MULTIPLY WARPED PRODUCT ON QUASI-EINSTEIN MANIFOLD WITH A SEMI-SYMMETRIC NON-METRIC CONNECTION

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Abstract. In this paper, we have studied warped products and multiply warped product on quasi-Einstein manifold with semi-symmetric non-metric connection. Then we have applied our results to generalized Robertson-Walker space times with a semi-symmetric non-metric connection.

1. Introduction

Let \((M^n, g), (n > 2)\) be a Riemannian manifold and \(U_S = \{x \in M : S \neq \frac{r}{n}g\text{ at } x\}\), then the manifold \((M^n, g)\) is said to be quasi-Einstein manifold [4, 6] if on \(U_S \subset M\), we have

\[
S - \alpha g = \beta A \otimes A,
\]

where \(A\) is a 1-form on \(U_S\) and \(\alpha\) and \(\beta\) some functions on \(U_S\). It is clear that the 1-form \(A\) as well as the function \(\beta\) are nonzero at every point on \(U_S\). The scalars \(\alpha, \beta\) are known as the associated scalars of the manifold. Also, the 1-form \(A\) is called the associated 1-form of the manifold defined by \(g(X, \rho) = A(X)\) for any vector field \(X, \rho\) being a unit vector field, called the generator of the manifold. Such an \(n\)-dimensional quasi-Einstein manifold is denoted by \((QE)_n\).

Let \((B, g_B)\) and \((F, g_F)\) be two Riemannian manifolds and \(f > 0\) is a differential function on \(B\). Consider the product manifold \(B \times F\) with its projections \(\pi: B \times F \to B\) and \(\sigma: B \times F \to F\). The warped product \(B \times_f F\) is the manifold \(B \times F\) with the Riemannian structure such that \(||X||^2 = ||\pi^*(X)||^2 + f^2(\pi(p))||\sigma^*(X)||^2\), for any vector field \(X\) on \(M\). Thus we have \(g_M = g_B + f^2 g_F\) holds on \(M\). Here \(B\) is called the base of \(M\) and \(F\) the fiber. The function \(f\) is called the warping function of the warped product [9].

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The concept of warped products was first introduced by Bishop and O’Neil [3] to construct examples of Riemannian manifold with negative curvature.

Now, we can generalize warped products to multiply warped products. A multiply warped product is the product manifold $M = B \times_{b_1} F_1 \times_{b_2} F_2 \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus b_3^2 g_{F_3} \cdots \oplus b_m^2 g_{F_m}$, where each $i \in \{1, 2, \ldots, m\}$, $b_i: B \to (0, \infty)$ is smooth and $(F_i, g_{F_i})$ is a pseudo-Riemannian manifold. In particular, when $B = (c, d)$, the metric $g_B = -dt^2$ is negative and $(F_i, g_{F_i})$ is a Riemannian manifold. We call $M$ as the multiply generalized Robertson-Walker space-time.

A multiply twisted product $(M, g)$ is a product manifold of the form $M = B \times_{b_1} F_1 \times_{b_2} F_2 \cdots \times_{b_m} F_m$ with the metric $g = g_B \oplus b_1^2 g_{F_1} \oplus b_2^2 g_{F_2} \oplus b_3^2 g_{F_3} \cdots \oplus b_m^2 g_{F_m}$, where each $i \in \{1, 2, \ldots, m\}$, $b_i: B \times F_i \to (0, \infty)$ is smooth.

In 1924, Friedmann and Schouten was introduced the notion of a semi-symmetric linear connection on a differential manifold [5]. The idea of metric connection with torsion on Riemannian manifold has given by Hayden (1932) in [7]. In 1970, Yano [15] was introduced a systematic study of semi-symmetric metric connection on Riemannian manifold. Later K. S. Amur and S. S. Pujar [1], C. S. Bagewadi [2], Sharafuddin and Hussain (1976) [11], S. Sular, C. Özgür [12], M. Tripathi [13] have also studied semi-symmetric metric connection on Riemannian manifold. In [10], S. Sular and C. Özgür has studied warped product on semi-symmetric non-metric connection. Y. Wang has considered multiply warped product with a semi-symmetric non-metric connection, then applied the results to generalized Robertson-Walker space-time in [14].

