

# Oscillatory Solutions Of The Classical Generator Equation\*

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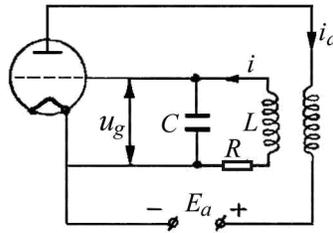
## Abstract

We present an approach for establishing an existence of oscillatory solutions for the classical generator equation. Instead of well known second order van der Poll differential equation we consider an integro-differential equation of first order and apply the fixed point method by choosing the appropriate functional space of oscillatory functions.

## 1 Introduction

In this paper we present a method of analysis of the classical generator circuit. Although the method is demonstrated on the classical example it could be applied to the analysis of wide class of more complicated problems, for instance, analysis of ladder oscillator, 2-dimensional low-pass multimode oscillator, etc. (cf. [1] - [3]).

It is known that [1]-[3] the generator regimes are approximated by the harmonic solutions. This means that voltages and currents are of the type  $U(t) = U_m \sin \omega t$ ,  $I(t) = I_m \sin \omega t$ . In many practical cases, however, this assumption is not adequate (cf. [4] - [11]). That is why we show an existence of oscillatory solutions with non-uniformly distributed zeros. We investigate the integro-differential equation corresponding to the oscillator circuit [4] - [11] (see the figure below), instead of second order van der Pol differential equation.



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Indeed, the second Kirchoff's law applied to the foregoing circuit yields

$$L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_{t_0}^t i(s) ds = M \frac{di_a}{dt} \quad (1)$$

with unknown function  $i = i(t)$  - the circuit current. The constants  $L$ ,  $R$ ,  $C$  and  $M$  have the usually accepted sense (cf. [4] - [11]) while  $i_a = i_a(t)$  is the anode current.

In view of the relation between the grid voltage  $u_g$  and current  $u_g(t) = \frac{1}{C} \int_{t_0}^t i(s) ds$  and taking into account the tube V-I characteristics  $i_a = i_a(u_g)$ , one can derive the second order differential equation:

$$LC \frac{d^2 u_g(t)}{dt^2} + \left( RC - M \frac{di_a}{du_g} \right) \frac{du_g}{dt} + u_g = 0. \quad (2)$$

If  $i_a = i_a(u_g)$  is a linear function, then obviously (2) becomes a linear differential equation. If, however, the tube V-I characteristic is approximated by a third order polynomial,  $i_a(u_g) = I_{a0} + Su_g - Qu_g^3$  ( $I_{a0}, S, Q$  are prescribed constants), then  $\frac{di_a}{du_g} = S - 3Qu_g^2$ . Consequently, substituting the foregoing expression in (2), we obtain a nonlinear second order differential equation:

$$LC \frac{d^2 u_g(t)}{dt^2} + (RC - MS + 3MQu_g^2) \frac{du_g}{dt} = 0, \quad (3)$$

which is the well known van der Pol equation considered in a lot of papers (cf. for instance [4] - [11]).

Our approach is based on the primary integro-differential equation (1). We assume the unknown function to be the current  $i = i(t)$ . Then in view of  $du_g/dt = i(t)/C$  and substituting in (1) we obtain

$$L \frac{di(t)}{dt} + Ri(t) + \frac{1}{C} \int_{t_0}^t i(s) ds = M \left[ S - 3Q \left( \frac{1}{C} \int_{t_0}^t i(\tau) d\tau \right)^2 \right] \frac{i(t)}{C}$$

or introducing the denotations

$$A_1 = \frac{MS - RC}{LC}, \quad A_2 = -\frac{1}{LC}, \quad A_3 = -\frac{3MQ}{LC^3},$$

we have

$$\frac{di(t)}{dt} = A_1 i(t) + A_2 \int_{t_0}^t i(s) ds + A_3 i(t) \left( \int_{t_0}^t i(s) ds \right)^2 \equiv F \left( i(t), \int_{t_0}^t i(s) ds \right) \quad (4)$$

where  $F(x, y) = A_1 x + A_2 y + A_3 x y^2$ .

