SECOND ORDER PARALLEL TENSORS ON LP-SASAKIAN MANIFOLDS WITH A COEFFICIENT $\alpha$

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Abstract. In 1926, Levy [3] had proved that a second order symmetric parallel nonsingular tensor on a space of constant curvature is a constant multiple of the metric tensor. Sharma [4] has proved that a second order parallel tensor in a Kähler space of constant holomorphic sectional curvature is a linear combination with constant coefficient of the Kählerian metric and the fundamental 2-form. In this paper, we have shown that a second order symmetric parallel tensor on Lorentzian Para Sasakian manifold (briefly LP-Sasakian) with a coefficient $\alpha$ (non zero Scalar function) is a constant multiple of the associated metric tensor and we have also proved that there is no non zero skew symmetric second order parallel tensor on a LP-Sasakian manifold.

1. Introduction

In 1923, Eisenhart [2] showed that a Riemannian manifold admitting a second order symmetric parallel tensor other than a constant multiple of metric tensor is reducible. In 1926 Levy [3] obtained the necessary and sufficient conditions for the existence of such tensors. Sharma [4] has generalized Levy’s result by showing that a second order parallel (not necessarily symmetric and non-singular) tensor on an $n$-dimensional ($n > 2$) space of constant curvature is a constant multiple of the metric tensor. Sharma has also proved in [4] that on a Sasakian manifold, there is no non zero parallel 2-form. In this paper we have defined LP-Sasakian manifold with a coefficient $\alpha$, (non zero scalar function) and have proved the following two theorems:

Theorem 1.1. On a LP- Sasakian manifold with a coefficient $\alpha$, a second order symmetric parallel tensor is a constant multiple of the associated positive definite Riemannian metric tensor.

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Theorem 1.2. On a LP-Sasakian manifold with a coefficient $\alpha$, there is no non zero parallel 2-form.

Let $M$ be an n-dimensional differentiable manifold of class $C^\infty$ endowed with

(1,1) tensor field $\Phi$, a contravariant vector field $T$, a covariant vector field $A$ and a Lorentzian metric $g$ on $M$ which makes $T$ a timelike unit vector field such that the following conditions are satisfied [1].

(1.1) $A(T) = -1$
(1.2) $\Phi(T) = 0$
(1.3) $A(\Phi X) = 0$
(1.4) $\Phi^2 X = X + A(X) T$
(1.5) $A(X) = g(X, T)$
(1.6) $g(\Phi X, \Phi Y) = g(X, Y) + A(X) A(Y)$
(1.7) $\Phi(X, Y) = g(X, \Phi Y) = g(Y, \Phi X) = \Phi(X, Y)$
(1.8) $\Phi(X, T) = 0$.

Then a manifold satisfying conditions (1.1)–(1.8) is called a LP-Sasakian structure $(\Phi, T, A, g)$ on $M$.

Definition 1.1. If in a LP-Sasakian manifold, the following relation

(1.9) $\Phi X = \frac{1}{\alpha}(\nabla_X T)$
(1.10) $\Phi(X, Y) = \frac{1}{\alpha} (\nabla_X A(Y)) = \frac{1}{\alpha} (\nabla_X A)(Y)$
(1.11) $\alpha(X) = \nabla_X \alpha$
(1.12) $g(X, \alpha) = \alpha(X)$
(1.13) $\nabla_X \Phi(Y, Z) = \alpha \{ \{g(X, Y) + \eta(Y) \eta(X)\} \eta(Z) + \{g(X, Z) + \eta(Z) \eta(X)\} \eta(Y)\}$.

hold, where $\nabla$ denotes the Riemannian connection of the metric tensor $g$, then $M$ is called a LP-Sasakian manifold with coefficient $\alpha$.

2. PROOFS OF THEOREM 1.1 AND 1.2

In proving Theorems 1.1 and 1.2 we need the following theorems.