In this paper, we have considered quasi-Einstein warped product manifolds endowed with semi-symmetric metric non-connection. First we have obtained the necessary and sufficient conditions of quasi-Einstein warped product manifold with semi-symmetric non-metric connection. Next we have established that under certain conditions Robertson-Walker space times would be converted to quasi-Einstein manifold with the above connection. Later we have shown that $(n-1)$-dimensional base is isometric to a $(n-1)$-dimensional sphere of a particular radius with respect to semi-symmetric non-metric connection. In the last section we have studied special multiply warped product with semi symmetric non-metric connection.

2. Preliminaries

Let $(M^n, g)$ be a Riemannian manifold with the Levi-civita connection $\nabla$. A linear connection $\tilde{\nabla}$ on $(M^n, g)$ is said to be semi-symmetric if its torsion tensor $T$ can be written as

$$T(X, Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X, Y],$$

satisfies the condition

$$T(X, Y) = \pi(Y)X - \pi(X)Y,$$
where \( \pi \) is an 1-form on \( M^n \) with the associated vector field \( P \) defined by 
\[ \pi(X) = g(X, P), \]
for all vector fields \( X \in \chi(M^n) \).

A connection \( \nabla \) is called semi-symmetric non-metric connection if \( \nabla g \neq 0 \).

The relation between semi-symmetric non-metric connection \( \tilde{\nabla} \) and the Levi-Civita connection \( \nabla \) of \( M^n \) and it is given by [14]
\[
(2.1) \quad \tilde{\nabla} X Y = \nabla X Y + \pi(Y)X,
\]
where \( g(X, P) = \pi(X) \).

Further, a relation between the curvature tensors \( R \) and \( \tilde{R} \) of type (1,3) of the connections \( \nabla \) and \( \tilde{\nabla} \) respectively is given by [14],
\[
(2.2) \quad \tilde{R}(X; Y) Z = R(X; Y) Z + g(Z, \nabla_X P) Y - g(Z, \nabla_Y P) X \\
+ \pi(Z)[\pi(Y)X - \pi(X)Y],
\]
for any vector field \( X, Y, Z \) on \( M^n \).

3. Generalized Robertson-Walker Space-times with a Semi-Symmetric Non-Metric Connection

In this section we have considered quasi-Einstein warped product manifolds with respect to semi-symmetric non-metric connection. Now, we have proved the following theorem.

**Theorem 3.1.** Let \((M, g)\) be a warped product \( I \times_f F \) where \( I \) is an open interval in \( \mathbb{R} \), \( \dim I = 1 \) and \( \dim F = \tilde{n} - 1 \), \( (\tilde{n} \geq 3) \). Then \((M, g)\) is a quasi-Einstein manifold with respect to semi-symmetric non-metric connection iff \( F \) is quasi-Einstein manifold for \( P \in \chi(B) \) with respect to the Levi-Civita connection or the warping function \( f \) is a constant on \( I \) for \( P \in \chi(F) \).

**Proof.** Assume that \( P \in \chi(B) \) and let \( g_I \) be the metric on \( I \). Taking \( f = e^{\frac{q^2}{4}} \) and by using the proposition use of [10] we get
\[
(3.1) \quad \tilde{S}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -\tilde{n} - 1 - [2q'' + (q')^2 - 4]g_I(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}),
\]
\[
(3.2) \quad \tilde{S}(\frac{\partial}{\partial t}, V) = 0,
\]
\[
(3.3) \quad \tilde{S}(V, W) = S^F(V, W) + e^q[\tilde{n} - 1 - q' - \tilde{n} - 3 - (q')^2]g_F(V, W),
\]
for any vector field \( V, W \) on \( F \).

