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An Algorithm For Detecting "Linear" Solutions Of Nonlinear Polynomial Differential Equations*

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Abstract

A symbolic computational algorithm which detects "linear" solutions of nonlinear polynomial differential equations of single functions, is developed in this paper.

1 Introduction

The problem of obtaining general solutions of differential equations via symbolic algorithms has been studied in the past by many authors (see e.g. [2],[4],[5],[6],[7],[8]). These algorithms allowed new calculation techniques to be accomplished, much more efficiently, faster and without approximation errors. In this paper we treat with differential equations of the form:

$$p(x, y(x), y'(x), y''(x), \ldots, y^{(n)}(x)) = 0 \tag{1}$$

where p is a polynomial function and $y(x)$ a complex function of a single variable. Our aim is to discover possible "linear" solutions of (1). By the term "linear" we mean solutions which can be obtained by solving linear differential equations of the form $\alpha_{-1}(x) + \alpha_0(x)y(x) + \alpha_1(x)y'(x) + \cdots + \alpha_n(x)y^{(n)}(x)$, where $\alpha_i(x), i = -1, \ldots, n$, are polynomials of the single variable x. Sometimes, the solutions of those linear differential equations are called holonomic. Our approach is focused on the construction of an algorithm which faces the problem symbolically. What this algorithm is essentially doing is that helps us to rewrite p as follows:

$$p = c_1 L_{0,0}^{j_{0,1}} [L_{1,1}^{(1)}]^{j_{1,1}} \cdots [L_{1,n-k}^{(n-k)}]^{j_{n-k,1}} + \cdots + c_\nu L_{0,0}^{j_{0,\nu}} [L_{\nu,1}^{(1)}]^{j_{1,\nu}} \cdots [L_{\nu,n-k}^{(n-k)}]^{j_{n-k,\nu}} + R$$

where $L_{a,b}$, ($L_{0,0}$ is a common factor), are differential polynomials of the form:

$$L_{a,b} = \frac{A_{0,a,b}(x)}{A_{-1,a,b}(x)} + \frac{A_{1,a,b}(x)}{A_{-1,a,b}(x)} y(x) + \frac{A_{2,a,b}(x)}{A_{-1,a,b}(x)} y'(x) + \cdots + y^{(k)}(x)$$

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and $A_{\lambda,a,b}$ are complex polynomials of a single variable x, of the form: $A_{\lambda,a,b} = \sum_{\iota=0}^{r} W_{\iota,\lambda,a,b} x^{\iota}$, $r \geq 0$. The quantities k and r are predetermined, n is the order of p, and $j_{a,b}$ are specific positive whole numbers. The quantities $W_{\iota,\lambda,\sigma,\varphi}$ are undetermined parameters which can take certain values, c_j are the coefficients, depending from the parameters $W_{\iota,\lambda,\sigma,\varphi}$ and R a rational function of the variables $x, y, y', \ldots, y^{(k-1)}$ and the parameters $W_{\iota,\lambda,\sigma,\varphi}$, called the remainder. Afterward, we seek for those values of the parameters which eliminate the remainder. If this is possible, then the linear differential polynomial $L_{0,0} = \dfrac{A_{0,0,0}(x)}{A_{-1,0,0}(x)} + \dfrac{A_{1,0,0}(x)}{A_{-1,0,0}(x)} y(x) + \cdots + y^{(k)}(x)$, with $W_{0,0,0,0}, W_{1,0,0,0}, \ldots, W_{\rho,-1,0,0}$ evaluated over those values which annihilates the remainder, is a factor of p, (where the operation of differentiation has been taken under consideration). Since $A_{-1,0,0}(x)$ is a common denominator, the above fact means that any solution of $A_{0,0,0}(x) + A_{1,0,0}(x)y(x) + \cdots + A_{-1,0,0}(x)y^{(k)} = 0$ is a solution of the equation (1), too. Since we do it for every $k = 0, \ldots, n$, and several values of r, we collect likewise, all the " linear " solutions of the equation $p = 0$. In the case where $r = 0$, the method will provide us with trivial linear solutions, these are solutions obtained by a linear differential equation with constant coefficients.

