MODULES OVER GROUP RINGS OF LOCALLY FINITE GROUPS WITH FINITENESS RESTRICTIONS

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Abstract. We study an $R_G$-module $A$, where $R$ is a ring, $A/C_A(G)$ is infinite, $C_G(A) = 1$, $G$ is a group. Let $\mathcal{L}_{\text{inf}}(G)$ be the system of all subgroups $H \leq G$ such that the quotient modules $A/C_A(H)$ are infinite. We investigate an $R_G$-module $A$ such that $\mathcal{L}_{\text{inf}}(G)$ satisfies either the weak minimal condition or the weak maximal condition as an ordered set. It is proved that if $G$ is a locally finite group then either $G$ is a Chernikov group or $G$ is a finite-finitary group of automorphisms of $A$.

1. Introduction

Important finiteness conditions in group theory are the weak minimal condition on subgroups and the weak maximal condition on subgroups. Let $G$ be a group, $\mathcal{M}$ be a set of subgroups of $G$. $G$ is said to satisfy the weak minimal condition on $\mathcal{M}$-subgroups if for a descending series of subgroups $G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n \geq G_{n+1} \geq \cdots$, $G_n \in \mathcal{M}$, $n \in \mathbb{N}$, there exists the number $m \in \mathbb{N}$ such that an index $|G_n : G_{n+1}|$ is finite for any $n \geq m$ [11]. Similarly $G$ is said to satisfy the weak maximal condition on $\mathcal{M}$-subgroups if for an ascending series of subgroups $G_0 \leq G_1 \leq G_2 \leq \cdots \leq G_n \leq G_{n+1} \leq \cdots$, $G_n \in \mathcal{M}$, $n \in \mathbb{N}$, there exists the number $m \in \mathbb{N}$ such that an index $|G_n : G_{n+1}|$ is finite for any $n \geq m$ [1].

These finiteness conditions were applied to investigate infinite dimensional linear periodic groups [9]. Also similar finiteness conditions were considered in [2].

Let $A$ be an $R_G$-module, $R$ be an associative ring, $G$ be a group. $G$ is a finite-finitary group of automorphisms of $A$ if $C_G(A) = 1$ and $A/C_A(g)$ is finite for any $g \in G$ [10]. Finite-finitary groups of automorphisms of $A$ with additional restrictions were studied in [10].

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Let $\mathcal{L}_{nf}(G)$ be the system of all subgroups $H$ of $G$ such that $A/C_A(H)$ is infinite. Previously, we studied an $RG$-module $A$ with some restrictions on subgroups of $\mathcal{L}_{nf}(G)$ [4, 3, 5].

In this paper we continue these investigations. We say that $G$ satisfies the condition $W_{\min-nf}$ if $G$ satisfies the weak minimal condition on $M$-subgroups where $M = \mathcal{L}_{nf}(G)$ and $G$ satisfies the condition $W_{\max-nf}$ if $G$ satisfies the weak maximal condition on $M$-subgroups where $M = \mathcal{L}_{nf}(G)$.

Next, we consider an $RG$-module $A$ with $C_G(A) = 1$. We investigate an $RG$-module $A$ such that $G$ satisfies either $W_{\min-nf}$ or $W_{\max-nf}$. Main results of the paper are Theorem 1 and Theorem 2.

2. Preliminary results

**Lemma 1.** Let $A$ be an $RG$-module, $R$ be an associative ring. Then the following conditions hold:

1. if $L \leq H \leq G$ and $A/C_A(H)$ is finite then $A/C_A(L)$ is finite also;
2. if $L, H \leq G$, $A/C_A(L)$ and $A/C_A(H)$ are finite then $A/C_A([L,H])$ is finite also.

**Corollary 1.** Let $A$ be an $RG$-module, $R$ be an associative ring, $FFD(G)$ be the set of all elements $x \in G$ such that $A/C_A(x)$ is finite. Then $FFD(G)$ is a normal subgroup of $G$.