Since, \( M \) is a quasi-Einstein manifold with respect to the semi-symmetric non-metric connection, we have
\[
\tilde{S}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = \alpha g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) + \beta \eta(\frac{\partial}{\partial t})\eta(\frac{\partial}{\partial t}),
\]
and
\[
\tilde{S}(V, W) = \alpha g(V, W) + \beta \eta(V)\eta(W),
\]
Then the last equations reduce to

\begin{equation}
\tilde{S}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = \alpha g_{I}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) + \beta \eta(\frac{\partial}{\partial t})\eta(\frac{\partial}{\partial t}),
\end{equation}

and

\begin{equation}
\tilde{S}(V, W) = \alpha e^{\gamma}g_{F}(V, W) + \beta \eta(V)\eta(W).
\end{equation}

Decomposing the vector field \( U \) uniquely into its components \( U_{I} \) and \( U_{F} \) on \( I \) and \( F \), respectively, then we have \( U = U_{I} + U_{F} \). Since \( \dim I = 1 \), we can take \( U_{I} = v\frac{\partial}{\partial m} \) which gives \( U = v\frac{\partial}{\partial m} + U_{F} \), where \( v \) is a function on \( M \). Then we can write

\begin{equation}
\eta(\frac{\partial}{\partial t}) = g(U, \frac{\partial}{\partial t}) = v.
\end{equation}

Using the equations (3.6), the equations (3.4), (3.5) reduce to

\begin{equation}
\tilde{S}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = \alpha + \beta v^{2},
\end{equation}

and

\begin{equation}
\tilde{S}(V, W) = \alpha e^{\gamma}g_{F}(V, W) + \beta \eta(V)\eta(W).
\end{equation}

Comparing the right hand sides of (3.1) and (3.7) we get,

\begin{equation}
\alpha + \beta v^{2} = -\frac{n - 1}{4}[2q'' + (q')^{2} - 4].
\end{equation}

Similarly comparing the right hand sides of (3.3) and (3.8) we obtain

\begin{equation}
S_{F}(V, W) = e^{\gamma}[\alpha + \frac{n - 3}{4}(q')^{2} - \frac{(n - 1)}{2}\frac{q'}{q'} - \frac{q''}{2}]g_{F}(V, W) + \beta \eta(V)\eta(W),
\end{equation}

which gives that \( F \) is a quasi-Einstein manifold with respect to the Levi-Civita connection for \( P \in \chi(B) \).

Now taking \( P \in \chi(F) \) and by use of \([10]\) we get,

\begin{equation}
\tilde{S}(\frac{\partial}{\partial t}, V) = (\tilde{n} - 1)\frac{q'}{2}\pi(V)g_{I}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})
\end{equation}

and

\begin{equation}
\tilde{S}(V, \frac{\partial}{\partial t}) = (1 - \tilde{n})\frac{q'}{2}\pi(V)g_{I}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}),
\end{equation}

for any vector field \( V \in \chi(F) \).

Since \( M \) is a quasi-Einstein manifold, we have

\begin{equation}
\tilde{S}(\frac{\partial}{\partial t}, V) = \tilde{S}(V, \frac{\partial}{\partial t}) = \alpha g(V, \frac{\partial}{\partial t}) + \beta \eta(V)\eta(\frac{\partial}{\partial t}).
\end{equation}

Now \( g(V, \frac{\partial}{\partial m}) = 0 \) as \( \frac{\partial}{\partial m} \in \chi(B) \) and \( V \in \chi(F) \).

Hence from the last equation we get

\begin{equation}
\tilde{S}(\frac{\partial}{\partial t}, V) = \tilde{S}(V, \frac{\partial}{\partial t}) = \beta \eta(V)\eta(\frac{\partial}{\partial t}).
\end{equation}
Therefore we have
\begin{equation}
\eta(V)\frac{\partial}{\partial t} = (\bar{n} - 1)\frac{q'}{2}\pi(V)g_B\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right).
\end{equation}
(3.15)
\begin{equation}
\eta(V)\frac{\partial}{\partial t} = (1 - \bar{n})\frac{q'}{2}\pi(V)g_B\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right).
\end{equation}
(3.16)

Comparing from (3.15), (3.16) we get
\[ q' = 0. \]
Hence, $q$ is constant. Therefore $f$ is constant. \hfill \Box

Now, we consider the warped product $M = B \times_f I$ with $\dim B = \bar{n} - 1$, $\dim I = 1$ ($\bar{n} \geq 3$). Under this assumption we have obtained the following theorem.

