^n(z)g(z + c)}{p(z)}.$$ 

Then $F$ and $G$ share \((1, 2)\) except for the zeros of $p(z)$. Now by Lemma 7, we see that one of the following three cases holds.

Case 1. Suppose

$$T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + S(r, F) + S(r, G).$$

Now by applying Lemmas 1 and 4, we have

$$T(r, F) \leq N_2(r, 0; f^n f(z + c)) + N_2(r, 0; g^n g(z + c)).$$
By Lemma 5, we have

\[ + N_2(r, \infty; f^n f(z + c)) + N_2(r, \infty; g^n g(z + c)) + S(r, f) + S(r, g) \]
\[ \leq N_2(r, 0; f^n) + N_2(r, 0; f(z + c)) + N_2(r, 0; g^n) + N_2(r, 0; g(z + c)) \]
\[ + N_2(r, \infty; f^n) + N_2(r, \infty; f(z + c)) + N_2(r, \infty; g^n) + N_2(r, \infty; g(z + c)) \]
\[ + S(r, f) + S(r, g) \]
\[ \leq 2 N(r, 0; f) + N(r, 0; f(z + c)) + 2 N(r, 0; g) + N(r, 0; g(z + c)) + 2 N(r, \infty; f) \]
\[ + N(r, \infty; f(z + c)) + 2 N(r, \infty; g) + N(r, \infty; g(z + c)) + S(r, f) + S(r, g) \]
\[ \leq 4 T(r, f) + N(r, 0; f) + N(r, \infty; f) + 4 T(r, g) + N(r, 0; g) + N(r, \infty; g) \]
\[ + S(r, f) + S(r, g) \]
\[ \leq 6 T(r, f) + 6 T(r, g) + S(r, f) + S(r, g). \]

By Lemma 5, we have

\[(n - 1) T(r, f) \leq 6 T(r, f) + 6 T(r, g) + S(r, f) + S(r, g) \leq 12 T_1(r) + S_1(r), \quad (8)\]

where \( T_1(r) \) is the maximum of \( T(r, f) \) and \( T(r, g) \) and \( S_1(r) \) denotes any quantity satisfying \( S_1(r) = o(T_1(r)) \) as \( r \to \infty \), outside of a possible exceptional set of finite logarithmic measure. Similarly we have

\[(n - 1) T(r, g) \leq 12 T_1(r) + S_1(r). \quad (9)\]

Combining (8) and (9) we get \((n - 1) T_1(r) \leq 12 T_1(r) + S_1(r)\), which contradicts with \( n \geq 14 \).

**Case 2.** \( F \equiv G \). Then we have

\[ f^n(z) f(z + c) \equiv g^n(z) g(z + c). \quad (10)\]

Let \( h = \frac{f}{g} \). Then by (10), we have

\[ h^n(z) \equiv \frac{1}{h(z + c)}. \quad (11)\]

Now by Lemmas 1, 2 and 4, we get

\[ n T(r, h) = T(r, h^n) + S(r, h) = T(r, \frac{1}{h(z + c)}) + S(r, h) \]
\[ \leq N(r, 0; h(z + c)) + m(r, \frac{1}{h(z + c)}) + S(r, h) \]
\[ \leq N(r, 0; h(z)) + m(r, \frac{1}{h(z)}) + S(r, h) \]
\[ \leq T(r, h) + S(r, h). \]

Since \( n \geq 2 \), we see that \( h \) is a constant. By (11), we have \( h^{n+1} = 1 \). Thus \( f(z) = t g(z) \) and \( t^{n+1} = 1 \).

**Case 3.** \( F G \equiv 1 \). Then we have \( f^n(z) f(z + c) g^n(z) g(z + c) \equiv p^2(z) \). Hence Theorem 1 follows by Lemma 6. This completes the proof.
PROOF OF THEOREM 2. Let $F(z) = \frac{f^n(z)f(z+c)}{p(z)}$ and $G(z) = \frac{g^n(z)g(z+c)}{p(z)}$. Then $F$ and $G$ share $(1, 1)$ except for the zeros of $p(z)$. We now consider the following two cases.