**Proof.** By Lemma 1 (2) $FFD(G)$ is a subgroup of $G$. Since $C_A(x^g) = C_A(x)g$ for each $x, g \in G$ then $FFD(G)$ is a normal subgroup of $G$. □

**Lemma 2.** Let $A$ be an $RG$-module, $R$ be an associative ring, $H$ be a subgroup of $G$. Suppose that $H$ contains a normal subgroup $K$ such that $A/C_A(K)$ is infinite. Then the following conditions hold:

1. if $G$ satisfies $W_{\min-nf}$ then $H/K$ satisfies the weak condition of minimality on subgroups;
2. if $G$ satisfies $W_{\max-nf}$ then $H/K$ satisfies the weak condition of maximality on subgroups.

**Lemma 3.** Let $A$ be an $RG$-module, $R$ be an associative ring, $L, K$ and $H$ be subgroups of $G$ with the the following properties:

1. $K$ is a normal subgroup of $L$;
2. $K$ and $L$ are $H$-invariant subgroups;
3. $L/K \cap HK/K = \langle 1 \rangle$;
4. $L/K = \text{Dr}_{n \in \mathbb{N}} L_n/K$, $L_n/K \neq \langle 1 \rangle$ is an $H$-invariant subgroup for any $n \in \mathbb{N}$.

Then the following conditions hold:

1. if $G$ satisfies $W_{\max-nf}$ then $A/C_A(HL)$ is finite;
2. if $G$ satisfies $W_{\min-nf}$ then $A/C_A(HK)$ is finite.

**Proof.** There are two infinite subsets $\Sigma$ and $\Delta$ of $\mathbb{N}$ such that $\Sigma \cup \Delta = \mathbb{N}$, $\Sigma \cap \Delta = \emptyset$. Since $\Delta$ is infinite then there is an infinite strongly ascending...
series of subsets of $\Delta$

$$\Delta(1) \subset \Delta(2) \subset \cdots \subset \Delta(k) \subset \cdots.$$ Also there is strongly descending series of subsets of $\Delta$

$$\Delta^*(1) \supset \Delta^*(2) \supset \cdots \supset \Delta^*(k) \supset \cdots,$$
such that the sets $\Delta(k+1) \setminus \Delta(k)$ and $\Delta^*(k) \setminus \Delta^*(k+1)$ are infinite for any $n \in \mathbb{N}$. Let

$$D_k/K = Dr_{t \in \Sigma \cup \Delta(k)} L_t/K$$

and

$$D^*_k/K = Dr_{t \in \Sigma \cup \Delta^*(k)} L_t/K.$$ At first we consider the strongly ascending series of subgroups

$$HD_1 < HD_2 < \cdots < HD_k < \cdots.$$ 

$|HD_{k+1} : HD_k|$ are infinite by construction. If $G$ satisfies $W_{\text{max-nf}}$ then there is $m \in \mathbb{N}$ such that $A/C_A(HD_m)$ is finite. Since $\langle H, L_t | t \in \Sigma \rangle \leq HD_m$ then $A/C_A(\langle H, L_t | t \in \Sigma \rangle)$ is finite by Lemma 1. Similarly we prove that $A/C_A(\langle H, L_t | t \in \Delta \rangle)$ is finite.

Since $\Sigma \cup \Delta = \mathbb{N}$ we obtain

$$\langle \langle H, L_t | t \in \Delta \rangle, \langle H, L_t | t \in \Sigma \rangle \rangle = \langle H, L_t | t \in \Sigma \cup \Delta \rangle = HL.$$ By Lemma 1 $A/C_A(HL)$ is finite.

Likewise we can construct the strongly descending series of subgroups

$$HD^*_1 > HD^*_2 > \cdots > HD^*_k > \cdots,$$
such that $|HD^*_k : HD^*_k+1|$ are infinite. If $G$ satisfies $W_{\text{min-nf}}$ then there is $m \in \mathbb{N}$ such that $A/C_A(HD^*_m)$ is finite. Since $HK \leq HD^*_m$ then $A/C_A(HK)$ is finite by Lemma 1. □

**Corollary 2.** Let $A$ be an $\mathbf{R}G$-module, $\mathbf{R}$ be an associative ring, $L$, $K$ and $H$ be subgroups of $G$ with the the following properties:

(i) $K$ is a normal subgroup of $L$;

(ii) $K$ and $L$ are $H$-invariant subgroups;

(iii) $L/K = Dr_{n \in \mathbb{N}} L_n/K$ where $L_n/K \neq \langle 1 \rangle$ is an $H$-invariant subgroup for any $n \in \mathbb{N}$;

(iv) the set $\mathbb{N} \setminus \text{Supp}(L/K \cap HK/K)$ is infinite.