Another example of integro-differential system arises in the analysis of line array of oscillators [1]:

$$I_k(t) = \frac{1}{L_0} \int_{t_0}^t [U_k(\tau) - U_{k+1}(\tau)] d\tau,$$

$$I_{k-1} - I_k = C \frac{dU_k}{d(t)} + \frac{1}{L} \int_{t_0}^t U_k(\tau) d\tau + gU_k(t) - g_1U_k(t) + g_3U_k^3(t), \quad k = 1, 2, \dots, N.$$

The usually accepted approach to solve the above system is to reduce it to the second order system of van der Pol type (cf. [1]):

$$\begin{aligned} 0 = & \frac{d^2U_k(t)}{dt^2} - \frac{g_1 - g}{C} \left( 1 - \frac{3g_3}{g_1 - g} U_k^2(t) \right) \frac{dU_k}{dt} + \left( \frac{1}{LC} + \frac{2}{L_0C} \right) U_k(t) \\ & - \frac{1}{L_0C} U_{k-1}(t) - \frac{1}{L_0C} U_{k+1}(t). \end{aligned}$$

Instead of the above system one can consider the following one:

$$\begin{aligned} \frac{dU_k(t)}{dt} = & \frac{1}{L_0C} \int_{t_0}^t [U_{k+1}(\tau) + U_{k-1}(\tau)] d\tau - \left( \frac{1}{LC} + \frac{2}{L_0C} \right) \int_{t_0}^t U_k(\tau) d\tau - \\ & - \frac{g - g_1}{C} U_k(t) - \frac{g_3}{C} U_k^3(t) \\ = & 0, \end{aligned}$$

which is of the type (4).

## 2 $\Phi$ -Contractive Mappings in Uniform Spaces

Here we recall fixed point theorems for  $\Phi$ -contractive mappings in uniform spaces introduced in [12]. The uniform spaces turn out a natural extension of the metric spaces. The particular case of the uniform spaces are the locally convex topological vector spaces whose topology is uniformizable (cf.[12]). Therefore the fixed point results from [12] are valid in locally convex spaces, too.

By  $(X, \mathfrak{S})$  we mean a Hausdorff ( $T_2$ -separated) sequentially complete uniform space whose uniformity is generated by a saturated family of pseudometrics

$$\mathfrak{S} = \{d_k(x, y) : k \in A\},$$

where  $A$  is an index set. Let  $(\Phi) = \{\Phi_k(t) : k \in A\}$  be a family of functions (which we shall call  $\Phi$ -contractive) with the properties:

( $\Phi 1$ ) for every  $k \in A$   $\Phi_k(t) : R_+^1 \rightarrow R_+^1$  is monotone increasing and continuous from the right;

( $\Phi 2$ ) for every  $k \in A$  it follows  $0 < \Phi_k(t) < t$  for  $t > 0$  (by right continuity  $\Phi_k(0) = 0$ ).

Let  $j : A \rightarrow A$  be a mapping of the index set  $A$  into itself. The iterations of  $j$  can be defined as follows:  $j^0(k) = k$ ,  $j^n(k) = j(j^{n-1}(k))$  ( $n = 1, 2, \dots$ ). A mapping  $T : X \rightarrow X$  is said to be  $\Phi$ -contractive if  $d_k(Tx, Ty) \leq \Phi_k(d_{j(k)}(x, y))$  for every  $x, y \in X$  and  $k \in A$ .

**THEOREM 2.1** ([12]). Let the following conditions be fulfilled:

1) the operator  $T : X \rightarrow X$  is  $\Phi$ -contractive;

2) for every  $k \in A$  there exists a function  $\bar{\Phi}_k(t)$  which possesses the properties of  $(\Phi)$  such that  $\sup \{\Phi_{j^n(k)}(t) : n = 0, 1, 2, \dots\} \leq \bar{\Phi}_k(t)$  and  $\bar{\Phi}_k(t)/t$  is monotone non-decreasing;

3) there is  $x_0 \in X$  such that  $d_{j^n(k)}(x_0, Tx_0) \leq q(k, x_0) < \infty$  ( $n = 0, 1, 2, \dots$ ) for some  $q > 0$ .

Then  $T$  has at least one fixed point in  $X$ .

A uniform space  $(X, \mathfrak{S})$  is said to be  $j$ -bounded if for every  $k \in A$  and  $x, y \in X$  there exists  $q = q(x, y, k)$  such that  $d_{j^n(k)}(x, y) \leq q(x, y, k) < \infty$  ( $n = 0, 1, 2, \dots$ ). It is easy to verify that  $j$ -boundedness of  $(X, \mathfrak{S})$  implies condition 3) of the last Theorem 2.1.