Theorem 2.1. On a LP-Sasakian manifold with coefficient $\alpha$ the following holds

(2.1) $A(R(X, Y) Z) = \alpha^2 [g(Y, Z) A(X) - g(X, Z) A(Y)]$

$\quad - [\alpha(X) \Phi(Y, Z) - \alpha(Y) \Phi(X, Z)]$

Proof. On differentiating (1.10) covariantly and using (1.11), (1.12) and (1.13) the proof follows immediately. \qed
Theorem 2.2. For a LP-Sasakian manifold with coefficient $\alpha$, we have:

\[(2.2) \quad R(T, X) Y = \alpha^2 [A(Y) X + g(X, Y) T] + \alpha (Y) \Phi X - \overline{\alpha} \Phi (X, Y),\]

where $g(X, \overline{\alpha}) = \alpha(X)$.

Proof. The proof follows in an obvious manner after making use of (1.12) and (2.1).

Theorem 2.3. For a LP-Sasakian manifold, with a coefficient $\alpha$ the following holds:

\[(2.3) \quad R(T, X) T = \beta \phi x + \alpha^2 [X + A(X) T]\]

Proof. In view of equation (3.2), the proof follows immediately.

Proof of Theorem 1.1. Let $J$ denote a $(0, 2)$-tensor field on a LP-Sasakian manifold $M$ with a coefficient $\alpha$ such that $\nabla J = 0$, then it follows that

\[(2.4) \quad J(R(W, X) Y, Z) + J(Y, R(W, X) Z) = 0\]

holds for arbitrary vector fields $X, Y, Z, W$ on $M$. Substituting $W = Y = Z = T$ in (2.4) we get

\[(2.5) \quad J(R(T, X) T, T) + J(T, R(T, X) T) = 0.\]

On using Theorem 3.3, the equation (2.5) becomes

\[(2.6) \quad 2\beta J(\Phi X, T) + 2\alpha^2 J(X, T) + 2\alpha^2 g(X, T) J(T, T) = 0.\]

On simplifying (2.6), we get

\[(2.7) \quad -g(X, T) J(T, T) - J(X, T) - \frac{\beta}{\alpha^2} J(\Phi X, T) = 0.\]

Replacing $X$ by $\Phi Y$ in (2.7) we get

\[(2.8) \quad J(\Phi Y, T) = g(\Phi Y, T) J(T, T) + \frac{\beta}{\alpha^2} J(\Phi^2 Y, T)\]

Using (1.4) and (1.5) in the above equation we get

\[(2.9) \quad J(\Phi Y, T) = -\frac{\beta}{\alpha^2} [J(T, T) A(Y) + J(Y, T)]\]

Using (2.7) and (2.9) we get

\[(2.10) \quad J(T, T) A(Y) + J(Y, T) = 0 \text{ if } \alpha^4 + \beta^2 \neq 0.\]

Differentiating (2.10) covariantly with respect to $g$ we get

\[(2.11) \quad J(T, T) g(X, \Phi Y) + 2g(X, T) J(\Phi Y, T) + J(X, \Phi Y) = 0\]

From the above equation and (1.9) we obtain

\[(2.12) \quad J(T, T) g(X, \Phi Y) = -J(X, \Phi Y)\]

Replacing $\Phi y$ by $y$ in (2.12) we get

\[(2.13) \quad J(X, Y) = -J(T, T) g(X, Y)\]
In view of the fact that $J(T, T)$ is constant which can be checked by differentiating it along any vector field on $M$. Thus we have proved the theorem. □

**Proof of Theorem 1.2.** Let $J$ be a parallel 2-form on a LP-Sasakian manifold $M$ with a coefficient $\alpha$. Then putting $W = Y = T$ in (2.4) and using Theorem 3.3 and equations (1.1)–(1.6) we get

\begin{equation}
\beta J(\Phi X, Z) + \alpha^2 J(X, Z) + \alpha^2 J(T, Z) A(X) + \alpha^2 J(T, X) A(Z) + J(T, \Phi X) \alpha(Z) - J(\pi, T) \Phi(X, Z) = 0
\end{equation}