Our method is an extension of a similar procedure, introduced by the author and applied in the study of difference equations and feedback design [1],[9]. Its main merit is its computational orientation. It turns to be a useful tool, implemented on a computer machine and gives useful results. Moreover, despite our method resembles with the approach of differential algebra, [5], there are some differences. Concretely, (i) We are working with a single differential polynomial whilst Ritt's algorithm deals with sets of differential polynomials. (ii) The existence of the parameters $W_{\iota,\lambda,\sigma,\varphi}$ permits us to find classes of linear solutions. We can then select among them, these particular solutions which satisfy additional conditions. For instance, we can search for those values of $W_{\iota,\lambda,\sigma,\varphi}$, if any, which do not only eliminate the remainder but also yield stable linear factors. (iii) In the classical Ritt's approach we find the minimum number of differential polynomials which generate a differential ideal. In our method we check if a given polynomial belongs to a differential ideal, produced by linear differential polynomials. Throughout the text, $\mathbf{C}$, $\mathbf{R}$ and $\mathbf{Z}^{+}$ will denote the sets of complex numbers, real numbers and positive integers, respectively.

2 The Algebraic Framework

Let $\mathbf{C}[x]$ be the ring of polynomials of a single variable with complex coefficients. This polynomial ring is a differential ring too, with the usual derivation [2],[5]. Let $y(x)$ be a complex function and $y^{(i)}(x)$, $i = 0, 1, 2, \ldots$ its derivatives. A differential polynomial p, in $y(x)$ or shortly in y, is a polynomial in y and its derivatives with coefficients in $\mathbf{C}[x]$. p can be written as follows:

$$p = \sum_{\lambda=1}^{\varphi} s_{\lambda} x^{a_{\lambda}} \prod_{i=0}^{n} [y^{(i)}(x)]^{\theta_{i,\lambda}}$$

where $s_{\lambda} \in \mathbf{C}$ and some of the exponents $a_{\lambda}, \theta_{i,\lambda} \in \mathbf{Z}^{+}$ are not equal to zero. The number n, which represents the highest order derivative of $y(x)$, is called the order of

p. An equation of the form $p = 0$, with $y(x)$ as unknown function, is called a polynomial differential equation. Any function which satisfies it, is called a solution or a general solution. An expression of the form: $L = \sum_{i=0}^{n} a_i(x) y^{(i)}(x)$, $a_i(x) \in \mathbf{C}[x]$, is called a linear differential polynomial and the equation $L = 0$ a linear differential equation. Its solutions are called "linear" or holonomic solutions of order n.

Let $p_1 = s_1 x^{a_1} \prod_{i=0}^{n} [y^{(i)}(x)]^{\theta_{i,1}}$, $p_2 = s_2 x^{a_2} \prod_{i=0}^{n} [y^{(i)}(x)]^{\theta_{i,2}}$ be two, not identical, terms of p. This means that there is at least one index k, $1 \leq k \leq n$, such that $\theta_{k,1} \neq \theta_{k,2}$ or $\theta_{i,1} = \theta_{i,2}$, $i = 0, 1, \ldots n$ and $a_1 \neq a_2$. We say that the term p_2 is ordered higher than p_1 with respect to *lexicographical* order and we write $p_1 \prec p_2$, if either there is an index s such that $\theta_{s,1} < \theta_{s,2}$ and $\theta_{j,1} = \theta_{j,2}$, $j = s+1, \ldots, n$ or $\theta_{i,1} = \theta_{i,2}$, $i = 0, 1, \ldots, n$ and $a_1 < a_2$. By means of this rank we can order all the terms of p in an ascending way. The term which is ordered higher, is called the *maximum* term of p. The i-derivative of a polynomial p is denoted by $p^{(i)}$. The *differential ideal*, generated by a finite set of differential polynomials: $\mathbf{\Phi} = \{\Phi_1, \Phi_2, \ldots, \Phi_m\}$ and denoted by $[\mathbf{\Phi}]$ is a set which consists of all differential polynomials that can be formed of elements in $\mathbf{\Phi}$ by multiplication with arbitrary polynomials, addition and differentiation.