**Theorem 3.2.** Let $(M, g)$ be a warped product $B \times_f I$, where $\dim I = 1$ and $\dim B = \bar{n} - 1$ ($\bar{n} \geq 3$).

i) If $(M, g)$ is a quasi-Einstein manifold with scalars $\alpha, \beta$ respect to the semi-symmetric non-metric connection, $P \in \chi(B)$ is parallel on $B$ with respect to the Levi-Civita connection on $B$ and $f$ is a constant on $B$, then $\alpha = 0$.

ii) If $(M, g)$ is a quasi-Einstein manifold with respect to the semi-symmetric non-metric connection for $P \in \chi(F)$, then $f$ is a constant on $B$.

iii) If $f$ is a constant on $B$ and $B$ is a quasi-Einstein manifold with respect to the Levi-Civita connection for $P \in \chi(F)$, then $M$ is an quasi-Einstein manifold with respect to the semi-symmetric non-metric connection.

**Proof.** Assume that $(M, g)$ is a quasi-Einstein manifold with respect to the semi-symmetric non-metric connection. Then we write
\begin{equation}
\tilde{S}(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y).
\end{equation}
(3.17)

Decomposing the vector field $U$ uniquely into its components $U_B$ and $U_I$ on $B$ and $I$, respectively, then we have
\begin{equation}
U = U_B + U_I.
\end{equation}
(3.18)

Since $\dim I = 1$, we can take $U_I = v\frac{\partial}{\partial t}$ which gives $U = v\frac{\partial}{\partial t} + U_B$, where $v$ is a function on $M$. From (3.17), (3.18) and from the proposition of [10], we have,
\begin{equation}
\tilde{S}^B(X, Y) = \alpha g_B(X, Y) + \beta g_B(X, U_B)g_B(Y, U_B) - \frac{H^I(X, Y)}{f} - g(Y, \nabla_X P) + \pi(X)\pi(Y).
\end{equation}
(3.19)

By contraction over $X$ and $Y$ and we get
\begin{equation}
\tilde{r}^B = \alpha(\bar{n} - 1) + \beta g_B(U_B, U_B) - \frac{\Delta f}{f} + \pi(P) - \sum_{i=1}^{\bar{n}-1} g(e_i, \nabla e_i P).
\end{equation}
(3.20)
Also from (3.17) we have

(3.21) \[ \tilde{r}^M = \alpha \tilde{n} + \beta g_B(U_B, U_B), \]

So, by the use of (3.21) in (3.20) we get

(3.22) \[ \tilde{r}^B = \tilde{r}^M - \alpha - \frac{\Delta f}{f} + \pi(P) - \sum_{i=1}^{\tilde{n}-1} g(e_i, \nabla_{e_i} P) \]

Also from the proposition of [10] we get

\[ \tilde{r}^M = \tilde{r}^B - 2 \frac{\Delta f}{f} - \pi(P) + 2 \sum_{i=1}^{\tilde{n}-1} g(e_i, \nabla_{e_i} P) + (\tilde{n} - 1) \frac{Pf}{f}. \]

Therefore, from above two relation we get

\[ \alpha + \frac{\Delta f}{f} - \pi(P) + \sum_{i=1}^{\tilde{n}-1} g(e_i, \nabla_{e_i} P) = -2 \frac{\Delta f}{f} - \pi(P) + \sum_{i=1}^{\tilde{n}-1} g(e_i, \nabla_{e_i} P) + (\tilde{n} - 1) \frac{Pf}{f}. \]

Since \( P \in \chi(B) \) is parallel and \( f \) is a constant on \( B \), then we get \( \alpha = 0 \).

ii) Let \( P \in \chi(F) \). By the use of the proposition of [10] we get,

\[ \tilde{S}(X, V) = (\tilde{n} - 1) \pi(V) \frac{Xf}{f} \]

and

\[ \tilde{S}(V, X) = (1 - \tilde{n}) \pi(V) \frac{Xf}{f}, \]

for any vector field \( X \in \chi(B) \) and \( V \in \chi(F) \). Since \( F = I \), then taking \( V = P \) we have

