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SEMI-SLANT PSEUDO-RIEMANNIAN SUBMERSIONS FROM INDEFINITE ALMOST CONTACT 3-STRUCTURE MANIFOLDS ONTO PSEUDO-RIEMANNIAN MANIFOLDS

S. S. SHUKLA AND UMA SHANKAR VERMA

Abstract. In this paper, we introduce the notion of a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold onto a pseudo-Riemannian manifold. We investigate the geometry of foliations determined by horizontal and vertical distributions and provide a non-trivial example. We also find a necessary and sufficient condition for a semi-slant submersion to be totally geodesic. Moreover, we check the harmonicity of such submersions.

1. Introduction

The theory of Riemannian submersions was introduced by O’Neill [14] in 1966 and Gray [9] in 1967. Several geometers studied Riemannian submersions between Riemannian manifolds equipped with some additional structures such as almost complex, almost contact etc. [7, 8, 14, 15]. It is well known that Riemannian submersions are related with physics and have their applications in Kaluza-Klein theory [4, 10, 11], Yang-Mills theory [5, 22], the theory of robotics [1], supergravity and superstring theories [11, 13].

In 1976, B. Watson defined almost Hermitian submersions between almost Hermitian manifolds and gave some differential geometric properties among fibres, base manifolds and total manifolds [23]. In 2010, Sahin introduced anti-invariant and semi-invariant Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds [18, 20]. He also gave the notion of a slant submersion from an almost Hermitian manifold onto a Riemannian manifold as a generalization of Hermitian submersions and anti-invariant submersions [19]. K. S. Park also studied slant and semi-slant submersions and obtained several interesting results [16, 17].

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In the present paper, our aim is to study semi-slant pseudo-Riemannian submersions from indefinite almost contact 3-structure manifolds onto pseudo-Riemannian manifolds. The composition of the paper is as follows. In section 2, we collect some basic definitions, results on indefinite almost contact 3-structure manifolds and pseudo-Riemannian submersions. In section 3, we define semi-slant pseudo-Riemannian submersions from indefinite almost contact 3-structure manifolds onto pseudo-Riemannian manifolds giving an example. We investigate the geometry of foliations which arise from the definition of above submersions and find a necessary and sufficient condition for submersions to be totally geodesic. We also check the harmonicity of such submersions.

2. Preliminaries:

2.1. Indefinite Almost Contact 3-structure Manifolds: Let $M$ be a $(4n + 3)$-dimensional Riemannian manifold and for $i = 1, 2, 3$, $\phi_i$ be $(1,1)$-type tensor fields, $\xi_i$ vector fields, called characteristic vector fields and $\eta_i$ 1-forms on $M$. Then $(\phi_i, \xi_i, \eta_i), i = 1, 2, 3,$ is called an almost contact 3-structure on $M$ if it satisfies [2, 3, 12]

\begin{align}
\phi_i \xi_j &= -\phi_j \xi_i = \xi_k, \\
\phi_i \circ \phi_j - \eta_j \otimes \xi_i &= -\phi_j \circ \phi_i + \eta_i \otimes \xi_j = \phi_k, \\
\eta_i(\xi_j) &= 0, \\
\eta_i(\phi_j) &= -\eta_j(\phi_i) = \eta_k,
\end{align}

for any triad $(i, j, k)$ of cyclic permutation in symmetric group $S_3$.

$M$ is said to be an almost contact 3-structure manifold, if it is equipped with an almost contact 3-structure. Again, $M$ is called an indefinite almost contact 3-structure manifold, if it is endowed with a pseudo-Riemannian metric $g$ such that

\begin{align}
g(\phi_i X, \phi_i Y) &= g(X, Y) - \varepsilon_i \eta_i(X) \eta_i(Y), \\
g(\xi_i, \xi_i) &= \varepsilon_i, \\
g(\xi_i, X) &= \varepsilon_i \eta_i(X), \\
g(\phi_i X, Y) &= -g(X, \phi_i Y),
\end{align}

for any $X, Y \in \Gamma(TM)$ and $\forall i = 1, 2, 3$.

An indefinite almost contact 3-structure manifold is called

(a) cosymplectic 3-structure manifold if $\nabla \phi_i = 0$,

(b) Sasakian 3-structure manifold if

\begin{align}
(\nabla_X \phi_i) Y &= g(X, Y) \xi_i - \eta_i(Y) X, \text{ and } \\
\nabla_X \xi &= -\phi_i X,
\end{align}

for any $X, Y \in \Gamma(TM); i = 1, 2, 3$. 
2.2. **Pseudo-Riemannian Submersions:** Let \((\tilde{M}^m, g)\) and \((M^n, g)\) be two connected pseudo-Riemannian manifolds of indices \(\bar{s} (0 \leq \bar{s} \leq m)\) and \(s (0 \leq s \leq n)\) respectively, where \(m \geq n\) and \(\bar{s} > s\).

A pseudo-Riemannian submersion is a smooth surjective map \(f: \tilde{M}^m \to M^n\) which satisfies the following conditions \([7, 8, 9, 14]\):

(i) the derivative map \(f_{\ast p}: T_p \tilde{M} \to T_{f(p)} M\) is surjective at each point \(p \in \tilde{M}\);

(ii) the fibres \(f^{-1}(q)\) of \(f\) over \(q \in M\) are pseudo-Riemannian submanifolds of \(\tilde{M}\);

(iii) \(f\) preserves the length of horizontal vectors.

A vector field on \(\tilde{M}\) is called vertical if it is always tangent to fibres and it is called horizontal if it is always orthogonal to fibres. We denote by \(\mathcal{V}\) the vertical distribution and by \(\mathcal{H}\) the horizontal distribution. Also, we denote vertical and horizontal projections of a vector field \(E\) on \(\tilde{M}\) by \(vE\) and by \(hE\) respectively. A horizontal vector field \(\tilde{X}\) on \(\tilde{M}\) is said to be basic if \(\tilde{X}\) is \(f\)-related to a vector field \(X\) on \(M\) i.e. \(f_{\ast} \tilde{X} = X \circ f\). Thus, every vector field \(X\) on \(M\) has a unique horizontal lift \(\tilde{X}\) on \(\tilde{M}\).

We recall the following lemma for later use:

**Lemma 2.1** ([7, 15]). If \(f: \tilde{M} \to M\) is a pseudo-Riemannian submersion and \(\tilde{X}, \tilde{Y}\) are basic vector fields on \(\tilde{M}\) that are \(f\)-related to the vector fields \(X, Y\) on \(M\) respectively, then we have the following properties

(i) \(\tilde{g}(\tilde{X}, \tilde{Y}) = g(X, Y) \circ f\),

(ii) \(h[X, \tilde{Y}]\) is a vector field and \(f_{\ast} h[\tilde{X}, \tilde{Y}] = [X, Y] \circ f\),

(iii) \(h(\nabla_{\tilde{X}} \tilde{Y})\) is a basic vector field \(f\)-related to \(\nabla_X Y\), where \(\nabla\) and \(\nabla\) are the Levi-Civita connections on \(\tilde{M}\) and \(M\) respectively,

(iv) \([E, U] \in \mathcal{V}, \) for any \(U \in \mathcal{V}\) and for any vector field \(E \in \Gamma(T\tilde{M})\).

