On A Second-Order Differential Inclusion With Constraints*

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Abstract

We prove the existence of viable solutions to the Cauchy problem $x'' \in F(x, x') + f(t, x, x'), x(0) = x_0, x'(0) = y_0, x(t) \in K$, where $K \subset \mathbb{R}^n$ is a closed set, $F$ is a set-valued map contained in the Fréchet subdifferential of a $\phi$-convex function of order two and $f$ is a Carathéodory map.

1 Introduction

In this note we consider the second order differential inclusions of the form

$$x'' \in F(x, x') + f(t, x, x'), \quad x(0) = x_0, x'(0) = y_0,$$

where $F(, .) : D \subset \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a given set-valued map, $f(, ., .) : D_1 \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a given function and $x_0, y_0 \in \mathbb{R}^n$.

Existence of solutions of problem (1.1) that satisfy a constraint of the form $x(t) \in K$, $\forall t$, well known as viable solutions, has been studied by many authors, mainly in the case when the multifunction is convex valued and $f \equiv 0$ ([2], [6], [8], [10] etc.).

Recently in [1], the situation when the multifunction is not convex valued is considered. More exactly, in [1] it is proved the existence of viable solutions of the problem (1) when $F(, .)$ is an upper semicontinuous, compact valued multifunction contained in the subdifferential of a proper convex function. The result in [1] extends the result in [9] obtained for problems without constraints (i.e., $K = \mathbb{R}^n$).

The aim of this note is to prove existence of viable solutions of the problem (1) in the case when the set-valued map $F(, .)$ is upper semicontinuous compact valued and contained in the Fréchet subdifferential of a $\phi$-convex function of order two.

On one hand, since the class of proper convex functions is strictly contained into the class of $\phi$-convex functions of order two, our result generalizes the result in [1]. On the other hand, our result may be considered as an extension of our previous viability result for second-order nonconvex differential inclusions in [5] obtained for a problem without perturbations (i.e., $f \equiv 0$).

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The proof of our result follows the general ideas in [1] and [5]. We note that in the proof we pointed out only the differences that appeared with respect to the other approaches.

The paper is organized as follows: in Section 2 we recall some preliminary facts that we need in the sequel and in Section 3 we prove our main result.

2 Preliminaries

We denote by $\mathcal{P}(\mathbb{R}^n)$ the set of all subsets of $\mathbb{R}^n$ and by $R_+$ the set of all positive real numbers. For $\epsilon > 0$ we put $B(x, \epsilon) = \{ y \in \mathbb{R}^n; ||y - x|| < \epsilon \}$ and $\overline{B}(x, \epsilon) = \{ y \in \mathbb{R}^n; ||y - x|| \leq \epsilon \}$. With $B$ we denote the unit ball in $\mathbb{R}^n$. By $cl(A)$ we denote the closure of the set $A \subset \mathbb{R}^n$, by $co(A)$ we denote the convex hull of $A$ and we put $||A|| = \sup\{||a||; a \in A\}$.

Let $\Omega \subset \mathbb{R}^n$ be an open set and let $V : \Omega \rightarrow R \cup \{+\infty\}$ be a function with domain $D(V) = \{x \in \mathbb{R}^n; V(x) < +\infty\}$.

**DEFINITION 2.1.** The multifunction $\partial_F V : \Omega \rightarrow \mathcal{P}(\mathbb{R}^n)$, defined as:

$$\partial_F V(x) = \{ \alpha \in \mathbb{R}^n, \liminf_{\gamma \to x, \gamma \in \Omega} \frac{V(y) - V(x) - \langle \alpha, y - x \rangle}{||y - x||} \geq 0 \} \text{ if } V(x) < +\infty$$

and $\partial_F V(x) = \emptyset$ if $V(x) = +\infty$ is called the Fréchet subdifferential of $V$.

According to [4] the values of $\partial_F V$ are closed and convex.