(3.23) \[ \tilde{S}(X, P) = (\tilde{n} - 1) \pi(P) \frac{Xf}{f}, \]

and

(3.24) \[ \tilde{S}(P, X) = (1 - \tilde{n}) \pi(P) \frac{Xf}{f}. \]

Since \( M \) is a quasi-Einstein manifold, we have

\[ \tilde{S}(X, P) = \tilde{S}(P, X) = \alpha g(P, X) + \beta \eta(P) \eta(X). \]

Again we have \( g(P, X) = 0 \) for \( X \in \chi(B) \) and \( P \in \chi(F) \). Hence, we have \( Xf = 0 \). This implies that \( f \) is constant.

iii) Assume that \( B \) is a quasi-Einstein manifold with respect to Levi-Civita connection. Then we have

(3.25) \[ S^B(X, Y) = \alpha g(X, Y) + \beta \eta(X) \eta(Y), \]

for any vector field \( X, Y \) tangent to \( B \).

\[ \tilde{S}^M(X, Y) = S^B(X, Y) + \frac{H^f(X, Y)}{f}, \]
for any vector field $P \in \chi(F)$. Since $f$ is a constant, then $H^f(X, Y) = 0$ for all $X, Y \in \chi(F)$.

The above equation reduces to

\[ S^M(X, Y) = S^B(X, Y). \]

By the use of (3.25) in (3.26) we get

\[ S^M(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y). \]

which shows that $M$ is a quasi-Einstein manifold with respect to the semi-symmetric non-metric connection.

**Theorem 3.3.** Let $(M, g)$ be a warped product $I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F$, $P = \frac{\partial}{\partial t}$, $\dim F = l$. Then $(M, g)$ is a quasi-Einstein manifold with respect to semi-symmetric non-metric connection $\tilde{\nabla}$ with constant associated scalars $\alpha$ and $\beta$, if and only if the following conditions are satisfied.

i) $(F, g_F)$ is quasi-Einstein manifold with scalar $\alpha_F, \beta_F$.

ii) \( l(1 - \frac{f''}{f}) = \alpha - \nu^2\beta, \)

iii) \( \alpha_F + (1 - l)f'^2 - \alpha f^2 + f''f + lf'f = 0 \) and $\beta = \beta_F$.

**Proof.** By the proposition of [10] we have

\[ \tilde{S}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = l(\frac{f''}{f} - 1), \]

\[ \tilde{S}(\frac{\partial}{\partial t}, V) = \tilde{S}(V, \frac{\partial}{\partial t}) = 0, \]

\[ \tilde{S}(V, W) = S^F(V, W) + g_F(V, W)\{f f'' - (l - 1)f'^2 + lf'f\}. \]

Then by the quasi-Einstein condition, we get the theorem 3.3. \hfill \Box

From the theorem 3.3. Putting $\dim F = 1$ we get the following corollary.

**Corollary 3.1.** Let $(M, g)$ be a warped product $I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F$, $P = \frac{\partial}{\partial t}$, $\dim F = 1$. Then $(M, g)$ is a quasi-Einstein manifold with respect to semi-symmetric non-metric connection if and only if

\[ f'' + (\alpha - \nu^2\beta - 1)f = 0. \]

By the corollary 3.1. and elementary methods for ordinary differential equations we get

**Theorem 3.4.** Let $(M, g)$ be a warped product $I \times_f F$ with the metric tensor $-dt^2 + f(t)^2 g_F$, $P = \frac{\partial}{\partial t}$, $\dim F = 1$. Then $(M, g)$ is a quasi-Einstein manifold with respect to semi-symmetric non-metric connection if and only if

i) \( \text{when } \alpha - \nu^2\beta < 1, f(t) = c_1 e^{\sqrt{1 - (\alpha - \nu^2\beta)t}} + c_2 e^{-\sqrt{1 - (\alpha - \nu^2\beta)t}}, \)

ii) \( \text{when } \alpha - \nu^2\beta = 1, f(t) = c_1 + c_2 t, \)
iii) when $\alpha - \nu^2 \beta > 1$, we have that $f(t) = c_1 \cos((\sqrt{\alpha - \nu^2 \beta - 1})t) + c_2 \sin((\sqrt{\alpha - \nu^2 \beta - 1})t)$.