A pseudo-Riemannian submersion \(f: \tilde{M} \to M\) determines tensor fields \(\mathcal{T}\) and \(\mathcal{A}\) of type \((1, 2)\) on \(M\) defined by formulas \([7, 14, 15]\)

\[
\mathcal{T}(E, F) = \mathcal{T}_E F = h(\nabla_{\nabla_{\tilde{U}} F}) + v(\nabla_{\tilde{V}} F),
\]

\[
\mathcal{A}(E, F) = \mathcal{A}_E F = v(\nabla_{\nabla_{\tilde{U}} F}) + h(\nabla_{\tilde{V}} F),
\]

for any \(E, F \in \Gamma(T\tilde{M})\).

Let \(\tilde{X}, \tilde{Y}\) be horizontal vector fields and \(U, V\) be vertical vector fields on \(\tilde{M}\). Then, we have

\[
\mathcal{T}_U \tilde{X} = v(\nabla_U \tilde{X}),
\]

\[
\mathcal{T}_U V = h(\nabla_U V),
\]

\[
\nabla_U \tilde{X} = \mathcal{T}_U \tilde{X} + h(\nabla_U \tilde{X}),
\]

\[
\mathcal{T}_X F = 0,
\]

\[
\mathcal{T}_E F = \mathcal{T}_{\nabla_{U} F},
\]

\[
\nabla_U V = \mathcal{T}_U V + v(\nabla_U V),
\]
A_X Y = v(\nabla_X Y),
(2.20) A_X U = h(\nabla_X U),
(2.21) \nabla_X U = A_X U + v(\nabla_X U),
(2.22) A_U F = 0,
(2.23) A_E F = A_{hE} F,
(2.24) \nabla_X Y = A_X Y + h(\nabla_X Y),
(2.25) h(\nabla_U X) = h(\nabla_X U) = A_X U,
(2.26) A_X \tilde{Y} = \frac{1}{2} v[\tilde{X}, \tilde{Y}],
(2.27) A_X \tilde{Y} = -A_Y \tilde{X},
(2.28) \nabla_U V = \nabla_V U,
\forall E, F \in \Gamma(TM).

It can be easily shown that a Riemannian submersion \( f: \tilde{M} \to M \) has totally geodesic fibres if and only if \( T \) vanishes identically. By lemma (2.1), the horizontal distribution \( \mathcal{H} \) is integrable if and only if \( A = 0 \). Also, in view of equations (2.27) and (2.28), \( A \) is alternating on the horizontal distribution and \( T \) is symmetric on the vertical distribution.

Now, we recall the notion of harmonic maps between pseudo-Riemannian manifolds. Let \((\tilde{M}, \tilde{g})\) and \((M, g)\) be pseudo-Riemannian manifolds and let \( f: \tilde{M} \to M \) be a smooth map. Then the second fundamental form of the map \( f \) is given by

\[(\nabla f_\ast)(X, Y) = (\nabla_X f_\ast Y) \circ f - f_\ast(\nabla_X Y)\]

for all \( X, Y \in \Gamma(TM) \), where \( \nabla^f \) denotes the pullback connection of \( \nabla \) with respect to \( f \) and the tension field \( \tau \) of \( f \) is defined by

\[\tau(f) = \text{trace}(\nabla f_\ast) = \sum_{i=1}^{m} (\nabla f_\ast)(e_i, e_i),\]

where \( \{e_1, e_2, \cdots e_m\} \) is an orthonormal frame on \( \tilde{M} \).

It is known that \( f \) is harmonic if and only if \( \tau(f) = 0 \) [6].

In this paper, we study pseudo Riemannian submersions \( f: \tilde{M} \to M \) such that fibres \( f^{-1}(q) \) over \( q \in M \) be pseudo-Riemannian submanifolds admitting non-lightlike vector fields.

3. Semi-Slant Pseudo-Riemannian Submersions

**Definition 3.1.** Let \((\tilde{M}^{4m+3}, \tilde{\phi}_i, \tilde{\xi}_i, \tilde{\eta}_i, \tilde{g})\) be an indefinite almost contact 3-structure manifold and \((M^n, g)\) be a pseudo-Riemannian manifold. A pseudo-Riemannian submersion \( f: \tilde{M} \to M \) is called a semi-slant pseudo-Riemannian submersion if structure vector fields \( \tilde{\xi}_i; i = 1, 2, 3 \) are horizontal and there exists a distribution \( \mathcal{D} \subseteq \ker f_\ast\) such that

(i) \( \ker f_\ast = \tilde{\mathcal{D}} \oplus \mathcal{D}^\perp; \)

\[\]
(ii) $\tilde{\phi}_i(D) \subseteq \tilde{D}$; and
(iii) for any non-zero vector field $X_p \in \tilde{D}_p^\perp$, the angle $\theta_i$ between $\tilde{\phi}_i X_p$ and the space $\tilde{D}_p^\perp$ is constant.

We observe that if dimension $\tilde{D} = 0$, then a semi-slant pseudo-Riemannian submersion $f: M \to M$ is a slant pseudo-Riemannian submersion [19].

For any vector field $U \in \mathcal{V}$, we put

$$U = PU + QU,$$

where $PU \in \tilde{D}$ and $QU \in \tilde{D}^\perp$.

Also, for any vector field $U \in \tilde{D}^\perp$, we set

$$\tilde{\phi}_i U = \psi_i U + \omega_i U,$$

where $\psi_i U$ and $\omega_i U$ are horizontal and vertical components of $\tilde{\phi}_i U$ respectively.

For any vector field $X \in \mathcal{H}$, we put

$$\tilde{\phi}_i X = t_i X + n_i X,$$

where $t_i X$ and $n_i X$ are horizontal and vertical components of $\tilde{\phi}_i X$ respectively.