Let $\mathcal{W} = \{W_{\iota,\lambda,\sigma,\varphi}\}$ be a set of undetermined parameters, taking values in $\mathbf{C}$. A *Formal-(k,r)-Factorization* of p, denoted by $Formal(p,k,r)$, is an expression of p of the form:

$$Formal(p,k,r) = \sum_{\mu=1}^{\nu} c_\mu \prod_{i=0}^{n-k} \left[\left(\frac{A_{0,k+i,\mu}(x)}{A_{-1,k+i,\mu}(x)} + \frac{A_{1,k+i,\mu}(x)}{A_{-1,k+i,\mu}(x)} y(x) + \frac{A_{2,k+i,\mu}(x)}{A_{-1,k+i,\mu}(x)} y'(x) + \cdots + y^{(k)}(x) \right)^{(i)} \right]^{j_{i,\mu}} + R \qquad (2)$$

where the quantities $A_{\lambda,k+i,\mu}(x)$, $\lambda = -1, \ldots, k$ are r-degree polynomials of the single variable x and parametrical coefficients, that is: $A_{\lambda,k+i,\mu}(x) = \sum_{\iota=0}^{r} W_{\iota,\lambda,k+i,\mu} x^{\iota}$, the coefficients c_μ and the remainder R are rational functions of the terms $x, y'(x), \ldots, y^{(k-1)}(x)$, $W_{\iota,\lambda,k+i,\mu}$ only. Some of the exponents $j_{i,\mu} \in \mathbf{Z}^{+}$ may be equal to zero. Sometimes, (2) is written briefly as $Formal(p,k,r) = \sum_{\mu=1}^{\nu} c_\mu \prod_{i=0}^{n-k} [L_{i,\mu}^{(i)}]^{j_{i,\mu}} + R$, where we used the notation $L_{i,\mu}$ for the linear differential polynomial.

We can take different expressions of the $Formal(p,k,r)$ of a concrete differential polynomial p, by giving to the parameters $W_{\iota,\lambda,\sigma,\mu}$ certain values. Such procedures are called *evaluations* of the $Formal(p,k,r)$. A most rigorous approach is the following: Let $\mathcal{W} = \{W_{\iota,\lambda,\sigma,\mu}\}$ be the set of the variables, appeared in the Formal - (k,r) - Factorization of a given polynomial p. By arranging the parameters in an increasing order we form the vector $\mathcal{W} = (W_{\iota_h,\lambda_h,\sigma_h,\mu_h})_{h=1,2,\ldots,n}$. Let $\mathbf{s} = (a_h)_{h=1,2,\ldots,n}$ be a vector of complex numbers, which has the same length with the vector $\mathcal{W}$. We say that the parameters $\mathcal{W}$ follow the rule $\mathbf{s}$ and we write $\mathcal{W} \to \mathbf{s}$ if the following substitutions are valid: $W_{\iota_h,\lambda_h,\sigma_h,\mu_h} = a_h$, $h = 1, 2, \ldots, n$. Let M a set of rules, $M = \{\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_\omega, \ldots\}$ then $Formal(p,k,r) \big|_M$ is the set: $\{ Formal(p,k,r) \big|_{\mathbf{s}_1}, Formal(p,k,r) \big|_{\mathbf{s}_2}, \ldots, Formal(p,k,r) \big|_{\mathbf{s}_\omega}, \ldots\}$. The set of substitutions M, may be finite or infinite. When we evaluate $Formal(p,k,r)$ over M, the linear differential polynomials $L_{i,\mu}$ and the remainder R take specific values, these are denoted by $L_{i,\mu}\ \Big|_M$ and $R\ \Big|_M$. A case of particular interest is when we can find values of the parameters which eliminate the remainder R. Whenever this happens, p is a "combination" of linear differential polynomials or, in a more formal language, p is a member of the differential ideal produced by these linear differential polynomials. Relevant is the following theorem:

THEOREM 1. Let p be a differential polynomial of order n, let k be given and $Formal(p,k,r)=\sum_{\mu=1}^{\nu} c_\mu \prod_{i=0}^{n-k}[L_{i,\mu}^{(i)}]^{j_{i,\mu}}+R$ its Formal-(k,r)-Factorization. Let us suppose that there is a set of rules, denoted by $\mathcal{R}$, which eliminates the remainder R, i.e. $R\ \Big|_{\mathcal{R}}=0$, then $p\in[\ L_{i,\mu}\ \Big|_{\mathcal{R}}, i=0,\ldots,n-k,\mu=1,\ldots,\nu]$.

PROOF. The proof comes straightforward from the definition of the Formal-(k,r)-Factorization.

It is obvious that if the linear differential equations $L_{i,\mu}\ \Big|_{\mathcal{R}}=0$ have a common solution, this is a solution of the nonlinear equation $p=0$, too. This is the cornerstone of our approach.