**THEOREM 2.2** ([12]). If to the conditions of Theorem 2.1 we add the assumption that  $X$  is  $j$ -bounded, then  $T$  has a unique fixed point in  $X$ .

### 3 An Existence of Oscillatory Solutions

Now we are able to formulate the main problem for (4): to find an oscillatory solution of (4) on an interval  $[t_0, \infty)$ . Without loss of generality one can choose the initial value to be  $i(t_0) = i_0 = 0$ .

Let  $S = \{t_k\}_{k=0}^{\infty}$  be an increasing sequence of real numbers satisfying the following conditions:

(t1)  $\lim_{k \rightarrow \infty} t_k = \infty$  ;

(t2)  $0 < \inf \{t_{k+1} - t_k : k = 0, 1, 2, \dots\} \leq \sup \{t_{k+1} - t_k : k = 0, 1, 2, \dots\} \leq T_0 < \infty$ .

Let  $C_S[t_0, \infty)$  ( $C_S^1[t_0, \infty)$ ) be the set of all continuously (continuously differentiable) functions  $f(t) : [t_0, \infty) \rightarrow (-\infty, \infty)$  with zeros at  $S$ , that is,  $f(t_k) = 0$ , ( $k = 0, 1, \dots$ ). Introduce the sets

$$M = \left\{ f \in C_S[t_0, \infty) : \int_{t_k}^{t_{k+1}} f(t) dt = 0; k = 0, 1, 2, \dots \right\}$$

$$M_1 = \left\{ f \in C_S^1[t_0, \infty) : \int_{t_k}^{t_{k+1}} f(t) dt = 0; k = 0, 1, 2, \dots \right\}.$$

Then it is obvious every primitive function  $F(t) = \int_{t_0}^t f(\tau) d\tau$  has zeros at  $S$  provided  $f(\cdot) \in M$ .

**LEMMA 3.1.** Equation (4) has a continuously differentiable oscillatory solution  $i(\cdot) \in M_1$  iff the operator  $G$  has a fixed point in  $M$ , that is,

$$i(t) = (Gi)(t) \tag{5}$$

where  $(Gi)(t) =: i(t_{k+1}) + \int_{t_k}^t F \left( i(s), \int_{t_0}^s i(\tau) d\tau \right) ds \equiv \int_{t_k}^t F \left( i(s), \int_{t_0}^s i(\tau) d\tau \right) ds$ ,  $t \in [t_k, t_{k+1}]$  ( $k = 0, 1, 2, 3, \dots$ ) (cf.[13]).

**REMARK 1.** Note that  $\int_{t_0}^t i(s) ds = \int_{t_0}^{t_k} i(s) ds + \int_{t_k}^t i(s) ds = \int_{t_k}^t i(s) ds$ ,  $t \in [t_k, t_{k+1}]$ .

PROOF. Let  $i(\cdot) \in M_1$  be an oscillatory solution of (4). Then integrating (4) on every interval  $[t_k, t] \subset [t_k, t_{k+1}]$  ( $k = 0, 1, 2, \dots$ ) we obtain

$$i(t) = \int_{t_k}^t F \left( i(s), \int_{t_0}^s i(\tau) d\tau \right) ds \tag{6}$$

that is,  $G$  has a fixed point. In what follows we show that the fixed point belongs to  $M$ . But  $i(\cdot) \in M_1$  and substituting  $t = t_{k+1}$  we obtain

$$i(t_{k+1}) = \int_{t_k}^{t_{k+1}} F \left( i(s), \int_{t_0}^s i(\tau) d\tau \right) ds = 0.$$

Following [13] we obtain that

$$\frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \left[ i(s) + sF \left( i(s), \int_{t_0}^s i(\tau) d\tau \right) \right] ds = i(t_{k+1}) = 0 \quad (k = 0, 1, 2, \dots).$$