**Example 3.2.** Let $(\mathbb{R}^{15}_6; \tilde{\phi}_i, \tilde{\xi}_i, \tilde{\eta}_i, \tilde{\gamma}, i = 1, 2, 3)$ be an indefinite almost contact 3-structure manifold such that for any $(a_1, a_2, \ldots, a_{15})^t \in \mathbb{R}^{15}$,

$$\tilde{\phi}_1 ((a_1, a_2, \ldots, a_{15})^t) =$$

$$(-a_3, a_4, a_1, -a_2, -a_7, a_8, a_5, -a_6, -a_{11}, a_{12}, a_9, -a_{10}, 0, -a_{15}, a_{14})^t,$$

$$\tilde{\phi}_2 ((a_1, a_2, \ldots, a_{15})^t) =$$

$$(-a_4, -a_3, a_2, a_1, -a_8, -a_7, a_6, a_5, -a_{12}, -a_{11}, a_{10}, a_9, a_{15}, 0, -a_{13})^t,$$

$$\tilde{\phi}_3 ((a_1, a_2, \ldots, a_{15})^t) =$$

$$(-a_2, a_1, -a_4, a_3, -a_6, a_5, -a_8, a_7, -a_{10}, a_9, -a_{12}, a_{11}, -a_{14}, a_{13}, 0)^t,$$

$$\tilde{\xi}_1 = \frac{\partial}{\partial x_3}, \tilde{\xi}_2 = \frac{\partial}{\partial x_4}, \tilde{\xi}_3 = \frac{\partial}{\partial x_5}, \tilde{\eta}_1 = dx_{13}, \tilde{\eta}_2 = dx_{14}, \tilde{\eta}_3 = dx_{15},$$

signature of $\tilde{\gamma} = (-, -, +, +, +, +, +, +, +, +, +)$ and let $(\mathbb{R}^7_2; g)$ be a pseudo-Riemannian manifold.

Define a submersion $f: \{\mathbb{R}^{15}_6; (x_1, x_2, \ldots, x_{15})^t\} \to \{\mathbb{R}^7_2; (y_1, y_2, \ldots, y_7)^t\}$ by

$$f ((x_1, x_2, \ldots, x_{15})^t) \longmapsto (x_5 \sin \alpha - x_7 \cos \alpha, x_6 \sin \alpha + x_8 \cos \alpha, x_9 \sin \alpha - x_{11} \cos \alpha, x_{10} \sin \alpha + x_{12} \cos \alpha, x_{13}, x_{14}, x_{15})^t,$$

where $\alpha$ is a real number.

The vertical distribution $\mathcal{V}$ is

$$\text{Span} \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}, \cos \alpha \frac{\partial}{\partial x_5} + \sin \alpha \frac{\partial}{\partial x_7}, - \cos \alpha \frac{\partial}{\partial x_6} + \sin \alpha \frac{\partial}{\partial x_8}, \cos \alpha \frac{\partial}{\partial x_9} + \sin \alpha \frac{\partial}{\partial x_{11}}, - \cos \alpha \frac{\partial}{\partial x_{10}} + \sin \alpha \frac{\partial}{\partial x_{12}} \right\}.$$
\[ D = \text{Span} \left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right\} \subset \mathcal{V} \text{ and } \]
\[ \tilde{D}^\perp = \text{Span} \left\{ \cos \alpha \frac{\partial}{\partial x_5} + \sin \alpha \frac{\partial}{\partial x_7}, -\cos \alpha \frac{\partial}{\partial x_6} + \sin \alpha \frac{\partial}{\partial x_8}, \right. \]
\[ \left. \cos \alpha \frac{\partial}{\partial x_9} + \sin \alpha \frac{\partial}{\partial x_{11}}, -\cos \alpha \frac{\partial}{\partial x_{10}} + \sin \alpha \frac{\partial}{\partial x_{12}} \right\}. \]

Then, \( \tilde{D} \) is invariant with respect to \( \phi_i, i = 1, 2, 3 \) and semi-slant angles \( \theta_1 = \cos^{-1}(\tan 2\alpha), \theta_2 = \theta_3 = \frac{\pi}{7}. \)

**Proposition 3.3.** Let \( f : \tilde{M} \to M \) be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold \((\tilde{M}, \tilde{\phi}_i, \tilde{\xi}_i, \tilde{\eta}_i, \tilde{g})\) onto a pseudo-Riemannian manifold \((M, g)\). If \( U, V \) are vertical and \( X, Y \) are horizontal vector fields on \( M \), then we have
\[ g(\phi_iPU + \psi_iQV + \omega_iQU, V) = -\tilde{g}(U, \tilde{\phi}_iPV + \psi_iQV + \omega_iQV), \]
which gives
\[ \tilde{g}(\omega_iQU, V) = -\tilde{g}(U, \omega_iQV). \]
Similarly, for any vector fields \( U, V \in \mathcal{V} \) and \( \tilde{X}, \tilde{Y} \in \mathcal{H}, \) we have equations (3.5) and (3.6).

**Theorem 3.4.** Let \( f : M \to \tilde{M} \) be a pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold \((\tilde{M}, \tilde{\phi}_i, \tilde{\xi}_i, \tilde{\eta}_i, \tilde{g})\) onto a pseudo-Riemannian manifold \((M, g)\). Then, \( f \) is a semi-slant Riemannian submersion if and only if there exists \( \lambda \in [0, 1] \) such that
\[ (Q\omega_i)^2 = \lambda \tilde{\phi}_i^2. \]
Moreover, if \( \theta_i \) is semi-slant angle of the submersion, then \( \lambda = \cos^2 \theta_i; i = 1, 2, 3. \)

**Proof.** Let \( Z \) be any non-zero vector field in the distribution \( \tilde{D}^\perp. \) Then, we have
\[ \cos \theta_i = \frac{\tilde{g}(\tilde{\phi}_iZ, Q\omega_iZ)}{|\tilde{\phi}_iZ||Q\omega_iZ|}. \]
Again, we have
\[ \cos \theta_i = \frac{|Q\omega_iZ|}{|\phi_iZ|}. \]
Now, from equations (2.8), (3.2) and (3.8) we have

$$\cos \theta_i = \frac{\bar{g}(Z, Q\omega_i Q\omega_i Z)}{|Q\omega_i Z| |\phi_i Z|}.$$  

(3.10)

In view of equations (3.9) and (3.10), we have

$$\cos^2 \theta_i = \frac{\bar{g}(Z, (Q\omega_i)^2 Z)}{\bar{g}(Z, (\phi_i)^2 Z)}.$$  

(3.11)

Now, equation (3.11) implies that $\cos^2 \theta_i = \text{constant}$ if and only if $(Q\omega_i)^2$ and $\phi_i^2$ are conformally parallel.

Hence, $(Q\omega_i)^2 = \lambda(\phi_i)^2$, for some $\lambda \in [0, \infty)$.

Again, from equations (3.7) and (3.11), we have $\lambda = \cos^2 \theta_i$. Consequently, we have $\lambda \in [0, 1]$. □

**Corollary 3.5.** Let $f: \tilde{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold $(\tilde{M}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \bar{g})$ onto a pseudo-Riemannian manifold $(M, g)$. If $\theta_i$ is a semi-slant angle of the submersion, then for $U, V \in D^\perp; i = 1, 2, 3$, we have

$$\bar{g}(Q\omega_i U, Q\omega_i V) = (\bar{g}(U, V) - \varepsilon_i\tilde{\eta}(U)\tilde{\eta}(V)) \cos^2 \theta_i,$$  

(3.12)

$$\bar{g}(\psi_i U, \psi_i V) = (\bar{g}(U, V) - \varepsilon_i\tilde{\eta}(U)\tilde{\eta}(V)) \sin^2 \theta_i - \bar{g}(P\omega_i U, P\omega_i V).$$  

(3.13)

**Proof.** For any $U, V \in D^\perp$, using equations (3.1) and (3.2), we have

$$\bar{g}(Q\omega_i U, V) = -\bar{g}(U, Q\omega_i V).$$

On replacing $V$ by $Q\omega_i V$ and using equation (3.7), above equation implies

$$\bar{g}(Q\omega_i U, Q\omega_i V) = -\lambda\bar{g}(U, (\phi_i)^2 V).$$

Now, in view of equation (2.5), we have equation (3.12).