Case 1. $H \neq 0$. By Lemma 3, we have

$$\tilde{N}(r, 0; F) = \tilde{N}(r, 0; f^n f(z + c)) \leq \tilde{N}(r, 0; f^n) + \tilde{N}(r, 0; f(z + c))$$

$$\leq \tilde{N}(r, 0; f) + \tilde{N}(r, 0; f(z + c))$$

$$\leq N(r, 0; f) + N(r, 0; f(z + c)) \leq 2T(r, f) + S(r, f).$$

Similarly we have $\tilde{N}(r, \infty; F) \leq 2T(r, f) + S(r, f)$. Now by applying Lemmas 1, 4 and 8 we have

$$T(r, F)$$

$$\leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + \frac{1}{2} \tilde{N}(r, 0; F)$$

$$+ \frac{1}{2} \tilde{N}(r, \infty; F) + S(r, f) + S(r, g)$$

$$\leq 6T(r, f) + 6T(r, g) + T(r, f) + T(r, f) + S(r, f) + S(r, g)$$

$$\leq 8T(r, f) + 6T(r, g) + S(r, f) + S(r, g)$$

By Lemma 5, we have

$$(n - 1) T(r, f) \leq 8T(r, f) + 6T(r, g) + S(r, f) + S(r, g) \leq 14T_1(r) + S_1(r). \quad (12)$$

Similarly, we have

$$(n - 1) T(r, g) \leq 14T_1(r) + S_1(r). \quad (13)$$

Combining (12) and (13) we get $(n - 1) T_1(r) \leq 14T_1(r) + S_1(r)$, which contradicts with $n \geq 16$.

Case 2. $H \equiv 0$. In view of Lemmas 4 and 5, we get

$$\tilde{N}(r, 0; F) + \tilde{N}(r, 0; G) + \tilde{N}(r, \infty; F) + \tilde{N}(r, \infty; G)$$

$$\leq 4T(r, f) + 4T(r, g) + S(r, f) + S(r, g)$$

$$\leq \frac{4}{n - 1} T(r, F) + \frac{4}{n - 1} T(r, G) + S(r, F) + S(r, G) \leq \frac{8}{n - 1} T(r) + S(r).$$

Since $n > 12$, we have

$$\limsup_{r \to \infty} \frac{\tilde{N}(r, 0; F) + \tilde{N}(r, 0; G) + \tilde{N}(r, \infty; F) + \tilde{N}(r, \infty; G)}{T(r)} < 1$$

and so by Lemma 10, we have either $F \equiv G$ or $FG \equiv 1$. Hence Theorem 2 follows by the proof of Theorem 1. This completes the proof.

PROOF OF THEOREM 3. Let $F(z) = \frac{f^n(z)f(z+c)}{p(z)}$ and $G(z) = \frac{g^n(z)g(z+c)}{p(z)}$. Then $F$ and $G$ share $(1, 0)$ except for the zeros of $p(z)$. We now consider the following two cases.
**Case 1.** $H \neq 0$. By Lemma 3, we have
\[\tilde{N}(r, 0; F) \leq 2T(r, f) + S(r, f), \quad \tilde{N}(r, \infty; F) \leq 2T(r, f) + S(r, f),\]
\[\tilde{N}(r, 0; G) \leq 2T(r, g) + S(r, g)\]
and \[\tilde{N}(r, \infty; G) \leq 2T(r, g) + S(r, g).\]

Now by Lemmas 1, 4 and 9, we have
\[T(r, F) \leq N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + 2 \tilde{N}(r, 0; F) + \tilde{N}(r, 0; G) + 2T(r, f) + S(r, f) + S(r, g)\]
\[\leq 6 T(r, f) + 6 T(r, g) + 4T(r, f) + 2T(r, g) + 4T(r, f) + 2T(r, g) + S(r, f) + S(r, g)\]
\[\leq 14 T(r, f) + 10 T(r, g) + S(r, f) + S(r, g)\]

By Lemma 5, we have
\[(n - 1) T(r, f) \leq 14 T(r, f) + 10 T(r, g) + S(r, f) + S(r, g) \leq 24 T_1(r) + S_1(r). \quad (14)\]

Similarly we have
\[(n - 1) T(r, g) \leq 24 T_1(r) + S_1(r). \quad (15)\]

Combining (14) and (15) we get
\[(n - 1) T_1(r) \leq 24T_1(r) + S_1(r),\]
which contradicts with $n \geq 26$.

**Case 2.** $H \equiv 0$. Hence Theorem 3 follows from the proof of Theorems 1 and 2. This completes the proof.

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**References**