Consequently (5) can be written in the form:

$$\begin{aligned} i(t) &= \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \left[ i(s) + sF \left( i(s), \int_{t_0}^s i(\tau) d\tau \right) \right] ds + \\ &+ \int_{t_k}^t F \left( i(s), \int_{t_0}^s i(\tau) d\tau \right) ds, \end{aligned} \tag{7}$$

for  $t \in [t_k, t_{k+1}]$ , ( $k = 0, 1, 2, \dots$ ). Then integrating (7) we have:

$$\begin{aligned} &\int_{t_k}^{t_{k+1}} i(t) dt \\ &= \int_{t_k}^{t_{k+1}} \left[ i(s) + sF \left( i(s), \int_{t_0}^s i(\tau) d\tau \right) \right] ds + \int_{t_k}^{t_{k+1}} \int_{t_k}^{\tau} F \left( i(s), \int_{t_0}^s i(t) dt \right) ds d\tau \\ &= \int_{t_k}^{t_{k+1}} i(s) ds + \int_{t_k}^{t_{k+1}} sF \left( i(s), \int_{t_0}^s i(\tau) d\tau \right) ds \\ &\quad + \int_{t_k}^{t_{k+1}} \int_{\tau}^{t_{k+1}} F \left( i(s), \int_{t_0}^s i(t) dt \right) d\tau ds \\ &= \int_{t_k}^{t_{k+1}} sF \left( i(s), \int_{t_0}^s i(\tau) d\tau \right) ds + \int_{t_k}^{t_{k+1}} (t_{k+1} - \tau) F \left( i(\tau), \int_{t_0}^{\tau} i(t) dt \right) d\tau \\ &= t_{k+1} \int_{t_k}^{t_{k+1}} F(i(\tau), \int_{t_0}^{\tau} i(t) dt) d\tau \\ &= 0, \end{aligned}$$

which implies that  $i(\cdot) \in M$ .

Conversely, if  $G$  has a fixed point in  $M$  then this fixed point is an oscillatory solution of (4). Since  $F \left( i(s), \int_{t_0}^s i(\tau) d\tau \right)$  is continuous function on  $[t_k, t_{k+1}]$  and then  $i(t)$  from (5) is differentiable function and consequently differentiating  $i(t) = (Gi)(t)$  we obtain (4).

So the Lemma 3.1 is proved.

Introduce the set  $X = \{f(\cdot) \in M : |f(t)| \leq Ie^{\mu(t-t_k)}, t \in [t_k, t_{k+1}], k = 0, 1, 2, \dots\}$  for some  $\mu \in (0, \infty)$ . Here  $I > 0$  and  $\mu$  are constants which will be described below. The set  $X$  turns out into a uniform space with respect to the family of pseudometrics (cf. [12])

$$\rho_k(f, \bar{f}) = \sup \left\{ e^{-\mu(t-t_k)} |f(t) - \bar{f}(t)| : t \in [t_k, t_{k+1}] \right\}$$

The index set here is  $A = \{0, 1, \dots, k, \dots\}$ .

We apply the results from Section II.

**THEOREM 3.1.** Let the constants  $\mu, I, T_0 > 0$  be chosen in such a way that the following inequalities be fulfilled:

$$2T_0 \left[ \left( |A_1| + \frac{|A_2|}{\mu} \right) \frac{1}{2 - \mu T_0} + \frac{3|A_3|I^2}{\mu^2(2 - 3\mu T_0)} \right] = K < 1; \mu T_0 < \frac{2}{3}.$$

Then there exists a unique continuous oscillatory solution of (6) belonging to  $X$ . This solution can be obtained as a limit of successive approximations.

**PROOF.** Define the operator  $G : X \rightarrow X$  as in (7). First we show that  $G$  maps  $X$  into itself. It is easy to verify that  $(Gf)(t)$  is a continuous and oscillatory function. Indeed,  $(Gf)(t)$  is a continuous function as a composition of continuous functions. In the proof of Lemma 3.1 we have proved that  $(Gf)(t_k) = 0, (k = 0, 1, \dots)$ . Integrating (7) we have:

$$\begin{aligned} & \int_{t_k}^{t_{k+1}} (Gf)(t) dt \\ &= \int_{t_k}^{t_{k+1}} \left[ f(s) + sF(f(s), \int_{t_0}^s f(\tau) d\tau) \right] ds + \int_{t_k}^{t_{k+1}} \int_{t_k}^{\tau} F(f(s), \int_{t_0}^s f(t) dt) ds d\tau \\ &= \int_{t_k}^{t_{k+1}} f(s) ds + \int_{t_k}^{t_{k+1}} sF(f(s), \int_{t_k}^s f(\tau) d\tau) ds + \int_{t_k}^{t_{k+1}} \int_{\tau}^{t_{k+1}} F(f(s), \int_{t_k}^s f(t) dt) d\tau ds \\ &= \int_{t_k}^{t_{k+1}} sF(f(s), \int_{t_k}^s f(\tau) d\tau) ds + \int_{t_k}^{t_{k+1}} (t_{k+1} - \tau) F(f(\tau), \int_{t_k}^{\tau} f(t) dt) d\tau \\ &= t_{k+1} \int_{t_k}^{t_{k+1}} F(f(\tau), \int_{t_k}^{\tau} f(t) dt) d\tau = 0. \end{aligned}$$

It remains to show that if  $|i(t)| \leq Ie^{\mu(t-t_k)} \Rightarrow |(Gi)(t)| \leq e^{\mu(t-t_k)}$ . In view of the inequalities

$$\frac{e^h - 1}{h} = \frac{1}{h} \left( h + \frac{h^2}{2!} + \frac{h^3}{3!} + \dots \right) \leq 1 + \frac{h}{2} + \left( \frac{h}{2} \right)^2 + \dots \leq \frac{1}{1 - (h/2)} = \frac{2}{2 - h} \quad (8)$$

which holds for  $h < 2$  and we obtain for  $t \in [t_k, t_{k+1}]$  that

$$\begin{aligned} |(Gf)(t)| &\leq |A_1| \left| \int_{t_k}^t f(s) ds \right| + |A_2| \left| \int_{t_k}^t \int_{t_0}^{\tau} f(s) ds d\tau \right| + |A_3| \left| \int_{t_k}^t i(\tau) \left( \int_{t_0}^{\tau} f(s) ds \right)^2 d\tau \right| \\ &\leq |A_1| I \int_{t_k}^t e^{\mu(s-t_k)} ds + |A_2| I \int_{t_k}^t \int_{t_k}^{\tau} e^{\mu(s-t_k)} ds d\tau \\ &\quad + |A_3| I^3 \int_{t_k}^t e^{\mu(\tau-t_k)} \left( \int_{t_k}^{\tau} e^{\mu(s-t_k)} ds \right)^2 d\tau, \end{aligned}$$

hence,

$$\begin{aligned} |(Gf)(t)| &\leq |A_1| I \frac{e^{\mu(t-t_k)} - 1}{\mu} + |A_2| I \int_{t_k}^t \frac{e^{\mu(s-t_k)} - 1}{\mu} ds \\ &\quad + |A_3| I^3 \int_{t_k}^t e^{\mu(\tau-t_k)} \left( \frac{e^{\mu(\tau-t_k)} - 1}{\mu} \right)^2 d\tau \\ &\leq \frac{|A_1| I (e^{\mu(t_{k+1}-t_k)} - 1)}{\mu} + \frac{|A_2| I (e^{\mu(t_{k+1}-t_k)} - 1)}{\mu^2} + \frac{|A_3| I^3 |A_2| I (e^{3\mu(t_{k+1}-t_k)} - 1)}{3\mu} \\ &\leq \frac{|A_1| I (e^{\mu T_0} - 1)}{\mu T_0} T_0 + \frac{|A_2| I (e^{\mu T_0} - 1)}{\mu \mu T_0} T_0 + \frac{|A_3| I^3 |A_2| I (e^{3\mu T_0} - 1)}{\mu^2 3\mu T_0} T_0 \\ &\leq \frac{|A_1| I 2T_0}{2 - \mu T_0} + \frac{|A_2| I 2T_0}{\mu(2 - \mu T_0)} + \frac{|A_3| I^3 2T_0}{\mu^2 2 - 3\mu T_0} \\ &\leq I 2T_0 \left( \frac{|A_1| + |A_2|/\mu}{2 - \mu T_0} + \frac{|A_3| I^2}{\mu^2(2 - 3\mu T_0)} \right) \\ &\leq IK e^{\mu(t-t_k)} \\ &< I e^{\mu(t-t_k)}. \end{aligned}$$

Therefore the operator  $G$  maps  $X$  into itself.