Similarly, we can obtain equation (3.13). □

**Lemma 3.6.** Let $f: \tilde{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold $(\tilde{M}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \bar{g})$ onto a pseudo-Riemannian manifold $(M, g)$. Then, we have

$$\tilde{\phi}_i D^\perp \subseteq \mathcal{H} \oplus_{\text{ortho}} (\psi_i \bar{D}^\perp)^\perp,$$  

(3.14)

$$\tilde{\phi}_i^2 D^\perp \subseteq t_i \psi_i \bar{D}^\perp \oplus_{\text{ortho}} n_i \psi_i \bar{D}^\perp \oplus_{\text{ortho}} P^2(\omega_i \tilde{D}^\perp) \oplus_{\text{ortho}} Q^2(\omega_i \tilde{D}^\perp) \oplus_{\text{ortho}} P(Q\omega_i \tilde{D}^\perp);$$  

(3.15)

for $i = 1, 2, 3$.

**Proof.** The first result is obvious as $\psi_i \bar{D}^\perp \subset \mathcal{H}$.

From equations (3.1), (3.2) and (3.3), we have

$$\tilde{\phi}_i D^\perp \subseteq \psi_i \bar{D}^\perp \oplus P\omega_i \bar{D}^\perp \oplus Q\omega_i \bar{D}^\perp.$$  

Now, by using equations (3.1), (3.2) and (3.3), we get (3.15). □
Theorem 3.7. Let $f: \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold $(\bar{M}, \bar{\phi}, \bar{\xi}, \bar{n}, \bar{g})$ onto a pseudo-Riemannian manifold $(M, g)$. Then, $(\psi_i D^\perp)^{\perp}$ is invariant with respect to $\bar{\phi}_i; i = 1, 2, 3$.

Proof. Let $X \in (\psi_i D^\perp)^{\perp}, i = 1, 2, 3$. Then, for any $U \in D^\perp$, we have

$$\bar{g}(\bar{\phi}_i X, Q\omega_i U) = -\bar{g}(X, \bar{\phi}_i(Q\omega_i U)) = -\bar{g}(X, P\omega_i(Q\omega_i U) + (Q\omega_i)^2 U + \psi_i(Q\omega_i U)) = 0.$$

Again, for any $U \in \bar{D}^\perp$,

$$\bar{g}(\bar{\phi}_i X, U) = -\bar{g}(X, \bar{\phi}_i U) = -\bar{g}(X, \psi_i U + P\omega_i U + Q\omega_i U) = 0.$$

Thus, $\bar{\phi}_i X \in (\psi_i D^\perp)^{\perp}$. Hence, $\bar{\phi}_i(\psi_i D^\perp)^{\perp} \subset (\psi_i D^\perp)^{\perp}$. \hfill $\square$

Lemma 3.8. Let $f: \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold $(\bar{M}, \bar{\phi}, \bar{\xi}, \bar{n}, \bar{g})$ onto a pseudo-Riemannian manifold $(M, g)$. Then, for any $U, V \in \mathcal{V}, \bar{X}, \bar{Y} \in \mathcal{H}$ and $i = 1, 2, 3$, we have

\begin{align*}
(3.16) & \quad \bar{h}(\bar{\nabla}_{PU}(\bar{\phi}_i PV)) + v(\bar{\nabla}_{PU}(\bar{\phi}_i PV)) + h(\bar{\nabla}_{QU}(\bar{\phi}_i PV)) + v(\bar{\nabla}_{QU}(\bar{\phi}_i PV)) \\
& \quad + h(\bar{\nabla}_{PU}(\psi_i QV)) + \bar{T}_{PU}(\psi_i QV) + \bar{T}_{PU}(\omega_i QV) + v(\bar{\nabla}_{PU}(\omega_i QV)) \\
& \quad + h(\bar{\nabla}_{QU}(\psi_i QV)) + \bar{T}_{QU}(\psi_i QV) + h(\bar{\nabla}_{QU}(\omega_i QV)) + \bar{T}_{QU}(\omega_i QV) \\
& \quad = (\bar{\nabla}_U \bar{\phi}_i)V + t_i(\bar{T}_U V) + n_i(\bar{T}_U V) + \bar{\phi}_i(P(v\bar{\nabla}_U V)) + \psi_i(Q(v\bar{\nabla}_U V)) + \omega_i(Q(v\bar{\nabla}_U V)),
\end{align*}

\begin{align*}
(3.17) & \quad \bar{h}(\bar{\nabla}_X(t_i \bar{Y})) + \bar{A}_X(t_i \bar{Y}) + \bar{A}_X(n_i \bar{Y}) + v(\bar{\nabla}_X(n_i \bar{Y})) \\
& \quad = (\bar{\nabla}_X \bar{\phi}_i)Y + t_i(h\bar{\nabla}_X \bar{Y}) + n_i(h\bar{\nabla}_X \bar{Y}) + \bar{\phi}_i(P(\bar{A}_X \bar{Y})) + \psi_i(Q(\bar{A}_X \bar{Y})) + \omega_i(Q(\bar{A}_X \bar{Y})),
\end{align*}

\begin{align*}
(3.18) & \quad \bar{A}_X(\bar{\phi}_i PU) + v(\bar{\nabla}_X(\bar{\phi}_i PU)) + h(\bar{\nabla}_X(\psi_i QU)) \\
& \quad + A_X(\psi_i QU) + A_X(\omega_i QU) + v(\bar{\nabla}_X(\omega_i QV)) \\
& \quad = (\bar{\nabla}_X \bar{\phi}_i)U + t_i(A_X U) + n_i(A_X U) + \bar{\phi}_i(P(v\bar{\nabla}_X U)) + \psi_i(Q(v\bar{\nabla}_X U)) + \omega_i(Q(v\bar{\nabla}_X U)),
\end{align*}

\begin{align*}
(3.19) & \quad \bar{h}(\bar{\nabla}_U(t_i \bar{X})) + \bar{T}_U(t_i \bar{X}) + \bar{T}_U(n_i \bar{X}) + v(\bar{\nabla}_U(n_i \bar{X})) \\
& \quad = (\bar{\nabla}_U \bar{\phi}_i)X + t_i(h\bar{\nabla}_U \bar{X}) + n_i(h\bar{\nabla}_U \bar{X}) + \bar{\phi}_i(P(\bar{T}_U \bar{X})) + \psi_i(Q(\bar{T}_U \bar{X})) + \omega_i(Q(\bar{T}_U \bar{X})).
\end{align*}
Proof. For $U, V \in \mathcal{V}$, we have

\[
(\nabla_U \bar{\phi}_i)V + \bar{\phi}_i(\nabla_U V) = \nabla_{PU+QU}(\bar{\phi}_i(PV + QV)),
\]

which gives equation (3.16). Similarly, we can obtain other equations. \qed

Lemma 3.9. Let $f : \tilde{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite cosymplectic 3-structure manifold $(\tilde{M}, \bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g})$ onto a pseudo-Riemannian manifold $(M, g)$. Then, for any $U, V \in \mathcal{V}$, $Y \in \mathcal{H}$ and $\bar{\phi}_i, \bar{\xi}_i, \bar{\eta}_i, \bar{g}$.