EXAMPLE 1. Let us consider the differential polynomial $p=7xy''+8yy'$. We want to calculate the quantity $Formal(p,1,1)$. Since in this case $k=1$ and $r=1$, we shall form linear differential polynomials up to the first order and the polynomials $A_{\lambda,i,j}$ will be of degree one. Explicitly we have:

$$Formal(p,1,1)=7x*\left(\frac{W_{0,0,2,1}+W_{1,0,2,1}x}{W_{0,-1,2,1}+W_{1,-1,2,1}x}+\frac{W_{0,1,2,1}+W_{1,1,2,1}x}{W_{0,-1,2,1}+W_{1,-1,2.1}x}y(x)+y'(x)\right)'$$

$$+\left(\frac{-7x(W_{0,1,2,1}+W_{1,1,2,1}x)}{W_{0,-1,2,1}+W_{1,-1,2,1}x}+8y(x)\right)\cdot\left(\frac{W_{0,0,1,2}+W_{1,0,1,2}x}{W_{0,-1,2,1}+W_{1,-1,2,1}x}+\right.$$

$$\left.+\frac{W_{0,1,1,2}+W_{1,1,1,2}x}{W_{0,-1,2,1}+W_{1,-1,2,1}x}y(x)+y'(x)\right)+R$$

The remainder R is of the form $R=\Phi_0+\Phi_1 y+\Phi_2 y^2$, where Φ_0,Φ_1,Φ_2 are rational functions of the variable x and the $W_{\iota,\lambda,\sigma,\mu}$ parameters (not included because of their large size). The following rules eliminate the remainder: $\mathbf{s}_1=\{W_{0,1,2,1}=s, W_{1,-1,2,1}=\omega,\ W_{0,-1,2,1}=\varphi, W_{1,0,2,1}=k,\ W_{1,1,2,1}=\frac{s\omega}{\varphi}$, the other W-parameters $=0\}$ and $\mathbf{s}_2=\{W_{0,1,2,1}=s, W_{0,0,1,2}=\omega,\ W_{1,-1,2,1}=\varphi, W_{1,0,2,1}=k,\ W_{0,0,2,1}=\frac{-7s^2}{8\varphi},\ W_{1,-1,1,2}=\frac{8\omega\varphi}{7s}$, the other W-parameters $=0\}$, with $\omega,\varphi,s,k\in\mathbf{C}$. Indeed, for instance

$$Formal(p,1,1)\ \Big|_{\mathbf{r}_2}=7x\left(\frac{-7s^2+8x\varphi k}{8x\varphi^2}+\frac{s}{x\varphi}y+y'\right)'+\left(\frac{-7s}{\varphi}+8y\right)\left(\frac{7s}{8x\varphi}+y'\right)$$

and the differential ideal which contains p, is $\left[\left(\frac{-7s^2+8x\varphi k}{8x\varphi^2}+\frac{s}{x\varphi}y+y'\right),\left(\frac{7s}{8x\varphi}+y'\right)\right]$ where $\varphi,s,k,\in\mathbf{C}$. By setting $\varphi=s=1$ and $k=0$, we take the simplified ideal $\left[\frac{-7}{8x}+\frac{1}{x}y+y', \frac{7}{8x}+y'\right]$.

3 Detection of the Linear Solutions

The scope of this section is to present the algorithm which constructs for given k and r, special $Formal(p, k, r)$ with a linear differential polynomial as a common factor to each term. We denote this common factor by $L_{c,k}$. Afterward, by finding proper sets of values for the parameters $W_{\iota,\lambda,\sigma,\varphi}$, we eliminate the remainder. It is then clear that any solution of the linear equation $L_{c,k} = 0$, where the polynomial $L_{c,k}$ has been evaluated over this set, is a solution of the original system, too. By repeating the whole procedure for every $k = 0, \ldots, n$ and various values of r, we discover linear solutions. As we pointed out, the crucial issue is how can we eliminate the remainder. This is carried out by solving a system of algebraic equations. Finally, we have to elucidate that in this paper we do not take into account initial conditions. We are only focused on how we obtain general solutions, that is solutions which "contain" constants. We present now the algorithm upon discussion. Let us suppose that an algorithm which solves an algebraic system of polynomial equations, is available. These algorithms are classical in computational algebra and there are many of them in the literature [3]. We name such an algorithm as $SysAlgEqs$.