Now we show  $G$  is an  $\Phi$ -contractive operator. Indeed, for every  $f, \bar{f} \in X$  and  $t \in [t_k, t_{k+1}]$ , we have

$$\begin{aligned} &|(Gf)(t) - (G\bar{f})(t)| \\ &\leq |A_1| \int_{t_k}^t |f(\tau) - \bar{f}(\tau)| d\tau + |A_2| \int_{t_k}^t \int_{t_0}^{\tau} |f(\theta) - \bar{f}(\theta)| d\theta d\tau \\ &\quad + |A_3| \int_{t_k}^t \left| f(\tau) \left( \int_{t_0}^{\tau} f(\theta) d\theta \right)^2 - \bar{f}(\tau) \left( \int_{t_0}^{\tau} f(\theta) d\theta \right)^2 \right| d\tau \\ &\quad + |A_3| \int_{t_k}^t \left| \bar{f}(\tau) \left( \int_{t_0}^{\tau} f(\theta) d\theta \right)^2 - \bar{f}(\tau) \left( \int_{t_0}^{\tau} \bar{f}(\theta) d\theta \right)^2 \right| d\tau, \end{aligned}$$

hence

$$\begin{aligned}
& |(Gf)(t) - (G\bar{f})(t)| \\
\leq & \rho_k(f, \bar{f}) \left( \frac{|A_1|}{\mu} + \frac{|A_2|}{\mu^2} \right) \left[ e^{\mu(t-t_k)} - 1 \right] \\
& + |A_3| I^2 \int_{t_k}^t |f(\tau) - \bar{f}(\tau)| \left( \frac{e^{\mu(t-t_k)} - 1}{\mu} \right)^2 d\tau \\
& + |A_3| I \int_{t_k}^t e^{\mu(\tau-t_k)} \left| \left( \int_{t_k}^{\tau} f(\theta) d\theta \right)^2 - \left( \int_{t_k}^{\tau} \bar{f}(\theta) d\theta \right)^2 \right| d\tau \\
\leq & \rho_k(f, \bar{f}) \left( \frac{|A_1|}{\mu} + \frac{|A_2|}{\mu^2} \right) (e^{\mu T_0} - 1) + \rho_k(f, \bar{f}) \frac{|A_3| I^2}{\mu^2} \int_{t_k}^t e^{3\mu(\tau-t_k)} d\tau \\
& + 2|A_3| I^2 \int_{t_k}^t e^{\mu(\tau-t_k)} \frac{e^{\mu(\tau-t_k)} - 1}{\mu} \int_{t_k}^{\tau} |f(\theta) - \bar{f}(\theta)| d\theta d\tau \\
\leq & \rho_k(f, \bar{f}) \left( |A_1| + \frac{|A_2|}{\mu} \right) \frac{e^{\mu T_0} - 1}{\mu T_0} T_0 + \rho_k(f, \bar{f}) \frac{|A_3| I^2}{\mu^2} \frac{3e^{\mu T_0} - 1}{3\mu T_0} T_0 \\
& + \rho_k(f, \bar{f}) 2|A_3| I^2 \int_{t_k}^t e^{\mu(\tau-t_k)} \left( \frac{e^{\mu(\tau-t_k)} - 1}{\mu} \right)^2 d\tau \\
\leq & \rho_k(f, \bar{f}) \left( |A_1| + \frac{|A_2|}{\mu} \right) \frac{e^{\mu T_0} - 1}{\mu T_0} T_0 + \rho_k(f, \bar{f}) \frac{|A_3| I^2}{\mu^2} \frac{3e^{\mu T_0} - 1}{3\mu T_0} T_0 \\
& + \rho_k(f, \bar{f}) \frac{2|A_3| I^2}{\mu^2} \frac{3e^{\mu T_0} - 1}{3\mu T_0} T_0 \\
\leq & \rho_k(f, \bar{f}) e^{\mu(t-t_k)} \cdot 2T_0 \left[ \left( |A_1| + \frac{|A_2|}{\mu} \right) \frac{1}{2 - \mu T_0} + \frac{3|A_3| I^2}{\mu^2} \frac{1}{2 - 3\mu T_0} \right] \\
= & K \rho_k(f, \bar{f}) e^{\mu(t-t_k)}.
\end{aligned}$$