(3.20) \[ T_{PU}(\bar{\phi}_i PV) + h(\nabla_{QU}(\bar{\phi}_i PV)) + h(\nabla_{PU}(\bar{\psi}_i QV)) \]
\[ + T_{PU}(\omega_i QV) + h(\nabla_{QU}(\bar{\psi}_i QV)) + h(\nabla_{QU}(\omega_i QV)) \]
\[ = t_i(T_U V) + \bar{\phi}_i(Q(v \nabla_U V)); \]

(3.21) \[ v(\nabla_{PU}(\bar{\phi}_i PV)) + v(\nabla_{QU}(\bar{\phi}_i PV)) + T_{PU}(\bar{\psi}_i QV) \]
\[ + v(\nabla_{PU}(\omega_i QV)) + T_{QU}(\bar{\psi}_i QV) + T_{QU}(\omega_i QV) \]
\[ = n_i(T_U V) + \bar{\phi}_i(P(v \nabla_U V) + \bar{\phi}_i(Q(v \nabla_U V)). \]

(3.22) \[ h(\nabla_{A}(t_i Y)) + A_{A}(n_i Y) = t_i(h(\nabla_{A} Y) + \bar{\phi}_i(Q(A_{A} Y)); \]

(3.23) \[ A_{A}(t_i Y) + v(\nabla_{A}(n_i Y)) = n_i(h(\nabla_{A} Y) + \omega_i(Q(A_{A} Y)) + \bar{\phi}_i(P(A_{A} Y). \]

(3.24) \[ A_{A}(\bar{\phi}_i PU) + h(\nabla_{A}(\bar{\psi}_i QU)) + A_{A}(\omega_i QU) = t_i(A_{A} U) \]
\[ + \bar{\phi}_i(Q(v \nabla_{A} U)); \]

(3.25) \[ v(\nabla_{A}(\bar{\phi}_i PU)) + A_{A}(\bar{\psi}_i QU) + v(\nabla_{A}(\omega_i QU)) \]
\[ = n_i(A_{A} U) + \bar{\phi}_i(P(v \nabla_{A} U) + \omega_i(Q(v \nabla_{A} U). \]

(3.26) \[ h(\nabla_{A}(t_i X)) + T_{A}(n_i X) = t_i(h(\nabla_{A} X) + \bar{\phi}_i(Q(T_{A} X)); \]

(3.27) \[ T_{A}(t_i X) + v(\nabla_{A}(n_i X)) = n_i(h(\nabla_{A} X) + \bar{\phi}_i(P(T_{A} X) + \omega_i(Q(T_{A} X). \]

Proof. For $U, V \in \mathcal{V}$, we have

\[
(\nabla_U \bar{\phi}_i)V + \bar{\phi}_i(\nabla_U V) = \nabla_{PU+QU}(\bar{\phi}_i(PV + QV)),
\]

which gives

\[
t_i(T_U V) + n_i(T_U V) + \bar{\phi}_i(P(v \nabla_U V) + \bar{\phi}_i(Q(v \nabla_U V)) + \omega_i(Q(v \nabla_U V))
\]
\[ = T_{PU}(\bar{\phi}_i PV) + h(\nabla_{QU}(\bar{\phi}_i PV)) + h(\nabla_{PU}(\bar{\psi}_i QV)) \]
\[ + T_{PU}(\omega_i QV) + h(\nabla_{QU}(\bar{\psi}_i QV)) + h(\nabla_{QU}(\omega_i QV) \]
\[ + v(\nabla_{PU}(\bar{\phi}_i PV)) + v(\nabla_{QU}(\bar{\phi}_i PV)) + T_{PU}(\bar{\psi}_i QV) \]
\[ + v(\nabla_{PU}(\omega_i QV)) + T_{QU}(\bar{\psi}_i QV) + T_{QU}(\omega_i QV). \]
Let $U; V$ be a semi-slant pseudo-Riemannian submersion from an indefinite Sasakian 3-structure manifold $(\tilde{M}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ onto a pseudo-Riemannian manifold $(M, g)$. Then, for any $U, V \in \mathcal{V} X, Y \in \mathcal{H}$ and $i = 1, 2, 3$, we have

\begin{equation}
\begin{aligned}
\tilde{g}(U, V)\tilde{\xi}_i + t_i(\mathcal{T}_U V) + n_i(h\tilde{\nabla}_V V) + \tilde{\phi}_i P(v(\tilde{\nabla}_V V)) \\
+ \psi_i Q(v\tilde{\nabla}_V V) + \omega_i Q(v\tilde{\nabla}_V V) \\
= h(\nabla_{PU}(\tilde{\phi}_i PV)) + v(\nabla_{PU}(\tilde{\phi}_i PV)) \\
+ h(\nabla_{QU}(\tilde{\phi}_i PV)) + v(\nabla_{QU}(\tilde{\phi}_i PV)) \\
+ h(\nabla_{PU}(\psi_i QV)) + \mathcal{T}_{PU}(\psi_i QV) + \mathcal{T}_{PU}(\omega_i QV) + v(\tilde{\nabla}_{PU}(\omega_i QV)) \\
+ h(\nabla_{QU}(\psi_i QV)) + \mathcal{T}_{QU}(\psi_i QV) + h(\tilde{\nabla}_{QU}(\omega_i QV)) + \mathcal{T}_{QU}(\omega_i QV),
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
h(\tilde{\nabla}_X(t_i Y)) + A_X(t_i Y) + A_X(n_i Y) + v(\tilde{\nabla}_X(n_i Y)) \\
= \tilde{g}(X, Y)\tilde{\xi}_i - \varepsilon \tilde{\eta}_i (Y) X + t_i(h\tilde{\nabla}_X Y) + n_i(h\tilde{\nabla}_X Y) \\
+ \tilde{\phi}_i P(A_X Y) + \psi_i Q(A_X Y) + \omega_i Q(A_X Y),
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
A_X(\tilde{\phi}_i PU) + v(\tilde{\nabla}_X(\tilde{\phi}_i PU)) + h(\tilde{\nabla}_X(\psi_i QU)) \\
+ A_X(\psi_i QU) + A_X(\omega_i QU) + v(\tilde{\nabla}_X(\omega_i QU)) \\
= \tilde{g}(X, U)\tilde{\xi}_i + t_i(A_X U) + n_i(A_X U) \\
+ \tilde{\phi}_i P(v\tilde{\nabla}_X U) + \psi_i Q(v\tilde{\nabla}_X U) + \omega_i Q(v\tilde{\nabla}_X U),
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
h(\tilde{\nabla}_V(t_i X)) + \mathcal{T}_V(t_i X) + \mathcal{T}_V(n_i X) + v(\tilde{\nabla}_V(n_i X)) \\
= \tilde{g}(U, X)\tilde{\xi}_i - \varepsilon \tilde{\eta}_i (X) U + t_i(h\tilde{\nabla}_U X) + n_i(h\tilde{\nabla}_U X) \\
+ \tilde{\phi}_i P(\mathcal{T}_U X) + \psi_i Q(\mathcal{T}_U X) + \omega_i Q(v\tilde{\nabla}_U X).
\end{aligned}
\end{equation}