**DEFINITION 2.2.** Let $V : \Omega \rightarrow R \cup \{+\infty\}$ be a lower semicontinuous function. We say that $V$ is a $\phi$-convex of order 2 if there exists a continuous map $\phi_V : (D(V))^2 \times \mathbb{R}^2 \rightarrow R_+$ such that for every $x, y \in D(\partial_F V)$ and every $\alpha \in \partial_F V(x)$ we have

$$V(y) \geq V(x) + \langle \alpha, x - y \rangle - \phi_V(x, y, V(x), V(y))(1 + ||\alpha||^2)||x - y||^2. \quad (2)$$

In [4], [7] there are several examples and properties of such maps. For example, according to [4], if $M \subset \mathbb{R}^2$ is a closed and bounded domain, whose boundary is a $C^2$ regular Jordan curve, the indicator function of $M$

$$V(x) = I_M(x) = \begin{cases} 0, & \text{if } x \in M \\ +\infty, & \text{otherwise} \end{cases}$$

is $\phi$-convex of order 2.

In what follows we assume the next assumptions.

**HYPOTHESIS 2.3.** i) $\Omega = K \times 0$, where $K \subset \mathbb{R}^n$ is a closed set and $O \subset \mathbb{R}^n$ is a nonempty open set.

ii) $F(.,.) : \Omega \rightarrow \mathcal{P}(\mathbb{R}^n)$ is upper semicontinuous (i.e., $\forall z \in \Omega, \forall \epsilon > 0$ there exists $\delta > 0$ such that $||z - z'|| < \delta$ implies $F(z') \subset F(z) + \epsilon B$) with compact values.

iii) $f(.,.,.) : R \times \Omega \rightarrow \mathbb{R}^n$ is a Carathéodory function, i.e., $\forall (x, y) \in \Omega$, $t \rightarrow f(t, x, y)$ is measurable, for all $t \in R$ $f(t, ., .)$ is continuous and there exists $m(.) \in L^2(R, R_+)$ such that $||f(t, x, y)|| \leq m(t) \forall (t, x, y) \in R \times \Omega$. 

iv) For all \((t, x, v) \in R \times \Omega\), there exists \(w \in F(x, v)\) such that

\[
\liminf_{h \to 0} \frac{1}{h^2} d(x + hv + \frac{h^2}{2} w + \int_{t}^{t+h} f(s, x, v) ds, K) = 0.
\]

v) There exists a proper lower semicontinuous \(\phi\)-convex function of order two \(V : R^n \to R \cup \{+\infty\}\) such that

\[
F(x, y) \subset \partial_{\phi} V(y), \quad \forall (x, y) \in \Omega.
\]

### 3 The Main Result

In order to prove our result we need the following lemmas.

**Lemma 3.1 ([1])**. Assume that Hypotheses 2.3 i)-iv) are satisfied. Consider \((x_0, y_0) \in \Omega, r > 0\) such that \(B(x_0, r) \subset O, M := \sup\{||F(t, x)||; (t, x) \in \Omega_0 := [K \times \bar{B}(y_0, r)] \cap \bar{B}(y_0, r)\}, T_1 > 0\) such that \(\int_{0}^{T_1} (m(s) + M + 1) < \frac{r}{2}, T_2 = \min\{\frac{3r(1+T_1)}{4}, \frac{3r}{4(1+T_1)}\} \text{ and } T \in (0, \min\{T_1, T_2\})\). Then for every \(\epsilon > 0\) there exists \(\eta \in (0, \epsilon)\) and \(p \geq 1\) such that for all \(i = 1, \ldots, p - 1\) there exists \((h_i, (x_i, y_i), w_i) \in [\eta, \epsilon] \times \Omega_0 \times R^n\) with the following properties

\[
x_i = x_{i-1} + h_{i-1} y_{i-1} + \frac{h_{i-1}^2}{2} w_{i-1} + \int_{h_{i-1}}^{x_{i-2} + h_{i-1}} f(s, x_{i-1}, y_{i-1}) ds \in K,
\]

\[
y_i = y_{i-1} + h_{i-1} w_{i-1}, \quad w_i \in F(x_i, y_i) + \frac{\epsilon}{T} B,
\]

and

\[(x_i, y_i) \in \Omega_0, \quad \sum_{i=0}^{p-1} h_i < T \leq \sum_{i=0}^{p} h_i.
\]

Moreover, for \(\epsilon > 0\) sufficiently small we have \(\sum_{i=0}^{p-1} h_i^2 \leq \sum_{i=0}^{p-1} h_i < T\).