Multiplying by  $e^{-\mu(t-t_k)}$  and taking the supremum on  $[t_k, t_{k+1}]$  we obtain  $\rho_k(Gf, G\bar{f}) \leq K \rho_k(f, \bar{f})$ . Here  $j$  is the identity mapping. Then obviously the uniform space  $X$  is  $j$ -bounded. Indeed, for every  $f, \bar{f} \in X$  the following inequality holds  $\rho_{j^n(k)}(f, \bar{f}) = \rho_k(f, \bar{f}) < \infty$  ( $n = 0, 1, 2, \dots$ ). Therefore in view of Theorems 2.1 and 2.2 the operator  $G$  has a unique fixed point which is a solution of (5). Theorem 3.1 is thus proved.

## 4 Conclusions

We have:

$$i(t) = \int_k^t \left\{ A_1 i(\tau) + A_2 \int_{t_k}^{\tau} i(\theta) d\theta + A_3 i(\tau) \left[ \int_{t_k}^{\tau} i(\theta) d\theta \right]^2 \right\} d\tau, \quad t \in [t_k, t_{k+1}] \quad (k = 0, 1, \dots).$$

Let us choose the 0-approximation  $i^{(0)}(t) = I \sin \frac{2\pi(t-t_k)}{t_{k+1}-t_k}$ ,  $t \in [t_k, t_{k+1}]$ . Then for the first approximation we obtain:

$$\begin{aligned}
i^{(1)}(t) &= A_1 \int_{t_k}^t i^{(0)}(\tau) d\tau + A_2 \int_{t_k}^t \int_{t_k}^{\tau} i^{(0)}(\theta) d\theta d\tau + A_3 \int_{t_k}^t i^{(0)}(\tau) \left[ \int_{t_k}^{\tau} i^{(0)}(\theta) d\theta \right]^2 d\tau \\
&= A_1 \int_{t_k}^t I \sin \frac{2\pi(\tau-t_k)}{t_{k+1}-t_k} d\tau + A_2 \int_{t_k}^t \int_{t_k}^{\tau} I \sin \frac{2\pi(\theta-t_k)}{t_{k+1}-t_k} d\theta d\tau \\
&\quad + A_3 \int_{t_k}^t I \sin \frac{2\pi(\tau-t_k)}{t_{k+1}-t_k} \left[ \int_{t_k}^{\tau} I \sin \frac{2\pi(\theta-t_k)}{t_{k+1}-t_k} d\theta \right]^2 d\tau \\
&= \frac{A_1 I (t_{k+1}-t_k)}{2\pi} \left( 1 - \cos \frac{2\pi(t-t_k)}{t_{k+1}-t_k} \right) + \frac{A_2 I (t_{k+1}-t_k)(t-t_k)}{2\pi} \\
&\quad - \frac{A_2 I (t_{k+1}-t_k)^2}{(2\pi)^2} \sin \frac{2\pi(t-t_k)}{t_{k+1}-t_k} \\
&\quad - \frac{A_3 I^3 (t_{k+1}-t_k)^3}{(2\pi)^3} \left( \cos \frac{2\pi(t-t_k)}{t_{k+1}-t_k} - \cos^2 \frac{2\pi(t-t_k)}{t_{k+1}-t_k} + \frac{1}{3} \cos^3 \frac{2\pi(t-t_k)}{t_{k+1}-t_k} - \frac{1}{3} \right).
\end{aligned}$$

Then  $\rho_k(i^{(0)}, i^{(1)}) \leq \frac{A_1 I T_0}{\pi} + \frac{A_2 I T_0^2}{2\pi} \left( 1 + \frac{1}{2\pi} \right) + \frac{A_3 I^3 T_0^3}{(2\pi)^3} \left( 2 + \frac{2}{3} \right) + I$ . Therefore if  $i^*(t)$  is the solution then  $\rho_k(i^{(*)}, i^{(n)}) \leq \frac{K^n I}{1-K} \left[ \frac{A_1 T_0}{\pi} + \frac{A_2 T_0^2 (2\pi+1)}{4\pi^2} + \frac{A_3 I^2 T_0^3}{3\pi^3} + 1 \right]$ .

REMARK 2. The inequality from Theorem 3.1 implies that the convergence becomes faster provided the constants  $\mu$  and  $T_0$  to be chosen in a suitable way.

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