**Theorem 3.11.** Let $f: \tilde{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite cosymplectic 3-structure manifold $(\tilde{M}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ onto a pseudo-Riemannian manifold $(M, g)$. Then, the fibres of $f$ are totally geodesic if and only if

\begin{equation}
\tilde{\nabla}_U(\tilde{\phi}_i V) = \tilde{\phi}_i(v\tilde{\nabla}_U V),
\end{equation}

for any $U, V \in \mathcal{V}$ and $i = 1, 2, 3$.

**Proof.** Let $U, V \in \mathcal{V}$. For $i = 1, 2, 3$, using equation (3.1), we have

\[
\tilde{\nabla}_U(\tilde{\phi}_i V) = (\tilde{\nabla}_U \tilde{\phi}_i)V + \tilde{\phi}_i(\tilde{\nabla}_{PU} PV + \tilde{\nabla}_{PU} QV + \tilde{\nabla}_{QU} PV + \tilde{\nabla}_{QU} QV).
\]
Again, using equations (2.14) and (3.1), above equation gives
\[ \nabla_U(\phi_i V) = (\nabla_U\phi_i)V + \phi_i(T_U V) + \phi_i(v\nabla_U V). \]
As the manifold \((\widetilde{M}, \widetilde{\phi}_1, \widetilde{\xi}_i, \widetilde{\eta}_i, \widetilde{g})\) is cosymplectic, from above equation, we have
\[ \nabla_U(\phi_i V) = \phi_i(T_U V) + \phi_i(v\nabla_U V). \]
Now, fibres are totally geodesic if and only if \(T_U V = 0\). So the proof follows from equation (3.33).

**Theorem 3.12.** Let \(f : \widetilde{M} \to M\) be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold \((\widetilde{M}, \widetilde{\phi}_1, \widetilde{\xi}_i, \widetilde{\eta}_i, \widetilde{g})\) onto a pseudo-Riemannian manifold \((M, g)\). Then, the horizontal distribution \(H\) defines a totally geodesic foliation if and only if
\[ (3.34) \quad \tilde{g}(i(h\nabla_X \tilde{Y}), \psi_i Q) + \psi_i QU \]
\[ + \tilde{g}(n_i(h\nabla_X \tilde{Y}), \phi_i PU) + \tilde{g}(n_i(h\nabla_X \tilde{Y}), \phi_i PU) \]
\[ + \tilde{g}(n_i(h\nabla_X \tilde{Y}), \psi_i QU) + \tilde{g}(n_i(h\nabla_X \tilde{Y}), \psi_i QU) \]
\[ + \tilde{g}(\phi_i P(v\nabla_X \tilde{Y}), \phi_i PU) + \tilde{g}(\phi_i P(v\nabla_X \tilde{Y}), \phi_i PU) \]
\[ + \tilde{g}(\phi_i P(v\nabla_X \tilde{Y}), \psi_i QU) + \tilde{g}(\phi_i P(v\nabla_X \tilde{Y}), \psi_i QU) \]
\[ + \tilde{g}(\phi_i P(v\nabla_X \tilde{Y}), \psi_i QU) + \tilde{g}(\phi_i P(v\nabla_X \tilde{Y}), \psi_i QU) \]
\[ + \tilde{g}(\phi_i P(v\nabla_X \tilde{Y}), \psi_i QU) + \tilde{g}(\phi_i P(v\nabla_X \tilde{Y}), \psi_i QU) \]
\[ = \tilde{g}(t_i(h\nabla_X \tilde{Y}), \psi_i QU) + \tilde{g}(n_i(h\nabla_X \tilde{Y}), \phi_i PU) \]
\[ + \tilde{g}(n_i(h\nabla_X \tilde{Y}), \phi_i PU) + \tilde{g}(n_i(h\nabla_X \tilde{Y}), \phi_i PU) \]
\[ + \tilde{g}(\phi_i P(v\nabla_X \tilde{Y}), \phi_i PU) + \tilde{g}(\psi_i Q(v\nabla_X \tilde{Y}), \psi_i QU) \]
\[ + \tilde{g}(\phi_i P(v\nabla_X \tilde{Y}), \phi_i PU) + \tilde{g}(\psi_i Q(v\nabla_X \tilde{Y}), \phi_i PU) \]
\[ + \tilde{g}(\psi_i Q(v\nabla_X \tilde{Y}), \phi_i PU) + \tilde{g}(\omega Q(v\nabla_X \tilde{Y}), \omega_i QU) \]
\[ + \tilde{g}(\psi_i Q(v\nabla_X \tilde{Y}), \phi_i PU) + \tilde{g}(\omega Q(v\nabla_X \tilde{Y}), \omega_i QU) \]
\[ = \tilde{g}(t_i(h\nabla_X \tilde{Y}), \psi_i QU) + \tilde{g}(n_i(h\nabla_X \tilde{Y}), \phi_i PU) \]
\[ + \tilde{g}(n_i(h\nabla_X \tilde{Y}), \phi_i PU) + \tilde{g}(\psi_i Q(v\nabla_X \tilde{Y}), \psi_i QU) \]
\[ + \tilde{g}(\phi_i P(v\nabla_X \tilde{Y}), \phi_i PU) + \tilde{g}(\omega Q(v\nabla_X \tilde{Y}), \omega_i QU) \]
which implies \(\nabla_X \tilde{Y} \in H\) if and only if right side of above equation vanishes. So the proof follows from above equation. \(\square\)
Corollary 3.13. Let \( \tilde{M} \rightarrow M \) be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold \((\tilde{M}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) onto a pseudo-Riemannian manifold \((M, g)\). Then, following statements are equivalent:

(a) The horizontal distribution \( \mathcal{H} \) defines a totally geodesic foliation,