For \(k \geq 1\) and \(q = 1, \ldots, p\) denote by \(h_k^q\) the real number associated to \(\epsilon = \frac{1}{k}\) and \((t, x, y) = (h_k^q, x_q, y_q)\) given by Lemma 3.1. Define \(t_0 = 0, t_0^q = T, t_q = h_0^q + \ldots + h_{q-1}^q\) and consider the sequence \(x_k(\cdot) : [t_0^q, t_q] \to R^n, k \geq 1\) defined by

\[
x_k(0) = x_0,
\]

\[
x_k(t) = x_{q-1} + (t - t_{q-1}^q)y_{q-1} + \frac{1}{2}(t - t_{q-1}^q)^2 w_{q-1} + \int_{t_{q-1}^q}^{t}(t - s)f(s, x_{q-1}, y_{q-1}) ds.
\]

**Lemma 3.2 ([1])**. Assume that Hypotheses 2.3 i)-iv) are satisfied and consider \(x_k(\cdot)\) the sequence constructed above. Then there exists a subsequence, still denoted by \(x_k(\cdot)\) and an absolutely continuous function \(x(\cdot) : [0, T] \to R^n\) such that

i) \(x_k(\cdot)\) converges uniformly to \(x(\cdot)\),

ii) \(x_k'(\cdot)\) converges uniformly to \(x'(\cdot)\),

iii) \(x_k''(\cdot)\) converges weakly in \(L^2([0, T], R^n)\) to \(x''(\cdot)\),
iv) The sequence \( \left( \sum_{k=1}^{p} \int_{t_k^{q-1}}^{t_k^{q+1}} <x''_k(s), f(s, x_k(t_k^{q-1}), x'_k(t_k^{q-1})) > ds \right) \) converges to \( \int_{0}^{T} <x''(s), f(s, x(s), x'(s)) > ds \).

v) For every \( t \in (0, T) \) there exists \( q \in \{1, ..., p \} \) such that

\[
\lim_{k \to \infty} d((x_k(t), x'_k(t), x''_k(t)) - f(t, x_k(t_k^{q-1}), x'_k(t_k^{q-1}))), \text{graph}(F) = 0,
\]

vi) \( x(.) \) is a solution of the convexified problem

\[
x'' \in \co F(x, x') + f(t, x, x'), \quad x(0) = x_0, x'(0) = v_0.
\]

We are now able to prove our result.

THEOREM 3.3. Assume that Hypothesis 2.3 is satisfied. Then, for every \( (x_0, y_0) \in \Omega \) there exist \( T > 0 \) and \( x(.) : [0, T] \to \mathbb{R}^n \) a solution of problem (1) that satisfies \( x(t) \in K \ \forall t \in [0, T] \).

PROOF. Let \( (x_0, y_0) \in \Omega \) and consider \( r > 0, T > 0 \) as in Lemma 3.1 and \( x_k(.) : [0, T] \to \mathbb{R}^n, x(.) : [0, T] \to \mathbb{R}^n \) as in Lemma 3.2. Let \( \phi_V \) the continuous function appearing in Definition 2.2. Since \( V(.) \) is continuous on \( D(V) \) (e.g. [7]), by possibly decreasing \( r \) one can assume that for all \( y \in B(y_0, r) \cap D(V) \)

\[
|V(y) - V(y_0)| \leq 1.
\]

Set

\[
S := \sup \{\phi_v(y_1, y_2, z_1, z_2); y_i \in \overline{B}(y_0, r), z_i \in [V(y_0) - 1, V(y_0) + 1], i = 1, 2\}.
\]

From the statement vi) in Lemma 3.2 and Hypothesis 2.3 v) it follows that for almost all \( t \in [0, T] \),

\[
x''(t) - f(t, x(t), x'(t)) \in \partial F(x'(t)).
\]