Next the following theorem shows when base of quasi-Einstein warped product manifold is isometric to a sphere of a particular radius.

**Theorem 3.5.** Let $(M, g)$ be a warped product $B \times_f I$ connected with $(\bar{n} - 1)$-dimensional Riemannian manifold $B$ where $\bar{n} \geq 3$ and one-dimensional Riemannian manifold $I$. If $(M, g)$ is a quasi-Einstein manifold with constant associated scalars $\alpha$ and $\beta$, $U \in \chi(M)$ with respect to semi-symmetric non-metric connection, $P \in \chi(B)$ and the Hessian of $f$ is proportional to metric tensor $g_B$, then $(B, g_B)$ is a $(\bar{n} - 1)$-dimensional sphere of radius $\rho = \frac{\bar{n} - 1}{\sqrt{\bar{n} - 1}}$.

**Proof.** Let $M$ be a warped product manifold. Then from the proposition of [10] we have

$$\tilde{S}^M(X, Y) = \tilde{S}^B(X, Y) + \left[ \frac{H^I(X, Y)}{f} + g(\nabla_X P, Y) - \pi(X) \pi(Y) \right],$$

for any vector field $X, Y$ on $B$. Since $M$ is a quasi-Einstein manifold with respect to semi-symmetric non-metric connection, we have

$$\tilde{S}^M(X, Y) = \alpha g(X, Y) + \beta \eta(X) \eta(Y).$$

Decomposing the vector field $U$ uniquely into its components $U_B$ and $U_I$ on $B$ and $I$, respectively, then we have

$$U = U_B + U_I.$$ 

Since $\dim I = 1$, we can take $U_I = \nu \frac{\partial}{\partial t}$ which gives $U = \nu \frac{\partial}{\partial t} + U_B$, where $\nu$ is a function on $M$. Putting the value of (3.29), (3.30) in (3.28) we get

$$\tilde{S}^B(X, Y) = \alpha g_B(X, Y) + \beta g_B(X, U_B) g_B(Y, U_B)$$

$$- \left[ \frac{H^I(X, Y)}{f} + g(\nabla_X P, Y) - \pi(X) \pi(Y) \right].$$

By contraction over $X$ and $Y$ we get,

$$\tilde{\rho}^B = \tilde{\rho}^M - \alpha - \frac{\Delta f}{f} + \pi(P) - \sum_{i=1}^{\bar{n} - 1} g(e_i, \nabla e_i P).$$

Again from the proposition of [10] we get

$$\frac{\tilde{\rho}^M}{\bar{n}} = (\bar{n} - 1) \frac{P f}{f} - \frac{\Delta f}{f}.$$

From the last two equations we get

$$(\tilde{\rho}^B + \alpha) f = \bar{n} (\bar{n} - 1) P f - (\bar{n} + 1) \Delta f + f \pi(P) - \sum_{i=1}^{\bar{n} - 1} f g(e_i, \nabla e_i P).$$
Hence we get

\[
\frac{(\tilde{\tau}^B + \alpha)f}{\tilde{n}(\tilde{n} - 1)} = Pf - \frac{(\tilde{n} + 1)\Delta f}{\tilde{n}(\tilde{n} - 1)} + \frac{f\pi(P)}{\tilde{n}(\tilde{n} - 1)} - \sum_{i=1}^{\tilde{n}-1} \frac{f g(e_i, \nabla e_i P)}{\tilde{n}(\tilde{n} - 1)}.
\]

Since, the Hessian of \( f \) is proportional to metric tensor \( g_B \), then we have

\[
H^f(X,Y) = \frac{\tilde{n}}{\tilde{n} - 1} \left[ -Pf + \frac{(\tilde{n} + 1)\Delta f}{\tilde{n}(\tilde{n} - 1)} - \frac{f\pi(P)}{\tilde{n}(\tilde{n} - 1)} + \sum_{i=1}^{\tilde{n}-1} \frac{f g(e_i, \nabla e_i P)}{\tilde{n}(\tilde{n} - 1)} \right] g_B(X,Y).
\]