(b) \[
\bar{g}(t_i(h \bar{\nabla}_X \bar{Y}) + \psi_i Q(v \bar{\nabla}_X \bar{Y}), \psi_i QU) \\
+ \bar{g}(n_i(h \bar{\nabla}_X \bar{Y}) + \tilde{\phi}_i P(v \bar{\nabla}_X Y) + \omega_i Q(v \bar{\nabla}_X \bar{Y}), \tilde{\phi}_i PU) \\
+ \bar{g}(n_i(h \bar{\nabla}_X \bar{Y}) + \phi_i P(v \bar{\nabla}_X Y) + \omega_i Q(v \bar{\nabla}_X \bar{Y}), \omega_i QU) = 0,
\]

(c) \[
\bar{g}(\bar{\nabla}_X \bar{Y}, t_i \psi_i QU + \psi_i Q \omega_i QU) = 0,
\]

(d) \[
\bar{g}(n_i t_i(h \bar{\nabla}_X \bar{Y}) + \bar{\phi}_i P n_i(h \bar{\nabla}_X \bar{Y}) + \omega_i Q n_i(h \bar{\nabla}_X \bar{Y}) + \tilde{\phi}_i^2 P(v \bar{\nabla}_X \bar{Y}) \\
+ n_i \psi_i Q(v \bar{\nabla}_X \bar{Y}) + \bar{\phi}_i \psi_i Q(v \bar{\nabla}_X \bar{Y}) + \omega_i Q \psi_i Q(v \bar{\nabla}_X \bar{Y}), \ U) = 0,
\]

for all \( \bar{X}, \bar{Y} \in \mathcal{H}, \ U \in \mathcal{V} \) and \( i = 1, 2, 3 \).

Proof. In theorem (3.12), we have proved \( (a) \iff (b) \). Similarly, we can prove \( (b) \iff (c) \iff (d) \) and \( (d) \iff (a) \). \( \square \)

Theorem 3.14. Let \( f : \tilde{M} \rightarrow M \) be a semi-slant pseudo-Riemannian submersion from an indefinite cosymplectic 3-structure manifold \((\tilde{M}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) onto a pseudo-Riemannian manifold \((M, g)\). Then, the horizontal distribution \( \mathcal{H} \) defines a totally geodesic foliation if and only if

\[
\bar{g}(h \bar{\nabla}_X (t_i \bar{Y}) + h \bar{\nabla}_X (n_i \bar{Y}), \psi_i QU) + \bar{g}(v \bar{\nabla}_X (t_i \bar{Y}), \tilde{\phi}_i PU + \omega_i QU) \\
+ \bar{g}(v \bar{\nabla}_X (n_i \bar{Y}), \tilde{\phi}_i PU + \omega_i QU) = 0,
\]

for all \( \bar{X}, \bar{Y} \in \mathcal{H} \) and \( i = 1, 2, 3 \).

Proof. Let \( \bar{X}, \bar{Y} \in \mathcal{H}, \ U \in \mathcal{V} \). Then, we have

\[
\bar{g}(\bar{\nabla}_X \bar{Y}, \ U) = \bar{g}(\bar{\nabla}_X (\tilde{\phi}_i \bar{Y}), \tilde{\phi}_i U).
\]

By using equations (3.2) and (3.3), above equation implies

\[
\bar{g}(\bar{\nabla}_X \bar{Y}, \ U) = \bar{g}(h \bar{\nabla}_X (t_i \bar{Y}) + h \bar{\nabla}_X (n_i \bar{Y}), \psi_i QU) \\
+ \bar{g}(v \bar{\nabla}_X (t_i \bar{Y}), \tilde{\phi}_i PU + \omega_i QU) + \bar{g}(v \bar{\nabla}_X (n_i \bar{Y}), \tilde{\phi}_i PU + \omega_i QU).
\]

The distribution \( \mathcal{H} \) defines a totally geodesic foliation if and only if \( \bar{\nabla}_X \bar{Y} \in \mathcal{H} \).

So the proof follows from equation (3.36). \( \square \)

Theorem 3.15. Let \( f : \tilde{M} \rightarrow M \) be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold \((\tilde{M}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) onto a pseudo-Riemannian manifold \((M, g)\). Then, the vertical distribution \( \mathcal{V} \) defines a totally geodesic foliation if and only if
(3.37) \( \bar{g}(t_i(h\nabla_U V) + \psi_i Q(v\nabla_U V), t_i\bar{X}) + \bar{g}(n_i(h\nabla_U V) + \tilde{\phi}_i P(v\nabla_U V) + \omega_i Q(v\nabla_U V), n_i\bar{X}) = 0, \)

for all \( U, V \in \mathcal{V}, \bar{X} \in \mathcal{H} \) and \( i = 1, 2, 3. \)

**Proof.** Let \( U, V \in \mathcal{V}, \bar{X} \in \mathcal{H}. \) Using equation (2.5), we have

\[
\bar{g}(\nabla_U V, \bar{X}) = \bar{g}(\tilde{\phi}_i(\nabla_U V), \tilde{\phi}_i\bar{X}) + \tilde{\epsilon}_i \tilde{\eta}_i(\nabla_U V)\tilde{\eta}_i(\bar{X}).
\]

By splitting vertical and horizontal components in above equation and using equations (3.2) and (3.3), we get

(3.38) \( \bar{g}(\nabla_U V, \bar{X}) = \bar{g}(t_i(h\nabla_U V) + \psi_i Q(v\nabla_U V), t_i\bar{X}) + \bar{g}(n_i(h\nabla_U V) + \tilde{\phi}_i P(v\nabla_U V) + \omega_i Q(v\nabla_U V), n_i\bar{X}) + \tilde{\epsilon}_i \tilde{\eta}_i(\nabla_U V)\tilde{\eta}_i(\bar{X}). \)

The vertical distribution \( \mathcal{V} \) defines a totally geodesic foliation if and only if \( \nabla_U V \in \mathcal{V}. \) This completes the proof. \( \Box \)

**Theorem 3.16.** Let \( f : \bar{M} \to M \) be a semi-slant pseudo-Riemannian submersion from an indefinite cosymplectic 3-structure manifold \( (\bar{M}, \tilde{\phi}_i, \tilde{\xi}_i, \tilde{\eta}_i, \bar{g}) \) onto a pseudo-Riemannian manifold \( (M, g) \). Then, the vertical distribution \( \mathcal{V} \) defines a totally geodesic foliation if and only if

(3.39) \[ \bar{g}(h\nabla_U (\tilde{\phi}_i PV) + h\nabla_U (\psi_i QV) + h\nabla_U (\omega_i QV), t_i\bar{X}) + \bar{g}(n_i(h\nabla_U (\tilde{\phi}_i PV) + n_i\bar{X}) + \tilde{\epsilon}_i \tilde{\eta}_i(\nabla_U V)\tilde{\eta}_i(\bar{X}). \]

for all \( U, V \in \mathcal{V}, \bar{X} \in \mathcal{H}, i = 1, 2, 3. \)

**Proof.** Let \( U, V \in \mathcal{V} \) and \( \bar{X} \in \mathcal{H}. \) Using equation (2.5), we have

\[ \bar{g}(\nabla_U V, \bar{X}) = \bar{g}(\tilde{\phi}_i(\nabla_U V), \tilde{\phi}_i\bar{X}) + \tilde{\epsilon}_i \tilde{\eta}_i(\nabla_U V)\tilde{\eta}_i(\bar{X}). \]