Since the mapping \( x(.) \) is absolutely continuous, from (3) and Theorem 2.2 in [4] we deduce that there exists \( T_3 > 0 \) such that the mapping \( t \to V(x'(t)) \) is absolutely continuous on \([0, \min\{T, T_3\}]\) and

\[
(V(x'(t)))' = \langle x''(t), x''(t) - f(t, x(t), x'(t)) \rangle \quad \text{a.e.} \ [0, \min\{T, T_3\}].
\]

(4)

Without loss of generality we may assume that \( T = \min\{T, T_3\} \). From (4) we have

\[
V(x'(T)) - V(y_0) = \int_{0}^{T} ||x''(s)||^2 ds - \int_{0}^{T} \langle x''(s), f(s, x(s), x'(s)) \rangle ds
\]

(5)

On the other hand, for \( q = 1, ..., p \) and \( t \in [t_k^{q-1}, t_k^{q+1}] \)

\[
x''_k(t) - f(t, x_k(t_k^{q-1}), x'_k(t_k^{q-1})) \in F(x_k(t_k^{q-1}), x'_k(t_k^{q-1})) + \frac{1}{kT}B
\]

and therefore

\[
x''_k(t) - f(t, x_k(t_k^{q-1}), x'_k(t_k^{q-1})) \in \partial F(x'_k(t_k^{q-1})) + \frac{1}{kT}B.
\]
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We deduce the existence of $b_k^q \in B$ such that

$$x_k''(t) - f(t, x_k(t_k^{-1}), x_k'(t_k^{-1})) - \frac{b_k^q}{kT} \in \partial_F V(x_k(t_k^{-1})).$$

Taking into account Definition 2.2 we obtain

$$V(x_k(t_k^q)) - V(x_k(t_k^{-1})) \geq \left\langle x_k''(t) - f(t, x_k(t_k^{-1}), x_k'(t_k^{-1})), \frac{b_k^q}{kT}, \int_{t_k^{-1}}^{t_k^q} x_k(s)ds \right\rangle$$

$$- \phi_V \left( x_k(t_k^q), x_k'(t_k^{-1}), V(x_k(t_k^{-1})), V(x_k(t_k^{-1})) \right)$$

$$\times \left( 1 + \left\| x_k''(t) - f(t, x_k(t_k^{-1}), x_k'(t_k^{-1})) - \frac{b_k^q}{kT} \right\|^2 \right)$$

$$\times \left\| x_k(t_k^q) - x_k'(t_k^{-1}) \right\|^2.$$

Using the fact that $x_k''(\cdot)$ is constant on $[t_k^{-1}, t_k]$ one may write

$$V(x_k(t_k^q)) - V(x_k(t_k^{-1})) \geq \int_{t_k^{-1}}^{t_k^q} \left( x_k''(s), x_k''(s) \right) ds - \int_{t_k^{-1}}^{t_k^q} \left( x_k'(s), \frac{b_k^q}{kT} \right) ds$$

$$- \int_{t_k^{-1}}^{t_k^q} \left( x_k''(s), f(s, x_k(t_k^{-1}), x_k'(t_k^{-1})) \right) ds$$

$$- \phi_V \left( x_k'(t_k^q), x_k'(t_k^{-1}), V(x_k(t_k^{-1})), V(x_k(t_k^{-1})) \right)$$

$$\times \left( 1 + \left\| x_k''(t) - f(t, x_k(t_k^{-1}), x_k'(t_k^{-1})) - \frac{b_k^q}{kT} \right\|^2 \right)$$

$$\times \left\| x_k(t_k^q) - x_k'(t_k^{-1}) \right\|^2.$$

By adding on $q$ the last inequalities we get

$$V(x_k(T)) - V(y_0) \geq \int_0^T \left\| x_k''(s) \right\|^2 ds + a(k) + b(k)$$

$$- \sum_{q=1}^{p} \int_{t_k^{-1}}^{t_k^q} \left( x_k''(s), f(s, x_k(t_k^{-1}), x_k'(t_k^{-1})) \right) ds,$$  \hspace{1cm} (6)

where

$$a(k) = - \sum_{q=1}^{p} \frac{1}{kT} \int_{t_k^{-1}}^{t_k^q} \left( x_k''(s), b_k^q \right) ds,$$