Hence from the equations (3.35), (3.36) we get

\[
H^f(X,Y) + \frac{\tilde{n}^B + \alpha}{(\tilde{n} - 1)^2} f g_B(X,Y) = 0.
\]

So, \( B \) is isometric to the \((\tilde{n} - 1)-\)dimensional sphere of radius \(
\frac{\tilde{n}-1}{\sqrt{\tilde{n}^B + \alpha}} \)
[8]. Thus the theorem is proved. \( \square \)

4. Special Multiply Warped Product Manifolds with Semi-Symmetric Non-Metric Connection

Let \( M = B \times_b F_1 \times_b F_2 \ldots \times_b F_m \) be a multiply warped product with the metric tensor \(-dt^2 + b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}\) and \( I \) is an open interval in \( \mathbb{R} \) and \( b_i \in C^\infty(I) \).

Now, we prove the following theorem for multiply generalized Robertson-Walker space time.

**Theorem 4.1.** Let \( M = I \times_{b_1} F_1 \times_{b_2} F_2 \ldots \times_{b_m} F_m \) be a multiply warped product with the metric tensor \(-dt^2 + b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m}\) and \( P = \frac{\partial}{\partial t} \). Then \((M,g)\) is a quasi-Einstein manifold with respect to semi-symmetric non-metric connection \( \tilde{\nabla} \) with constant associated scalars \( \alpha \) and \( \beta \), if and only if the following conditions are satisfied.

i) \((F_i, g_{F_i})\) is quasi-Einstein manifold with scalar \( \alpha_{F_i}, \beta_{F_i}, i \in \{1,2,\ldots,m\} \),

ii) \(\sum_{i=1}^{m} l_i (1 - \frac{b_i''}{b_i'}) = \alpha - \nu^2 \beta \),

iii) \(\alpha b_i^2 - \alpha_{F_i} + b_i b_i' + (l_i - 1)b_i' + b_i l_i \sum_{j\neq i} \frac{b_j' b_i'}{b_j'} - b_i^2 \sum_{j=1}^{m} \frac{b_j b_i'}{b_j'} = 0 \) and \( \beta = \beta_{F_i} \).

**Proof.** By the proposition of [14] we have

\[
\tilde{S}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -\sum_{i=1}^{m} l_i (1 - \frac{b_i''}{b_i'}),
\]

\[
\tilde{S}(\frac{\partial}{\partial t}, V) = \tilde{S}(V, \frac{\partial}{\partial t}) = 0,
\]
(4.3) \[ S(V, W) = S^F_i(V, W) + g_{Fi}(V, W) \{ -b_i b''_i - (l_i - 1)b'_i \} + b_i b'_i \sum_{j \neq i} l^j_j b'_j \sum_{j=1}^m l^j_j b^j_j \].

Since, \( M \) is a quasi-Einstein manifold. So,

\[ \tilde{S}(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y). \]

Now,

\[ \tilde{S}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = \alpha g(U, \frac{\partial}{\partial t}) + \beta \eta(U)\eta(U) \]

Decomposing the vector field \( U \) uniquely into its components \( U_I \) and \( U_F \) on \( I \) and \( F \), respectively, then we have \( U = U_I + U_F \). Since \( \text{dim} I = 1 \), we can take \( U_I = v \frac{\partial}{\partial t} \) which gives \( U = v \frac{\partial}{\partial t} + U_F \), where \( v \) is a function on \( M \). Then we can write

\[ \eta(\frac{\partial}{\partial t}) = g(U, \frac{\partial}{\partial t}) = v. \]

Hence, we get

\[ \sum_{i=1}^m l_i(1 - \frac{b''_i}{b_i}) = \alpha - v \beta. \]

Again, \( \tilde{S}(V, W) = \alpha g(V, W) + \beta \eta(V)\eta(W) \).

From by the proposition of [14] and the equation (4.3) we get \((F_i, g_{Fi})\) is quasi-Einstein manifold.