As the manifold \( \bar{M} \) is cosymplectic, we have \( (\nabla_U \tilde{\phi}_i)V = 0. \) So, by using equations (3.2) and (3.3), we get

(3.40) \[ \bar{g}(\nabla_U V, \bar{X}) = \bar{g}(h\nabla_U (\tilde{\phi}_i PV) + h\nabla_U (\psi_i QV) + h\nabla_U (\omega_i QV), t_i\bar{X}) + \bar{g}(n_i(h\nabla_U (\tilde{\phi}_i PV) + n_i\bar{X}) + \tilde{\epsilon}_i \tilde{\eta}_i(\nabla_U V)\tilde{\eta}_i(\bar{X}). \]

The equation (3.40) implies that \( \nabla_U V \in \mathcal{V} \) if and only if equation (3.39) is satisfied. \( \Box \)

Now, using similar steps as in theorem 22 and theorem 24 of [21], we have

**Theorem 3.17.** Let \( f : \bar{M} \to M \) be a semi-slant pseudo-Riemannian submersion from indefinite cosymplectic 3-structure manifold \( (\bar{M}, \tilde{\phi}_i, \tilde{\xi}_i, \tilde{\eta}_i, \bar{g}) \) onto a pseudo-Riemannian manifold \( (M, g) \). Then, the submersion \( f \) is an affine map on \( \mathcal{H}; i = 1, 2, 3. \)

**Theorem 3.18.** Let \( f : \bar{M} \to M \) be a semi-slant pseudo-Riemannian submersion from indefinite cosymplectic 3-structure manifold \( (\bar{M}, \tilde{\phi}_i, \tilde{\xi}_i, \tilde{\eta}_i, \bar{g}) \) onto a pseudo-Riemannian manifold \( (M, g) \). Then, the submersion \( f \) is an affine map if and only if \( h(\nabla_E hF) + A_{hE} v F + T_{vE} v F \) is \( f \)-related to \( \nabla_X Y, \) for any \( E, F \in \Gamma(TM); i = 1, 2, 3. \)
Theorem 3.19. Let $f: \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from an indefinite almost contact 3-structure manifold $(\bar{M}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ onto a pseudo-Riemannian manifold $(M, g)$. Then, the submersion map $f$ is totally geodesic if and only if

\begin{equation}
\mathcal{T}_U V + A_X V + h \nabla_U \bar{Y} = 0,
\end{equation}

for any $U, V \in \mathcal{V}, \bar{X}, \bar{Y} \in \mathcal{H}$ and \(\forall i = 1, 2, 3\).

Proof. Let $E = \bar{X} + U$, $F = \bar{Y} + V \in \Gamma(T\bar{M})$. In view of equation (2.29) and theorem (3.17), by splitting $E, F$ in horizontal and vertical components, we have

\begin{equation}
(\nabla f_*)(E, F) = (\nabla f_*)(U, V) + (\nabla f_*)(\bar{X}, V) + (\nabla f_*)(U, \bar{Y})
= -f_*(h(\nabla_U V + \nabla_\bar{X} V + \nabla_U \bar{Y})),
\end{equation}

which gives

\begin{equation}
(\nabla f_*)(E, F) = -f_*(\mathcal{T}_U V + A_X V + h \nabla_U \bar{Y}).
\end{equation}

The submersion map $f$ is totally geodesic if and only if $\nabla f_*$ is 0. So the proof follows from equation (3.42). \(\square\)

Theorem 3.20. Let $f: \bar{M} \to M$ be a semi-slant pseudo-Riemannian submersion from indefinite almost contact 3-structure manifold $(\bar{M}^{4m+3}, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ onto a pseudo-Riemannian manifold $(M^n, g)$. If the fibres $f^{-1}(q)$ of $f$ over $q \in M$ are totally geodesic, then $f$ is a harmonic map.

Proof. The tension field $\tau(f)$ of the map $f: \bar{M} \to M$ is defined as

\begin{equation}
\tau(f) = \text{trace}(\nabla f_*).
\end{equation}

Let \(\{e_1, e_2, \ldots, e_{4m+3-n}, e_{4m+3-n+1} = \bar{e}_1, \bar{e}_2, \ldots e_{4m+3} = \bar{e}_n\}\) be an orthonormal basis of $\Gamma(T\bar{M})$, where \(\{e_1, e_2, \ldots, e_{4m+3-n}\}\) is an orthonormal basis of $\mathcal{V}$ and \(\{\bar{e}_1, \bar{e}_2, \ldots, e_{4m+3} = \bar{e}_n\}\) is an orthonormal basis of $\mathcal{H}$. Then, we have

\begin{equation}
\tau(f) = \sum_{i=1}^{4m+3-n} \bar{g}(e_i, e_i)(\nabla f_*)(e_i, e_i) + \sum_{j=1}^{n} \bar{g}(\bar{e}_j, \bar{e}_j)(\nabla f_*)(\bar{e}_j, \bar{e}_j).
\end{equation}

For any vertical vector fields $U, V \in \mathcal{V}$, using equation (2.14), we have

\begin{equation}
(\nabla f_*)(U, V) = (\nabla^f_* f_* V) \circ f - f_*(h \nabla_U V) = -f_*(h \nabla_U V),
\end{equation}

where $\nabla f$ is the pullback connection of $\nabla$ with respect to $f$. For any horizontal vector fields $\bar{X}, \bar{Y} \in \mathcal{H}$, which are $f$-related to $X, Y \in \Gamma(TM)$ respectively, Lemma 2.1 and Theorem 3.17 imply

\begin{equation}
(\bar{\nabla} f_*)(\bar{X}, \bar{Y}) = (\nabla^f_* f_* \bar{Y}) \circ f - f_*(h \nabla_{\bar{X}} \bar{Y})
= (\nabla f_* \bar{X} f_* \bar{Y}) \circ f - f_*(h \nabla_{\bar{X}} \bar{Y}) = 0.
\end{equation}
In view of equations (3.44), (3.45), (3.46) and Theorem 3.18, we get

\[
\tau(f) = -\sum_{i=1}^{4m+3-n} \bar{g}(e_{i}, e_{i}) f_{*}(T_{e_{i}}e_{i}).
\]

(3.47)

Now, if the fibres \( f^{-1}(q) \) of \( f \) over \( q \in M \) are totally geodesic, then \( T = 0 \). So the proof of the theorem follows from equation (3.47).

\[\square\]

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S. S. SHUKLA,
DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF ALLAHABAD,
ALLAHABAD, PIN-211002,
INDIA
E-mail address: sshukla.au@rediffmail.com

UMA SHANKAR VERMA,
DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF ALLAHABAD,
ALLAHABAD, PIN-211002,
INDIA
E-mail address: umashankarverma7@gmail.com