THE DIF-FORMAL ALGORITHM

Input:

- A differential polynomial p of order n.
- The upper bound of the degree r, of the polynomial coefficients . We denote it by ρ.

Output: The quantities $S_{k,r}$, $k = 0, \ldots, n$, $r = 0, \ldots, \rho$

FOR $k = 0$ **TO** n
FOR $r = 0$ **TO** ρ

Step 1: $R = p$, $\mu = 0$.

Step 2: **REPEAT** the following steps **UNTIL** R does not contain terms of order $\geq k$.

Step 2a: Set $\mu = \mu + 1$,

Step 2b: Find the maximum term of R, with respect to the lexicographical order. We denote it by

$$p_\mu = s_\mu \cdot x^{a_\mu} \cdot \prod_{i=0}^{n} [y^{(i)}(x)]^{\lambda_{i,\mu}}$$

where $a_\mu, \lambda_{i,\mu}$ are positive integers and $y^{(i)}(x)$ the derivatives of $y(x)$ of order i. At the first iteration s_μ is a constant, then it becomes a function of the free parameters $W_{\iota,\lambda,\sigma,\varphi}$ and x, as well.

Step 2c: Construct the linear formal differential polynomials:

$$L_{c,k} = \frac{A_{0,k}(x)}{A_{-1,k}(x)} + \sum_{j=0}^{k-1} \frac{A_{j+1,k}(x)}{A_{-1,k}(x)} y^{(j)} + y^{(k)}$$

$$L_{i,k}=\frac{A_{0,k+i,\mu}(x)}{A_{-1,k+i,\mu}(x)}+\sum_{j=0}^{k-1}\frac{A_{j+1,k+i,\mu}(x)}{A_{-1,k+i,\mu}(x)}y^{(j)}+y^{(k)},\quad i=1,\ldots,n-k$$

with $A_{\lambda,k}=\sum_{\iota=0}^{r}W_{\iota,\lambda,k}x^{\iota}$, $A_{\lambda,k+i,\mu}=\sum_{\iota=0}^{r}W_{\iota,\lambda,k+i,\mu}x^{\iota}$ and $\lambda=-1,\ldots,k$.

Step 2d: Execute the operation:

$$R=R-s_{\mu}\cdot x^{a_{\mu}}\cdot\prod_{i=0}^{k-1}[y^{(i)}(x)]^{\lambda_{i,\mu}}\cdot[L_{c,k}^{(0)}]^{\lambda_{k,\mu}}\cdot\prod_{i=1}^{n-k}[L_{i,k}^{(i)}]^{\lambda_{i,\mu}}$$

END of REPEAT

Step 3: By means of the *SysAlgEqs*-Algorithm we find the set $S_{k,r}$ of those values of the parameters, $W_{\iota,\lambda,\sigma,\varphi}$, which eliminate the remainder. In other words: $R\big|_{S_{k,r}}=0$.

END of FOR r

END of FOR k

It is obvious that the DIF-FORMAL algorithm terminates after a finite number of iterations.

THEOREM 2. Let $S_{k,r}$, be the outputs of the DIF-FORMAL Algorithm and $I\subset\{0,1,\ldots,n\}$, $J\subset\{0,1,\ldots,\rho\}$ subsets of indexes such that $S_{k,r}\neq\emptyset$ for $k\in I,r\in J$, then the solutions of the linear differential equations $L_{c,k}\big|_{S_{k,r}}=0$, $k\in I,r\in J$, are solutions of the nonlinear polynomial differential equation $p=0$, too.

PROOF. Let p a differential polynomial and k,r fixed. By substituting backwards the successive results of the step 2d we find that

$$p=\sum_{\mu=1}^{m}s_{\mu}x^{a_{\mu}}\prod_{i=0}^{k-1}[y^{(i)}]^{\lambda_{i,\mu}}\cdot[L_{c,k}^{(0)}]^{\lambda_{i,\mu}}\cdot\prod_{i=1}^{n-k}[L_{i,k}^{(i)}]^{\lambda_{i,\mu}}+R \qquad (3)$$