Also, after some calculation we can show that

\[ \alpha b_i^2 - \alpha F_i + b_i b''_i + (l_i - 1)b'_i + b_i b'_i \sum_{j \neq i} l^j_j b'_j - b_i^2 \sum_{j=1}^m l^j_j b^j_j = 0 \]

and \( \beta = \beta F_i \). \( \square \)

Next, we have obtained the following theorem with some condition of fibre and warping function with semi-symmetric non-metric connection.

**Theorem 4.2.** Let \( M = I \times_{b_1} F_1 \times_{b_2} F_2 \ldots \times_{b_m} F_m \) be a multiply warped product with the metric tensor \( -dt^2 \oplus b_1^2 g_{F_1} \oplus \cdots \oplus b_m^2 g_{F_m} \) with \( P \in \chi(F_r) \) and \( g_{F_i}(P, P) = 1 \) and \( \bar{n} \geq 3 \). Then \((M, g)\) is a quasi-Einstein manifold with respect to semi-symmetric non-metric connection \( \nabla \) with constant associated scalars \( \alpha \) and \( \beta \), if and only if the following conditions are satisfied.

i) \((F_i, g_{Fi}) \ (i \neq r)\) is quasi-Einstein manifold with scalar \( \alpha_{Fi}, \beta_{Fi} \), \( i \in \{1, 2, \ldots, m\} \).

ii) \[ \sum_{i=1}^m l_i \frac{b''_i}{b_i} = -\alpha + v^2 \beta. \]

iii) \[ \alpha F_i - b_i b''_i - (l_i - 1)b'_i - b_i b'_i \sum_{j \neq i} l^j_j b'_j - \alpha b_i^2 = 0 \] and \( \beta = \beta F_i \).
iv) 
\[S^{F_i}(V, W) - g_{F_i}(V, W)[b_i b''_i + (l_i - 1)b'^2_i + \alpha b'^2_i + b_i b'_i \sum_{j \neq i} l_j \frac{b'_j}{b''_j}] =
(n_i - 1)[\pi(V)\pi(W) - \frac{g(W, \nabla_V P) + g(V, \nabla_W P)}{2}],\]

for \(V, W \in \Gamma(TF_r), r = i.\)

Proof. By the proposition of [14] and \(g_{F_i}(P, P) = 1,\) we have that \(b_r\) is constant. So, we have
\[\tilde{S}(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = \sum_{i=1}^m l_i \frac{b''_i}{b_i} = -\alpha + \nu^2 \beta.\]

By variables separation, we have
\[\tilde{S}(V, W) = S^{F_i}(V, W) + b'_i g_{F_i}(V, W)[-\frac{b''_i}{b_i} - (l_i - 1) \frac{b'^2_i}{b'_i} - \sum_{j \neq i} l_j \frac{b'_j}{b''_j}] + (n_i - 1)[g(W, \nabla_V P) - \pi(V)\pi(W)].\]

When \(i \neq r,\) then \(\pi(V) = \nabla_P V = \nabla_P W = 0.\)

\[\tilde{S}(V, W) = S^{F_i}(V, W) + b'_i g_{F_i}(V, W)[-\frac{b''_i}{b_i} - (l_i - 1) \frac{b'^2_i}{b'_i} - \sum_{j \neq i} l_j \frac{b'_j}{b''_j}] = \alpha b'^2_i g_{F_i}(V, W) + \beta \eta(V)\eta(W).\]

By variables separation, we have \((F_i, g_{F_i}) (i \neq r)\) is quasi-Einstein manifold with scalar \(\alpha_{F_i}, \beta_{F_i}, i \in \{1, 2, \ldots, m\}.\)

When \(i = r,\) we get
\[S^{F_i}(V, W) - g_{F_i}(V, W)[b_i b''_i + (l_i - 1)b'^2_i + \alpha b'^2_i + b_i b'_i \sum_{j \neq i} l_j \frac{b'_j}{b''_j}] =
(n_i - 1)[\pi(V)\pi(W) - \frac{g(W, \nabla_V P) + g(V, \nabla_W P)}{2}]. \]

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References


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