If $G$ satisfies either $W_{\text{min-nf}}$ or $W_{\text{max-nf}}$ then $A/C_A(HK)$ is finite. In particular $A/C_A(H)$ is finite.

**Proof.** Let $\Delta = \mathbb{N} \setminus \text{Supp}(L/K \cap HK/K)$ and $T/K = Dr_{n \in \Delta} L_n/K$. Then $T/K \cap HK/K = \langle 1 \rangle$. We apply Lemma 3. □

**Corollary 3.** Let $A$ be an $\mathbf{R}G$-module, $\mathbf{R}$ be an associative ring, $L$, $K$ and $H$ be subgroups of $G$ with the the following properties:

(i) $K$ is a normal subgroup of $L$;

(ii) $K$ and $L$ are $H$-invariant subgroups;
(iii) \( L/K = \text{Dr}_{n \in \mathbb{N}} L_n/K, L_n/K \neq \langle 1 \rangle \) is an \( H \)-invariant subgroup for any \( n \in \mathbb{N} \).

If \( G \) satisfies either \( W_{\text{min-nf}} \) or \( W_{\text{max-nf}} \) then \( A/C_A(\langle h \rangle K) \) is finite for any \( h \in H \). In particular \( H \leq \text{FFD}(G) \).

Proof. Let \( h \in H \). Since \( L_n/K \) is an \( H \)-invariant subgroup for any \( n \in \mathbb{N} \) then \( L_n/K \) is an \( \langle h \rangle \)-invariant subgroup for any \( n \in \mathbb{N} \). In particular the set \( \text{Supp}(\langle h \rangle K/K \cap L/K) \) is finite. Then \( A/C_A(\langle h \rangle K) \) is finite by Corollary 2. \( \square \)

3. Main results

Obviously that a Chernikov group satisfies both the weak minimal condition on subgroups and the weak maximal condition on subgroups. It follows that if \( A \) is an \( R \mathbb{G} \)-module and \( G \) is Chernikov then \( G \) satisfies both \( W_{\text{min-nf}} \) and \( W_{\text{max-nf}} \).

Lemma 4. Let \( A \) be an \( R \mathbb{G} \)-module, \( R \) be an associative ring. Suppose that \( G \) satisfies either \( W_{\text{min-nf}} \) or \( W_{\text{max-nf}} \). Let \( K \) and \( H \) be subgroups of \( G \) such that \( K \) is a normal subgroup of \( H \) and \( H/K \) is an infinite elementary abelian \( p \)-group for some prime \( p \). Suppose that \( K \) and \( H \) are \( \langle g \rangle \)-invariant for some \( g \in G \). If \( g^k \in C_G(\langle h \rangle K) \) for some \( k \in \mathbb{N} \) then \( g \in \text{FFD}(G) \).

Proof. Let \( M = H/K \). We take \( 1 \neq b_1 \in M \). Put \( B_1 = \langle b_1 \rangle \). Since \( g \) induces the automorphism of finite order on \( M \) then \( B_1 \) is finite. \( M = B_1 \times C_1 \) is valid for some subgroup \( C_1 \).