$$b(k) = - \sum_{q=1}^{p} \phi_V \left( x_k'(t_k^q), x_k'(t_k^{-1}), V(x_k(t_k^{-1})), V(x_k(t_k^{-1})) \right)$$

$$\times \left( 1 + \left\| x_k''(t) - f(t, x_k(t_k^{-1}), x_k'(t_k^{-1})) - \frac{b_k^q}{kT} \right\|^2 \right) \left\| x_k(t_k^q) - x_k'(t_k^{-1}) \right\|^2.$$
On the other hand, one has

\[|a(k)| \leq \frac{1}{kT} \sum_{q=1}^{p} ||b_q^q|| \int_{t_{k-1}^q}^{t_k} ||x''(s)|| ds\]

\[\leq \frac{1}{kT} \int_0^T ||x''(s)|| ds \leq \frac{1}{kT} \int_0^T [M + \frac{1}{T} + m(s)] ds\]

and

\[|b(k)| \leq \sum_{q=1}^{p} S(1 + M^2) \int_{t_{k-1}^q}^{t_k} ||x''(s)||^2 ds\]

\[\leq S(1 + M^2) \int_{t_{k-1}^q}^{t_k} ||x''(s)||^2 ds \leq S(1 + M^2) \frac{1}{k} \int_0^T ||x''(s)||^2 ds\]

\[\leq \frac{1}{k} S(1 + M^2) \int_0^T [M + \frac{1}{T} + m(s)]^2 ds.\]

We infer that

\[\lim_{k \to \infty} a(k) = \lim_{k \to \infty} b(k) = 0.\]

Hence using also statement iv) in Lemma 3.2 and the continuity of the function \(V(.)\) by passing to the limit as \(k \to \infty\) in (6) we obtain

\[V(x'(T)) - V(y_0) \geq \limsup_{k \to \infty} \int_0^T ||x''(s)||^2 ds - \int_0^T \left< x''(s), f(s, x(s), x'(s)) \right> ds. \quad (7)\]

Using (4) we infer that

\[\limsup_{k \to \infty} \int_0^T ||x_k(t)||^2 dt \leq \int_0^T ||x''(t)||^2 dt\]

and, since \(x_k''(.)\) converges weakly in \(L^2([0, T], \mathbb{R}^n)\) to \(x''(.)\), by the lower semicontinuity of the norm in \(L^2([0, T], \mathbb{R}^n)\) (e.g. Prop. III.30 in [3]) we obtain that

\[\lim_{k \to \infty} \int_0^T ||x_k''(t)||^2 dt = \int_0^T ||x''(t)||^2 dt\]

i.e., \(x_k''(.)\) converges strongly in \(L^2([0, T], \mathbb{R}^n)\). Hence, there exists a subsequence (still denoted) \(x''(.)\) that converges pointwise to \(x''(.)\). From the statement v) in Lemma 3.2 it follows that

\[d((x(t), x'(t), x''(t) - f(t, x(t), x'(t))), graph(F)) = 0 \quad a.e. \ [0, T].\]

and since by Hypothesis 2.4 \(graph(F)\) is closed we obtain

\[x''(t) \in F(x(t), x'(t)) + f(t, x(t), x'(t)) \quad a.e. \ [0, T].\]
In order to prove the viability constraint satisfied by $x(.)$ fix $t \in [0, T]$. There exists a sequence $(t^k_q)_k$ such that $t = \lim_{k \to \infty} t^k_q$. But $\lim_{k \to \infty} ||x(t) - x_k(t^k_q)|| = 0$ and $x_k(t^k_q) \in K$. So the fact that $K$ is closed gives $x(t) \in K$ and the proof is complete.

REMARK 3.4. If $V(.) : \mathbb{R}^n \to \mathbb{R}$ is a proper lower semicontinuous convex function then (e.g. [7]) $\partial F V(x) = \partial V(x)$, where $\partial V(.)$ is the subdifferential in the sense of convex analysis of $V(.)$, and Theorem 3.3 yields the result in [1]. At the same time if in Theorem 3.3 $f \equiv 0$ then Theorem 3.3 yields the result in [5].

References