Let us define $\Phi^*$ to be a $(2, 0)$ tensor field metrically equivalent to $\Phi$ then contracting (2.14) with $\Phi^*$ and using the antisymmetry property of $J$ and the symmetry property of $\Phi^*$, we obtain in view of equations (1.3)–(1.6) and after simplifying the following:

\begin{equation}
J(\pi, T) = 0.
\end{equation}

Substituting (2.15) in (2.14) we get

\begin{equation}
\beta J(\Phi X, Z) + \alpha^2 [J(X, Z) + J(T, Z) A(X) + J(T, X) A(Z)] + J(T, \Phi X) \alpha(Z) = 0.
\end{equation}

On simplifying (2.16) we get

\begin{equation}
\beta J(\Phi Z, X) + \alpha^2 [J(Z, X) + J(T, X) A(Z) + J(T, Z) A(X)] + J(T, \Phi Z) \alpha(X) = 0.
\end{equation}

On making use of (1.4) in the above equation, we get the following equation:

\begin{equation}
- \beta[J(Z, \Phi Y) + J(\Phi Y, \Phi Z)] - \alpha(\Phi Y) J(\Phi Z, T) - \alpha(Z) J(\Phi Y, T) = 0.
\end{equation}

In view of (2.20) and (2.21) and after simplifying we obtain

\begin{equation}
\beta[J(T, Z) A(Y) + J(T, Y) A(Y)] + \alpha(Z) J(T, Y) + J(T, \Phi Z) \alpha(\Phi Y) + \alpha(Y) J(Z, T) + \alpha(\Phi Z) J(T, \Phi Y) = 0.
\end{equation}
Putting \( Y = \overline{\alpha} \) in (2.22) and using (2.15) we get
\[
\beta J (T, Z) A (\overline{\alpha}) + J (T, \Phi Z) \alpha (\Phi \overline{\alpha}) + \alpha (\overline{\alpha}) J (Z, T) = 0
\]
(2.23)

Let us put \( \alpha \overline{\alpha} = \widehat{\alpha} \) and \( \widehat{\beta} = \alpha (\Phi, \overline{\alpha}) \) in (2.23) we get
\[
J (Z, T) [\beta A (\overline{\alpha}) - \alpha (\overline{\alpha})] = J(T, \Phi Z) \widehat{\beta}.
\]
(2.24)

Replacing \( Z \) by \( \Phi Z \) in (2.24) we get
\[
J (\Phi Z, T) [\beta^2 - \overline{\alpha}] = \widehat{\beta} J (T, Z).
\]
(2.25)

Replacing \( Z \) by \( \Phi Z \) in (2.25) we get
\[
J (\Phi^2 Z, T) = \frac{\widehat{\beta}}{\alpha - \beta^2} J (\Phi Z, T).
\]
(2.26)

On making use of (2.25) and (1.4) in (2.26) we get
\[
\frac{\alpha - \beta^2}{\beta} J (Z, T) = \frac{\widehat{\beta}}{\alpha - \beta^2} J (Z, T).
\]
(2.27)

From (2.27) it follows immediately that
\[
J (Z, T) = 0 \text{ unless } (\overline{\alpha} - \beta^2)^2 - (\widehat{\beta})^2 \neq 0.
\]
(2.28)

Using (2.28) in (2.28) we get
\[
\beta J (Z, \Phi X) + \alpha^2 J (Z, X) = 0
\]
(2.29)

Differentiating (2.28) covariantly along \( Y \) and using the fact that \( \nabla J = 0 \) we get
\[
J (Z, \Phi Y) = 0.
\]
(2.30)

In view of (2.30) and (2.29), we see that \( J (Y, Z) = 0 \). \( \square \)

REFERENCES


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