Let

\[ \{C_1^\sigma | y \in \langle g \rangle \} = \{U_1, \ldots, U_m\}. \]

It follows that the \( \langle g \rangle \)-invariant subgroup

\[ D_1 = U_1 \cap \cdots \cap U_m \]

has finite index in \( M \). Let \( 1 \neq b_2 \in D_1 \) and \( B_2 = \langle b_2 \rangle \). Then \( \langle B_1, B_2 \rangle = B_1 \times B_2 \). As before we conclude that \( M = (B_1 \times B_2) \times C_2 \) for some subgroup \( C_2 \). Similarly we can construct the infinite set \( \{B_n | n \in \mathbb{N} \} \) of non-trivial \( \langle g \rangle \)-invariant subgroups such that \( \langle B_n | n \in \mathbb{N} \rangle = \text{Dr}_{n \in \mathbb{N}} B_n \). By Corollary 3 we have that \( g \in \text{FFD}(G) \). \( \square \)

Let \( \pi(G) \) be the set of all prime divisors of orders of elements of \( G \).

Corollary 4. Let \( A \) be an \( R \mathbb{G} \)-module, \( R \) be an associative ring. Suppose that \( G \) satisfies either \( W_{\text{min-nf}} \) or \( W_{\text{max-nf}} \). Let \( K \) and \( H \) are subgroups of \( G \) such that \( K \) is a normal subgroup of \( H \), \( H/K \) is a periodic almost locally solvable group. If \( H/K \) is not Chernikov then \( H \leq \text{FFD}(G) \).

Proof. Let \( L/K \) be a locally solvable normal subgroup of \( H/K \) of finite index. Since \( H/K \) is not Chernikov then \( L/K \) is not Chernikov too. Let \( g \) be an element of \( H \). Then \( L/K \) contains an abelian \( \langle g \rangle \)-subgroup \( C/K \) which is not Chernikov [12]. If the set \( \pi(C/K) \) is infinite then \( g \in \text{FFD}(G) \) by Corollary
3. If \( \pi(C/K) \) is finite then there is the prime \( p \) such that Sylov \( p \)-subgroup \( P/K \) of \( C/K \) is not Chernikov. It follows that the lower layer \( B/K \) of \( P/K \) is infinite. Therefore \( L/K \) contains a \( \langle g \rangle \)-invariant infinite elementary abelian subgroup \( B_1/K \). Then \( g \in FFD(G) \) by Lemma 4.

**Corollary 5.** Let \( A \) be an \( RG \)-module, \( R \) be an associative ring. Suppose that \( G \) satisfies either \( W_{\text{min-nf}} \) or \( W_{\text{max-nf}} \). Let \( K \) and \( H \) be subgroups of \( G \) such that \( K \) is a normal subgroup of \( H \), \( H/K \) is a locally finite group. If \( H/K \) is not Chernikov then \( H \) is an almost locally solvable group \([6]\) and \( g \in FFD(G) \) by Corollary 4. We have that \( H \) is solvable.

It follows that Theorem 1 is valid.

**Theorem 1.** Let \( A \) be an \( RG \)-module, \( R \) be an associative ring, \( G \) be a locally finite group. If \( G \) satisfies either \( W_{\text{min-nf}} \) or \( W_{\text{max-nf}} \) then either \( G \) is a Chernikov group or \( G \) is a finite-finitary group of automorphisms of \( A \).

**Lemma 5.** Let \( A \) be a \( RG \)-module, \( R \) be an associative ring, \( G \) be a locally solvable group. Suppose that \( A/C_A(G) \) is finite. Then \( G \) is almost abelian.

**Proof.** Let \( C = C_A(G) \). Then \( A \) has the series of \( RG \)-submodules \( \langle 0 \rangle \leq C \leq A \), where \( A/C \) is a finite \( R \)-module. Since \( G \leq C_G(C) \) then \( G/C_G(C) \) is trivial. Hence, \( G/C_G(A/C) \) is finite.

Let \( H = C_G(C) \cap C_G(A/C) \). Each element of \( H \) acts trivially on every factor of the series \( \langle 0 \rangle \leq C \leq A/C \). By Kaluzhnin Theorem (p. 144 [7]) \( H \) is abelian. By Remak’s Theorem

\[
G/H \leq G/C_G(C) \times G/C_G(A/C).
\]

It follows that \( G/H \) is finite. Then \( G \) is an almost abelian group.