This is a special Formal-(k,r)-Factorization of p with $c_{\mu}=s_{\mu}x^{a_{\mu}}\prod_{i=0}^{k-1}[y^{(i)}]^{\lambda_{i,\mu}}$ and $L_{c,k}$, as a common factor in all the terms but the remainder. We denote this Formal Factorization by $CFormal(p,k)$. Let us now suppose that there are subsets of indexes I,J, such that $S_{k,r}\neq\emptyset$, $k\in I,r\in J$. This means that $R\big|_{S_{k,r}}=0$ and thus, the linear differential polynomial $L_{c,k}$, evaluated over $S_{k,r}$, is a common factor of every term of $p=\ CFormal(p,k)\big|_{S_{k,r}}$. This implies that any solution of the linear differential equation $L_{c,k}\big|_{S_{k,r}}=0$ is also a solution of the nonlinear equation $p=0$. Since this argument is true for any k,r, the theorem has been proved.

The above result can be restated, using ideals, in the following way.

COROLLARY 1. Let $S_{k,r}$, be the outputs of the DIF-FORMAL Algorithm and $I\subset\{0,1,\ldots,n\}$, $J\subset\{0,1,\ldots,\rho\}$ subsets of indexes such that $S_{k,r}\neq\emptyset$ for $k\in I,r\in J$, then $p\in[\ L_{c,k}\big|_{S_k}]$, $k\in I,r\in J$.

In order to eliminate the remainder we have to solve a system of algebraic polynomial equations. This can be done via several methods. Groebner basis, [3], is a popular powerful tool, with satisfactory results.

REMARK 1. Generally speaking, the upper bound of the degree of the polynomial coefficients, ρ, may be chosen freely. Nevertheless, is meaningless to be arbitrary large, since this fact will create a lot of unnecessary terms, which will be eliminated immediately by putting their coefficients equal to zero. Therefore, an interesting question is if we can estimate a proper value for ρ, (a lower upper bound), which will minimize the number of terms to be left over. To give a first answer to this complex problem, we work as follows: Let p be a differential polynomial, k fixed and $Formal(p,k,r)$ as in (3) with $R=0$, (The DIF-Algorithm has been applied at p). By $\deg(t_\mu, x, k, r)$ we denote the degree of x at the μ-term of the $Formal(p,k,r)$. Since the degree of x at $L^{(i)}_{i,\mu}$ is $2ri$, $i \neq 0$, (this is due to the differentiation rules), we get $\deg(t_\mu, x, k, r) = a_\mu + r\lambda_{0,\mu} + \sum_{i=1}^{n-k} 2ri\lambda_{i,\mu}$. After some manipulations, we can see that the minimization of the number of the redundant terms is accomplished if the next relation holds:

$$
\begin{aligned}
\rho &= \min_{r\in\mathbf{Z}^+}\{\min_{\mu,\nu}|\deg(t_\mu,x,k,r)-\deg(t_\nu,x,k,r)|\} = \\
&= \min_{r\in\mathbf{Z}^+}\left\{\min_{\mu,\nu}\left|(a_\mu-a_\nu)+r\left[(\lambda_{0,\mu}-\lambda_{0,\nu})+2\sum_{i=1}^{n-k} i(\lambda_{i,\mu}-\lambda_{i,\nu})\right]\right|\right\}
\end{aligned}
$$

Actually, we ask the differences among the various values of the degree of x to be as small as possible, so that, we achieve the number of the unused x or xy or $xy' \ldots$ terms to be small. This problem can be faced via various optimization techniques and can provide us a satisfactory value for the lowest upper bound of the degree of the polynomial coefficients.

EXAMPLE 2. Let us consider the differential equation $xyy'' - x(y')^2 - yy' = 0$ or $p = 0$. By means of classical techniques we can find a solution of the form $y = ae^{bx^2}$, a, b constants. Now, to clarify our ideas and to indicate how the algorithm works in practice, we shall present the case $k=1, r=1$ in details. These equalities mean two things, first that we are going to detect linear polynomials of first order, included into the original equation, and second the polynomial coefficients will be of degree one. In other words the common linear differential polynomial will be:

$$
L_{c,2} = \frac{W_{0,0,1}+W_{1,0,1}x}{W_{0,-1,1}+W_{1,-1,1}x} + \frac{W_{0,1,1}+W_{1,1,1}x}{W_{0,-1,1}+W_{1,-1,1}x}y + y'
$$

The order of p is $n=2$ and its maximum term xyy''. At the first iteration we execute the subtraction: $p_1 = p - xy \cdot L'_{c,2}$. This operation will eliminate the y'' term. In the next iteration $-x(y')^2$ is the maximum term and we calculate $p_2 = p_1 + x \cdot (L_{c,2})^2$.