Let \( G_{\Sigma} \) be the intersection of all normal subgroups \( K \) of \( G \) such that \( G/K \) is solvable. If \( G \) is a solvable group then we denote the step of solvability of \( G \) by \( s(G) \).

**Theorem 2.** Let \( A \) be an \( RG \)-module, \( R \) be an associative ring, \( G \) be a locally solvable periodic group. If \( G \) satisfies either \( W_{\text{min-nf}} \) or \( W_{\text{max-nf}} \) then \( G/G_{\Sigma} \) is solvable.

**Proof.** Otherwise \( H = G/G_{\Sigma} \) is unsolvable. Let \( F_1 \) be a finite subgroup of \( H \). Since \( H \) is approximated by solvable subgroups then there is a normal subgroup \( K_1 \) of \( H \) such that \( F_1 \cap K_1 = \langle 1 \rangle \) and \( H/K_1 \) is solvable. It follows that \( K_1 \) is unsolvable. Therefore the steps of solvability of finite subgroups of \( K_1 \) not limited by the number. Then \( K_1 \) contains a finite subgroup \( D_1 \) such
that \( s(F_1) < s(D_1) \). Since \( F_1 \) and \( D_1 \) are finite then they are solvable. Let \( F_2 = D_1^{F_1} \). Then \( F_2 \) is a finite \( F_1 \)-invariant subgroup such that \( s(F_1) < s(F_2) \). Since \( F_1 F_2 \) is finite there is a normal subgroup \( K_2 \) of \( H \) such that \( F_1 F_2 \cap K_2 = \langle 1 \rangle \) and \( H/K_2 \) is solvable. Since \( K_2 \) is unsolvable then we can choose a finite \( F_1 F_2 \)-invariant subgroup \( F_3 \) of \( K_2 \) such that \( s(F_2) < s(F_3) \). Continuing our reasoning, we construct the strongly ascending series of finite subgroups
\[
F_1 < F_1 F_2 < \cdots < F_1 F_2 \cdots F_n < \cdots
\]
with the the following properties:

(i) \( F_n \) is an \( F_j \)-invariant subgroup for \( j < n \);
(ii) \( s(F_j) < s(F_n) \) for \( j < n \);
(iii) \( F_1 F_2 \cdots F_n \cap \langle F_j \mid j > n \rangle = \langle 1 \rangle \) for any \( n \in \mathbb{N} \).

It follows that \( \langle F_j \mid j \in \Delta \rangle \) is decomposed in the direct product of \( F_j, j \in \Delta \),
for an infinite subset \( \Delta \) of \( \mathbb{N} \). Therefore \( \langle F_j \mid j \in \Delta \rangle \) is unsolvable.

At first we suppose that \( G \) satisfies \( W_{\text{min-nf}} \). There is an infinite strictly descending series of subsets
\[
\mathbb{N} \supset \Delta(1) \supset \Delta(1) \supset \cdots \supset \Delta(k) \supset \cdots
\]
such that \( \Delta(k) \setminus \Delta(k+1) \) is infinite for any \( k \in \mathbb{N} \). Let \( L_k = \langle F_j \mid j \in \Delta(k) \rangle \) for any \( k \in \mathbb{N} \). We obtain the strongly descending series of subgroups \( L_1 > L_2 > \cdots > L_k > \cdots \) of \( H \). Let \( M_k \) be the preimage of \( L_k \) in \( G \). Then \( M_1 > M_2 > \cdots > M_k > \cdots \) is the strongly descending series of subgroups of
\( G \) such that \( |M_k : M_{k+1}| \) are infinite. Hence there is \( t \in \mathbb{N} \) such that \( A/C_A(M_t) \) is finite. \( M_t \) is solvable by Lemma 5. It follows that \( L_t = M_t/G_{\Delta} \) is solvable. Previously, we proved that \( L_t = \langle F_j \mid j \in \Delta(t) \rangle \) is unsolvable. Contradiction.

If \( G \) satisfies \( W_{\text{max-nf}} \) we construct an infinite strictly ascending series of subsets of \( \mathbb{N} \) and conduct similar reasoning.

When writing the paper the author used the methods of [9].

References


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