Working this way we finally get:

$$
\begin{aligned}
& Formal(p,1,1) \\
= \quad & xy\left(\frac{W_{0,0,1}+W_{1,0,1}x}{W_{0,-1,1}+W_{1,-1,1}x}+\frac{W_{0,1,1}+W_{1,1,1}x}{W_{0,-1,1}+W_{1,-1,1}x}y+y'\right)' \\
& -x\left(\frac{W_{0,0,1}+W_{1,0,1}x}{W_{0,-1,1}+W_{1,-1,1}x}+\frac{W_{0,1,1}+W_{1,1,1}x}{W_{0,-1,1}+W_{1,-1,1}x}y+y'\right)^2 \\
& +\left(\frac{2x(W_{0,0,1}+W_{1,0,1}x)-W_{0,-1,1}y+x(W_{0,1,1}-W_{1,-1,1}+W_{1,1,1}x)y}{W_{0,-1,1}+W_{1,-1,1}x}\right) \\
& +\left(\frac{W_{0,0,1}+W_{1,0,1}x}{W_{0,-1,1}+W_{1,-1,1}x}+\frac{W_{0,1,1}+W_{1,1,1}x}{W_{0,-1,1}+W_{1,-1,1}x}y+y'\right)+R
\end{aligned}
$$

The rules $\mathbf{s}_1 = (W_{1,1,1} = h, W_{0,-1,1} = l$, the other W-parameters $= 0)$ h, l arbitrary constants, and $\mathbf{s}_2 = ($ all the W-parameters $= 0)$, eliminate the remainder. By substituting them to $L_{c,2}$ we get the linear differential equations $xhy - ly' = 0$, $y' = 0$. The general solutions of those equations will be solutions of the original nonlinear differential equation too. Therefore $y(x) = ce^{-x^2h/2l}$, $y(x) = c$ are solutions of the equation $p = 0$, too. Working similarly, we obtain for $k = 0$,$k = 1, k = 2$ and $r = 0, r = 1, r = 2$, (we used the method of Remark 1), analogous results. What we have actually proved is that p is a member of the differential ideal $[xhy - ly', y']$.

EXAMPLE 3. We consider the differential equation: $(x^2 - x)y' + xy'' - x^2y''' + (-2x^3 - 3x^2 + 3x) = 0$. We investigated it by giving to the order k and to the degree r several values. Actually, we set $k = 0, 1, 2$ and $r = 0, 1, 2, 3$. The DIF-FORMAL Algorithm gave the following solutions: $y = 5x^2 + x + c$, $y = 5x^2 + x + c + c_1e^x$, $y = 5x^2 + x + c - 3e^{-x}c_1 - 2e^{-x}xc_1$, $y = 5x^2 + x + c + c_1e^x - \frac{1}{4}e^{-x}(3 + 2x)c_2$, c, c_1, c_2 constants.

References

[1] S. Kotsios, On detecting solutions of polynomials nonlinear difference equations, J. Difference Eq. Appl., 8(6)(2002), 551–571.

[2] I. Kaplansky, Differential Algebra, Herman Paris, 1957.

[3] D. Cox, J. Little, D. O'Shea, Ideals, Varieties and Algorithms, Springer-Verlag, New York, 1997.

[4] D. Behloul and S. S. Cheng, Computation of all polynomial solutions of a class of nonlinear differential equations, Computing, 77(2)(2006), 163–177.

[5] J. F. Ritt, Differential Algebra, American Mathematical Society, Providence, RI, 1950.

[6] W. R. Li, S. S. Cheng and T. T. Lu, Closed form solutions of iterative functional differential equations, Appl. Math. E-Notes, 1(2001),1–4.

[7] S. T. Glad, Differential algebraic modeling of nonlnear systems, In proceedings of MTNS-89, Amsterdam, (1989), 97–105.

[8] E. S. Cheb-Terrab, L. G. S. Duarte and L.A.C.P. da Mota, Computer algebra solving of first order ODEs using symmetry methods, Comput. Phys. Comm. 101(3)(1997), 254–268.

[9] S. Kotsios, A symbolic computational algorithm for designing feedback stabilizers of nonlinear systems by checking positivity of polynomials, preprint.